ADDENDUM TO "*FK*-SPACES CONTAINING c_0"

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1. Introduction. In this note we shall improve some of the results of our paper [2], in particular Theorems 16 and 24. The former result is improved by using a refinement of the closed graph theorem [2; Theorem 13] (see also [4]), which enables us to replace the assumption of separability of F by the assumption that F contains no subspace isomorphic to m. In the latter result we can replace m_0 by any dense subspace of m, and this leads to an interesting extension of the Bachelis-Rosenthal theorem on biorthogonal systems in separable Fréchet spaces.

We shall use the same notation as that of [2].

2. A closed graph theorem and applications. Let E and F be locally convex spaces and let $T : E \to F$ be a linear map. We shall say that T is *subcontinuous* if whenever $\sum_{i=1}^{\infty} x_i$ is subseries convergent in E, then $\sum_{i=1}^{\infty} Tx_i$ converges in F and

$$\sum_{i=1}^{\infty} Tx_i = T\left(\sum_{i=1}^{\infty} x_i\right).$$

Thus the Orlicz-Pettis theorem may be interpreted as saying the identity map is subcontinuous from the weak topology on E to the original topology.

THEOREM 1. *Let F be a fully complete space containing no subspace isomorphic to m, and suppose $T : E \to F$ has closed graph. Then T is subcontinuous.*

Proof. If T fails to be subcontinuous, then, since T has closed graph, there exists a series $\sum x_i$ which is subseries convergent in E but such that $\sum Tx_i$ fails to converge in F. Since F is complete, we may further suppose that for some continuous semi-norm p on F we may have $p(Tx_n) \geq 1$ for all n. Hence there is an equicontinuous sequence f_n of linear functionals on F such that $f_n(Tx_n) \geq 1$ for all n. We define a map $R : F \to m$ by $Ry = \{f_n(y)\}_{n=1}^{\infty}$; hence R is continuous.

We also define a map $S : m_0 \to E$ by $S(a) = \sum_{i=1}^{\infty} a_i x_i$; the map S is continuous for the norm topology on m. It follows that TS has closed graph, $TS : m_0 \to F$. Since F is fully complete and m_0 is barrelled [3], we may conclude that TS is continuous [5; 116]. As F is complete we may extend TS to a continuous linear map $V : m \to F$. Now consider $RV : m \to m$; we have $\|RV(e^{(n)})\| \geq 1$ for all n. Hence by the Orlicz-Pettis theorem $\sum_{n=1}^{\infty} RVe^{(n)}$ cannot converge weakly subseries, and it follows easily that RV is not weakly compact. We

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now quote Corollary 1.4 of Rosenthal [7] which says that RV is an isomorphism on some subspace H of m which is itself isomorphic to m. Hence $V(H)$ is isomorphic to m and F contains a subspace isomorphic to m, contrary to the assumption.

PROPOSITION 1. *Let E be an FK-space containing c_0, and let F be a locally convex space; let $T : W_E \cap m \to F$ be a linear map which is subcontinuous for the topology $\sigma(W_E \cap m, l)$ on $W_E \cap m$. Then T is weakly continuous.*

Proof. Let $\psi \in F'$; then ψT is subcontinuous on $(c_0, \sigma(c_0, l))$ and this implies that ψT is continuous on c_0 in the norm topology. Hence if $\psi T(e^{(n)}) = f_n$, then $f = (f_n) \in l$. Now the construction in [2; Lemma 4] combined with Theorem 2 implies that if $x \in W \cap m$, there is a series $x^{(n)}$ with $x^{(n)} \in \phi$ such that $x = \sum_{n=1}^{\infty} x^{(n)}(\sigma(W \cap m, l))$ subseries. Hence

$$\begin{aligned}\psi T(x) &= \sum_{n=1}^{\infty} \psi T(x^{(n)}) \\ &= \sum_{n=1}^{\infty} f(x^{(n)}) \\ &= f(x),\end{aligned}$$

where $f(x) = \sum_{i=1}^{\infty} f_i x_i$. Therefore $\psi T = f \in l$ and so T is weakly continuous.

THEOREM 2. *Let E be an FK-space containing c_0 and let F be an FK-space containing no subspace isomorphic to m. If $W_E \cap m \subseteq F$, then $W_E \cap m \subseteq W_F$.*

Proof. It follows from Theorem 1 and Proposition 1 that the inclusion mapping $W_E \cap m \to F$ is continuous for the topologies $\sigma(W_E \cap m, l)$ and $\sigma(F, F')$ and so the result follows.

We have specifically improved [2; Theorem 16], but the same improvement may be made in each of Theorems 15-23. In each case the assumption of separability may be replaced by the assumption that the space contains no subspace isomorphic to m.

3. FK-spaces containing m. As remarked in [2] and proved in [3], the assumption of separability of E in Theorem 24 is unnecessary since m_0 is barrelled in m in the norm topology. However if E is separable, we do not require this special property of m_0; indeed any dense subspace of m will suffice.

THEOREM 3. *Let E be a separable FK-space containing ϕ; suppose $\overline{E \cap m}$ has separable quotient in m. Then $m \subseteq E$.*

Proof. Let $F = c_0 + E$; then F is also a separable FK-space and $\overline{F \cap m}$ has separable quotient in m. Let $\{x^{(n)} : n = 1, 2, \cdots\}$ be a sequence in m such that lin $(F \cap m, x^{(1)}, x^{(2)}, \cdots)$ is dense in m.

We shall show that $\sigma(l, F \cap m)$ and the norm topology have the same convergent sequences. Suppose $a^{(n)} \in l$ and $a^{(n)} \to 0$ $(\sigma(l, F \cap m))$ but $\|a^{(n)}\|_1 \geq 1$ for all n. Then $a^{(n)} \to 0$ $(\sigma(l, c_0))$ and so $\sup_n \|a^{(n)}\| < \infty$; thus by selection of a subsequence we may suppose that $\lim_{n\to\infty} \sum_{i=1}^{\infty} a_i^{(n)} x_i^{(m)}$ exists for each $m = 1, 2, \cdots$. It follows that for x in a dense subset of m, $\lim_{n\to\infty} \sum_{i=1}^{\infty} a_i^{(n)} x_i$

converges and that $\sup_n \|a^{(n)}\|_1 < \infty$. Hence $a^{(n)}$ is $\sigma(l, m)$-Cauchy and so $a^{(n)} \to a$ in the norm topology on l, where $a \in l$. Clearly $a = 0$ and we have a contradiction.

The conclusions are that $\tau(F \cap m, l)$ is the restriction of $\tau(m, l)$ to $F \cap m$ and that $\sigma(l, F \cap m)$ is sequentially complete. Consider the inclusion map $F \cap m \to F$; then by [2; Theorem 13] we conclude that this map is continuous for the topology $\tau(F \cap m, l)$ on $F \cap m$. Now if $x \in m$, then $P_n x$ is $\tau(m, l)$-Cauchy and hence is $\tau(F \cap m, l)$-Cauchy in $F \cap m$. Therefore $P_n x$ is Cauchy in F and so $x \in F$; therefore $m \subseteq F$.

Then we have $E + c_0 \supseteq m$ and by [2; Theorem 25], $E \supseteq m$.

The assumption that E is separable is crucial here, for we may construct a BK-space E with $c_0 \subset E \subset m$ and with $E \neq m$ such that E is dense in m. For Rosenthal [6] has shown that c_0 is quasi-complemented in m; suppose X is a quasi-complement, then $c_0 + X$ is a BK-space (in the direct sum norm) and dense in m. However $c_0 + X \neq m$ since c_0 is not complemented in m.

This has an obvious application to the Bachelis-Rosenthal theorem [1] on biorthogonal systems in separable Fréchet spaces.

THEOREM 4. *Let E be a separable Fréchet space with a total biorthogonal system $\{\langle x_i, f_i\rangle\}_{i=1}^{\infty}$; let $x \in E$ and let $M(x)$ be the space of bounded sequences $\{t_n\}$ such that for some $y \in E$, $f_n(y) = t_n f_n(x)$. Then if $M(x)$ is dense in m, then $M(x) = m$ and $\sum_{i=1}^{\infty} f_i(x)x_i = x$ unconditionally.*

The distinction between this result and the result of Bachelis and Rosenthal is that they require weaker conditions on E (that E contains no subspace isomorphic to m) but stronger conditions on $M(x)$ (that $M(x) \supseteq m_0$). The stronger conditions on $M(x)$ are necessary as they use the fact that m_0 is barrelled.

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