$G(x, y, y') = xy' - y$, in which case all successive separants vanish at $x = 0$. Our proof goes as follows:

First, if $\frac{\partial^k G}{\partial (y')^k}(x, y(x), y'(x)) = 0$ for all $x$ in $I$ and for all $k$, then we are reduced to the case of a lower-order equation since $G$ is then independent of $y'$ (that is a simple version of our proof of equation (3.3) in Section 3).

If not, then let $k$ be the smallest integer such that $\frac{\partial^k G}{\partial (y')^k}(x, y(x), y'(x))$ is not identically 0 in $I$. Then since $y$ is analytic, $\frac{\partial^k G}{\partial (y')^k}(x, y(x), y'(x))$ is analytic and thus has finitely many zeros in $I$. This partitions $I$ into finitely many subintervals on which one considers the equation $\frac{\partial^{k-1} G}{\partial (y')^{k-1}}(x, y(x), y'(x)) = 0$ and proceeds as usual.

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# Zeros of Entire Functions and Characteristic Determinants

C. M. EOFF
*Department of Mathematics, University of Arkansas, Fayetteville, Arkansas 72701*
and
N. J. KALTON[1]
*Department of Mathematics, University of Missouri-Columbia, Columbia, Missouri 65211*

We study certain classes of entire functions which arise as characteristic determinants of trace-class operators on a Hilbert space. In particular, we show that the property of being a finite linear combination of commutators of Hilbert–Schmidt operators can be characterized by a simple growth condition on the characteristic determinant. We give applications both to entire functions and operator theory.

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## 1. INTRODUCTION

The motivation for this paper comes from the study of commutators of operators on a Hilbert space. It is shown in [5] (see also [1] and [9]) that a trace-class operator $T$ on a complex Hilbert space can be written as a finite sum of commutators of Hilbert–Schmidt operators if and only if the non-zero eigenvalues $(\lambda_n)$ of $T$, arranged so that $|\lambda_1| \geqslant |\lambda_2| \geqslant \cdots$, and repeated according to multiplicity, satisfy:

$$\sum_{n=1}^{\infty} \frac{|\lambda_1 + \cdots + \lambda_n|}{n} < \infty. \tag{1}$$

(If $T$ has only finitely many non-zero eigenvalues we define the remaining $\lambda_n = 0$.) Note that condition (1) implies, using Lidskii's theorem [4] that $\sum \lambda_n = \operatorname{tr} T = 0$. The set of all finite combinations of such commutators is denoted by $\mathscr{C}h_1$.

It is natural to try to relate the above characterization to the characteristic determinant of $T$ (cf. [4]) which is given by

$$D_T(z) = \prod_{n=1}^{\infty} (1 - \lambda_n z). \tag{2}$$

If $T$ is trace-class then $D_T$ is an entire function of minimal type and convergence class

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(see Boas [2] or Levin [7]). As we shall see the condition $T \in \mathscr{C}h_1$ is then equivalent to:

$$\int_0^\infty \int_0^{2\pi} \log_+ |D_T(re^{i\theta})| \, d\theta \frac{dr}{r^2} < \infty. \tag{3}$$

We shall also see that the behavior of $D_T$ on the real axis determines whether $\Re T = \frac{1}{2}(T + T^*) \in \mathscr{C}h_1$. In fact $\Re T \in \mathscr{C}h_1$ if and only if

$$\int_{-\infty}^\infty \log_+ |D_T(x)| \frac{dx}{x^2} < \infty. \tag{4}$$

Condition (4) is also equivalent to the condition that the operator $T$ defined on a real Hilbert space can be written as a sum of Hilbert–Schmidt commutators.

As will be seen, our results are obtained by simply relating growth rates of entire functions to the distribution of their zeros, and no detailed knowledge of [5] is necessary to read the paper. We also give some applications to the theory of entire functions which we hope are of some interest. These show a connection between the theory of certain classes of entire functions and the sequence space $h_1^{\mathrm{sym}}$ introduced in [5] and defined below.

## 2. PRELIMINARIES

Let $f$ be an entire function of exponential type (order one and finite type) with non-zero zeros $(a_n)$ repeated according to multiplicity and arranged so that $|a_1| \leqslant |a_2| \leqslant \cdots$. Let $\lambda_n = a_n^{-1}$ and let $\lambda_n = 0$ eventually if $f$ has only finitely many zeros. By Hadamard's theorem $f$ has a factorization of the form:

$$f(z) = z^m e^{\alpha + \beta z} \prod_{n=1}^\infty (1 - \lambda_n z) e^{\lambda_n z}. \tag{5}$$

The product converges uniformly on compact sets. We then define $n(r)$ to be the number of $a_n$ such that $|a_n| \leqslant r$, and we also set

$$S(r) = \beta + \sum_{|a_n| \leqslant r} \lambda_n. \tag{6}$$

$f$ is said to be of convergence class if $\sum |\lambda_n| < \infty$ or equivalently if $\int n(r) \frac{dr}{r^2} < \infty$. In this case it is possible to write:

$$f(z) = z^m e^{\alpha + \gamma z} \prod_{n=1}^\infty (1 - \lambda_n z)$$

where $\gamma = \beta + \sum \lambda_n = \lim_{r \to \infty} S(r)$. Notice that, if $\gamma = 0$ then $f$ is of minimal type (i.e. $|f(z)| = O(e^{\varepsilon |z|})$ for every $\varepsilon > 0$). See Boas [2] and Levin [7] for further details.

Unfortunately the collection of functions of minimal type and convergence class is not a linear space (see Boas [2]). We now introduce a closely related linear space of entire functions. We say that an entire function $f$ is of Valiron class ($f \in \mathscr{V}$) if



and only if

$$\int_0^\infty \int_0^{2\pi} \log_+ |f(re^{i\theta})| \, d\theta \frac{dr}{1 + r^2} < \infty. \tag{7}$$

If $M(f, r) = \max_{|z| = r} |f(z)|$ then

$$\log_+ M(f, r) \leqslant \frac{3}{2\pi} \int_0^{2\pi} \log_+ |f(2re^{i\theta})| \, d\theta$$

and so $f \in \mathscr{V}$ if and only if

$$\int_1^\infty \log_+ M(f, r) \frac{dr}{r^2} < \infty.$$

Thus we obtain (see Levin [7], Valiron [8]):

THEOREM 2.1 *$f \in \mathscr{V}$ if and only if $f$ is of exponential type, convergence class and*

$$\int_1^\infty \frac{|S(r)|}{r} dr < \infty.$$

*In particular, $f$ is of minimal type* $\left(\text{since } \lim_{r \to \infty} S(r) = 0\right)$.

Notice that $\mathscr{V}$ is a linear space which can be studied as an $F$-space (complete metric linear space). See Eoff [3] for details.

We shall say that an entire function $f$ is in the Cartwright class $\mathscr{C}$ [6] if $f$ is of exponential type and

$$\int_{-\infty}^\infty \frac{\log_+ |f(x)|}{1 + x^2} dx < \infty.$$

Obviously $\mathscr{V} \subset \mathscr{C}$ and $\mathscr{C}$ is also a linear space.

Finally we let $h_1^{\mathrm{sym}}$ be the space of all complex sequences $\lambda = (\lambda_n)$ such that $\sum |\lambda_n| < \infty$ and

$$\sup_{\phi \in \mathscr{L}} \left| \sum_{n=1}^\infty \lambda_n \phi(\log |\lambda_n|) \right| < \infty \tag{8}$$

where $\mathscr{L}$ is the class of all bounded functions $\phi: \mathbb{R} \to \mathbb{R}$ satisfying the Lipschitz estimate $|\phi(s) - \phi(t)| \leqslant |s - t|$. We make the convention that $\phi(-\infty) = 0$.

It is shown in [5] that $h_1^{\mathrm{sym}}$ is a linear space which becomes a quasi-Banach space under the quasi-norm

$$\|\lambda\| = \sum |\lambda_n| + \sup_{\phi \in \mathscr{L}} \left| \sum \lambda_n \phi(\log |\lambda_n|) \right|$$

and that $\lambda \in h_1^{\mathrm{sym}}$ if and only if $\lambda \in l_1$ and

$$\sum_{n=1}^\infty \frac{|\tilde{\lambda}_1 + \cdots + \tilde{\lambda}_n|}{n} < \infty \tag{9}$$

where $(\tilde{\lambda}_n)$ is any rearrangement of $(\lambda_n)$ such that $|\tilde{\lambda}_1| \geqslant |\tilde{\lambda}_2| \geqslant \cdots$. Notice that for any sequence $\lambda \in h_1^{\text{sym}}$ we have $\sum \lambda_n = 0$. One simple fact will be used frequently below: If $(\mu_n) \in l_1$ then the sequence $(\mu_1, -\mu_1, -\mu_2, \ldots) \in h_1^{\text{sym}}$.

## 3. ZEROS OF ENTIRE FUNCTIONS

Our first result will allow us to restate Theorem 2.1.

PROPOSITION 3.2 *Let $(\lambda_n)$ be a sequence of complex numbers and let $(s_n)$ be a sequence of non-negative real numbers such that $|\lambda_n| \leqslant s_n$ and $\sum s_n < \infty$. Let $\beta = -\sum \lambda_n$ and let*

$$\sigma(r) = \beta + \sum_{rs_n \geqslant 1} \lambda_n.$$

*Then $(\beta, \lambda_1, \lambda_2, \ldots) \in h_1^{\text{sym}}$ if and only if*

$$\int_1^\infty \frac{|\sigma(r)|}{r}\, dr < \infty.$$

*Proof* For convenience we may rearrange $(s_n)$ and $(\lambda_n)$ so that $(s_n)$ is monotone decreasing.

If $g \in L_\infty(\mathbb{R})$ is such that $\|g\|_\infty \leqslant 1$ and

$$\sup_{-\infty < t < \infty} \left| \int_0^t g(e^{-x})\, dx \right| < \infty \tag{9}$$

then for $\alpha \in \mathbb{R}$ we can define $\phi \in \mathscr{L}$ by

$$\phi(t) = \alpha - \int_0^t g(e^{-x})\, dx. \tag{10}$$

Conversely given $\phi \in \mathscr{L}$ we can let $\alpha = \phi(0)$ and find $g$ satisfying $\|g\|_\infty \leqslant 1$ and (9) and (10).

Suppose then that $\phi \in \mathscr{L}$ and $g \in L_\infty(\mathbb{R})$ are related by (10). Let $p$ be the first index such that $s_p < 1$. For $\tau > 1$ let $q$ be the last index such that $s_q > \tau^{-1}$. Then

$$\int_1^\tau \frac{\sigma(r)g(r)}{r}\, dr = -\sum_{k=p}^{q} \lambda_k \phi(\log s_k) - \sigma(1)\phi(0) + \sigma(s_q^{-1})\phi(-\log \tau).$$

Since $\phi$ is bounded and $\sigma(\tau) \to 0$ as $\tau \to \infty$ we have

$$\int_1^\infty \frac{\sigma(r)g(r)}{r}\, dr = -\sum_{k=p}^{\infty} \lambda_k \phi(\log s_k) - \sigma(1)\phi(0)$$
$$= -\sum_{k=p}^{\infty} \lambda_k \phi(\log s_k) - \phi(0)\left(\sum_{k=1}^{p-1} \lambda_k + \beta\right).$$



For convenience let us set $\lambda_0 = \beta$, $s_0 = |\beta|$. Then

$$\left| \int_1^\infty \frac{\sigma(r)g(r)}{r}\, dr + \sum_{k=0}^{\infty} \lambda_k(\phi(\log s_k)) \right| = \left| \sum_{k=0}^{p-1} \lambda_k(\phi(\log s_k) - \phi(0)) \right|$$
$$\leqslant \sum_{k=0}^{p-1} s_k |\log s_k|.$$

Now

$$\left| \sum_{k=0}^{\infty} \lambda_k(\phi(\log s_k) - \phi(\log|\lambda_k|)) \right| \leqslant \sum_{k=0}^{\infty} |\lambda_k| \log\left(\frac{s_k}{|\lambda_k|}\right)$$
$$\leqslant \frac{1}{e} \sum_{k=0}^{\infty} s_k$$
$$< \infty.$$

Thus

$$\left| \int_1^\infty \frac{\sigma(r)g(r)}{r}\, dr + \sum_{k=0}^{\infty} \lambda_k \phi(\log|\lambda_k|) \right| \leqslant C \tag{11}$$

where $C$ is a constant independent of $g$ and $\phi$.

Suppose first that $(\beta, \lambda_1, \lambda_2, \ldots) \in h_1^{\text{sym}}$. Then we may allow $g$ to vary over functions of compact support in the above equation (11) to obtain

$$\int_1^\infty \frac{|\sigma(r)|}{r}\, dr < \infty.$$

Conversely if $\displaystyle\int_1^\infty \frac{|\sigma(r)|}{r}\, dr < \infty$ then (11) implies that $(\beta, \lambda_1, \lambda_2, \ldots) \in h_1^{\text{sym}}$.

THEOREM 3.2 *Let $f$ be an entire function of experimental type with Hadamard factorization given by* (5), *i.e.*

$$f(z) = z^m e^{\alpha + \beta z} \prod_{n=1}^{\infty} (1 - \lambda_n z) e^{\lambda_n z}.$$

*Then $f \in \mathscr{V}$ if and only if $(\beta, \lambda_1, \lambda_2, \ldots) \in h_1^{\text{sym}}$.*

*Remark* The hypotheses imply $\sum |\lambda_n| < \infty$ and $\beta + \sum \lambda_n = 0$ so that

$$f(z) = z^m e^\alpha \prod (1 - \lambda_n z).$$

*Proof* If $f \in \mathscr{V}$ then $f$ is in the convergence class, while similarly if $(\beta, \lambda_1, \lambda_2, \ldots) \in h_1^{\text{sym}}$ then at least $f$ is convergence class.

Now Proposition 3.1 implies that $(\beta, \lambda_1, \ldots) \in h_1^{\text{sym}}$ if and only if

$$\int_1^\infty \frac{|S(r)|}{r}\, dr < \infty$$

(take $s_n = |\lambda_n|$ and $\sigma(r) = S(r)$ in the Proposition). The result is now an immediate corollary of Theorem 2.1.

We now turn to the real analog of Theorem 3.2. We shall need a lemma due to Levin [7, Lemma 3, p. 11].

LEMMA 3.3 *Let $\lambda_n \in \mathbb{C}$ be such that $\sum |\lambda_n| < \infty$. Let $n(t)$ denote the cardinality of the set $\{k: |\lambda_k| > t^{-1}\}$. Put*

$$F(z) = \prod_{|\lambda_n z| > 1} (1 - \lambda_n z) \prod_{|\lambda_n z| > 1} (1 - \lambda_n z) e^{\lambda_n z}.$$

*Then there is a constant $M$ such that for all $z$*

$$\log|F(z)| \leqslant M\left(\int_0^r \frac{n(t)}{t}\, dt + r^2 \int_r^\infty \frac{n(t)}{t^3}\, dt\right)$$

*where $r = |z|$.*

We also require the following classical result (Boas [2, p. 85]).

THEOREM 3.4 *Let $f$ be an entire function of exponential type. Then $f \in \mathscr{C}$ if and only if*

$$\int_{-\infty}^{\infty} \frac{|\log|f(x)||}{1+x^2}\, dx < \infty.$$

THEOREM 3.2 *Let $f$ be an entire function of exponential type with Hadamard factorization given by (5). Suppose further that $f$ is in the convergence class, i.e. $\sum |\lambda_n| < \infty$. Then the following conditions are equivalent:*

(i) $f \in \mathscr{C}$,

(ii) $\displaystyle\int_1^\infty \frac{|\Re S(r)|}{r}\, dr < \infty$,

(iii) $(\Re\beta, \Re\lambda_1, \Re\lambda_2, \ldots) \in h_1^{\text{sym}}$.

*Remark* Under these hypotheses $\Re(\beta + \sum \lambda_n) = 0$ so that

$$f(z) = z^m e^\alpha e^{i\gamma z} \prod_{n=1}^{\infty} (1 - \lambda_n z)$$

where $\gamma \in \mathbb{R}$. See Levin [7, p. 251].

For other classical results on the distribution of zeros of functions in the Cartwright class see Levin [7].

*Proof* We first note that it suffices to prove the theorem for the case when $\alpha = m = 0$, $|\lambda_1| \leqslant 1$ and $\beta \in \mathbb{R}$. Thus we suppose that

$$f(z) = e^{\beta z} \prod_{n=1}^{\infty} (1 - \lambda_n z) e^{\lambda_n z}. \tag{12}$$

For $x \in \mathbb{R}$ let

$$F(x) = \prod_{|\lambda_n x| > 1} (1 - \lambda_n x) \prod_{|\lambda_n x| \leqslant 1} (1 - \lambda_n x) e^{\lambda_n x}$$

so that

$$f(x) = \exp(xS(|x|))F(x). \tag{13}$$



Now we employ Lemma 3.3:

$$\begin{aligned}
\int_{|x| \geqslant 1} \frac{\log_+|F(x)|}{x^2}\, dx &\leqslant 2M\left(\int_1^\infty \int_1^r \frac{n(t)}{tr^2}\, dt\, dr + \int_1^\infty \int_r^\infty \frac{n(t)}{t^3}\, dt\, dr\right) \\
&\leqslant 2M\left(\int_1^\infty \frac{n(t)}{t} \int_t^\infty \frac{dr}{r^2}\, dt + \int_1^\infty \int_1^t \frac{n(r)}{t^3}\, dr\, dt\right) \\
&\leqslant 4M \int_1^\infty \frac{n(t)}{t^2}\, dt \\
&\leqslant 4M \sum_{n=1}^\infty |\lambda_n| < \infty. \qquad (14)
\end{aligned}$$

Now we prove (ii) $\Rightarrow$ (i). From (13) we have

$$\log|f(x)| = x\Re S(|x|) + \log|F(x)|. \tag{15}$$

Hence if (ii) holds then by (14) and (15) we immediately get

$$\int_{|x| \geqslant 1} \frac{\log_+|f(x)|}{x^2}\, dx < \infty,$$

and (i) holds.

Next we turn to (i) $\Rightarrow$ (ii). Since $f(z)f(-z) \in \mathscr{C}$ we may use Theorem 3.4 to deduce that

$$\int_{|x| \geqslant 1} \frac{|\log|F(x)F(-x)||}{x^2}\, dx = \int_{|x| \geqslant 1} \frac{|\log|f(x)f(-x)||}{x^2}\, dx < \infty. \tag{16}$$

Observe that

$$\log_+ \frac{1}{|F(x)|} \leqslant \log_+ \frac{1}{|F(x)F(-x)|} + \log_+|F(-x)|$$

so that (14) and (16) together imply

$$\int_{|x| \geqslant 1} \log_+\left(\frac{1}{|F(x)|}\right) \frac{dx}{x^2} < \infty$$

and hence

$$\int_{|x| \geqslant 1} \frac{|\log|F(x)||}{x^2}\, dx < \infty.$$

Now Theorem 3.4 and equation (15) give the conclusion

$$\int_{|x| \geqslant 1} \frac{|\Re S(|x|)|}{x^2} < \infty,$$

and (ii) holds.

Finally we note that (ii) $\Leftrightarrow$ (iii) follows from Proposition 3.1 upon setting $s_n$ to be $|\lambda_n|$ and replacing $\lambda_n$ by its real part.

## 4. SOME APPLICATIONS TO ENTIRE FUNCTIONS

In this section we give some simple applications of the results of the previous section.

THEOREM 4.1 *Suppose $0<\theta<\pi$ and $f$ is an entire function. Then $f\in\mathscr{V}$ if and only if $f(z)\in\mathscr{C}$ and $f(e^{i\theta}z)\in\mathscr{C}$.*

*Proof* We suppose $f$ is given by (5). Then (Boas [2, p. 134]) since $f\in\mathscr{C}$ we may $\sum|\mathfrak{I}\lambda_n|<\infty$. Similarly we have $\sum|\mathfrak{I}(e^{i\theta}\lambda_n)|<\infty$. Combining these statements we have that $f$ is in the convergence class. Now Theorem 3.5 allows us to deduce that both $(\mathfrak{R}\beta, \mathfrak{R}\lambda_1, \ldots)$ and $(\mathfrak{R}(e^{i\theta}\beta), \mathfrak{R}(e^{i\theta}\lambda_1, \ldots)$ are in $h_1^{\text{sym}}$. Since $h_1^{\text{sym}}$ is a linear space we have $(\beta, \lambda_1, \ldots)\in h_1^{\text{sym}}$ so that by Theorem 3.2, $f\in\mathscr{V}$.

THEOREM 4.2 *If $f$ is an entire function of exponential type and convergence class which is real on the real axis then $f\in\mathscr{C}$ if and only if $f\in\mathscr{V}$.*

*Proof* If $f$ is given by (5) then the $\lambda_n$ occur in complex conjugate pairs so that we may suppose that $\lambda_{2n-1}=\bar{\lambda}_{2n}$ for all $n$. It follows quickly that $e^{\alpha}\in\mathbb{R}$, $\beta+\sum\lambda_n\in\mathbb{R}$, and $\mathfrak{I}\beta=0$. Thus the sequence $(\mathfrak{I}\beta, \mathfrak{I}\lambda_1, \mathfrak{I}\lambda_2, \ldots)$ is in $h_1^{\text{sym}}$. Now the hypothesis that $f\in\mathscr{C}$ implies that $(\mathfrak{R}\beta, \mathfrak{R}\lambda_1, \ldots)\in h_1^{\text{sym}}$ by Theorem 3.5. Hence $(\beta, \lambda_1, \ldots)\in h_1^{\text{sym}}$ and Theorem 3.2 gives that $f\in\mathscr{V}$.

THEOREM 4.3 *Suppose $f\in\mathscr{C}$ is given by $f(z)=\sum_{n=0}^{\infty} a_n z^n$. Define $\mathfrak{R}f$ by $\mathfrak{R}f(z)=\sum_{n=0}^{\infty}\mathfrak{R}a_n z^n$. Then $f\in\mathscr{V}$ if and only if $f$ and $\mathfrak{R}f$ are both in the convergence class.*

*Proof* For convenience let $g=\mathfrak{R}f$ and let $h=\mathfrak{I}f=\sum\mathfrak{I}a_n z^n$. Obviously, if $f\in\mathscr{V}$ we can compute

$$M(g,r)\leqslant\sum_{n=0}^{\infty}|a_n|r^n\leqslant 2M(f,2r)$$

so that $g\in\mathscr{V}$.

Conversely, suppose $f$ and $g$ are in the convergence class, and set

$$f^*(z)=\sum_{n=0}^{\infty}\bar{a}_n z^n.$$

Then $f^*f=g^2+h^2$. Now since $|f^*(x)|=|f(x)|$ for real $x$ and $f^*$ is of exponential type we have $f^*\in\mathscr{C}$ and hence $f^*f\in\mathscr{C}$. Since $f^*f$ is real on the real axis and is clearly also convergence class we can apply Theorem 4.2 to deduce that $f^*f\in\mathscr{V}$. Similarly since $g$ is of exponential type and $|g(x)|\leqslant|f(x)|$ for real $x$ we have $g\in\mathscr{C}$ and, again by Theorem 4.2, $g\in\mathscr{V}$. Thus $h^2=f^*f-g^2\in\mathscr{V}$ since the Valiron class is linear. Finally note that $M(h^2,r)=(M(h,r))^2$ so that $h\in\mathscr{V}$ and $f=g+ih\in\mathscr{V}$.

## 5. APPLICATIONS TO OPERATOR THEORY

THEOREM 5.1 *Let $\mathscr{H}$ be a complex Hilbert space. Then a trace-class operator $T$ on $\mathscr{H}$ can be expressed as a finite linear combination of commutators of Hilbert–Schmidt operators (i.e. $T\in\mathscr{C}h_1$) if and only if the characteristic determinant $D_T$ of $T$ satisfies:*

$$\int_0^{\infty}\int_0^{2\pi}\frac{\log_+|D_T(re^{i\theta})|}{r^2}\,d\theta\,dr<\infty \tag{17}$$

*(or, equivalently, $D_T\in\mathscr{V}$ and $D_T'(0)=0$).*

*Proof* If $(\lambda_n)$ are the non-zero eigenvalues of $T$ then

$$D_T(z)=\prod_{n=1}^{\infty}(1-\lambda_n z).$$

The convergence of the integral in (17) is equivalent to $D_T\in\mathscr{V}$ and $D_T'(0)=\sum\lambda_n(=\text{tr }T)=0$. Thus Theorem 3.2 shows that it is equivalent to $(\lambda_n)\in h_1^{\text{sym}}$. The result now follows from Theorem 7.4 of [5].

LEMMA 5.2 *Let $A$ and $B$ be trace-class operators on a complex Hilbert space $\mathscr{H}$. Let $(\lambda_n)$, $(\mu_n)$ be enumerations, counting algebraic multiplicity, of the non-zero eigenvalues of $A$, $B$ respectively (where, as usual zeros are adjoined to the sequences when the number of non-zero eigenvalues is finite). Let $(\nu_n)$ be the eigenvalues of $A+B$. Then $(\nu_n)\oplus(-\lambda_n)\oplus(-\mu_n)\in h_1^{\text{sym}}$.*

Here $(\nu_n)\oplus(-\lambda_n)\oplus(-\mu_n)$ denotes any sequence obtained by forming the union of the three sequences.

*Proof* The operator $(A+B)\oplus(-A)\oplus(-B)$ on $\mathscr{H}\oplus\mathscr{H}\oplus\mathscr{H}$ is easily seen to be a commutator of a bounded operator and a trace-class operator, and thus (see [5]) belongs to $\mathscr{C}h_1$. The lemma follows immediately.

THEOREM 5.3 *Let $T$ be a trace-class operator on the complex Hilbert space $\mathscr{H}$. Let $H$ and $K$ be the real and imaginary parts of $T$, i.e. $H=\frac{1}{2}(T+T^*)$, $K=\frac{1}{2i}(T-T^*)$. Then $H\in\mathscr{C}h_1$ if and only if*

$$\int_{-\infty}^{\infty}\frac{\log_+|D_T(x)|}{x^2}\,dx<\infty. \tag{18}$$

*Proof* Let $(\lambda_n)$ denote the eigenvalues of $T$, $(\mu_n)$ the eigenvalues of $H$ and $(\nu_n)$ the eigenvalues of $K$. Then $(\lambda_n)\oplus(-\mu_n)\oplus(-i\nu_n)\in h_1^{\text{sym}}$. Hence $(\mathfrak{R}\lambda_n)\oplus(-\mu_n)\in h_1^{\text{sym}}$, and so $H\in\mathscr{C}h_1$ if and only if $(\mathfrak{R}\lambda_n)\in h_1^{\text{sym}}$. Now Theorem 3.5 yields the conclusion.

THEOREM 5.4 *Suppose $T$ is a trace-class operator whose matrix with respect to a given orthonormal basis is real. Then $T\in\mathscr{C}h_1$ if and only if (18) holds.*

*Proof* Since $D_T$ is real on the real axis this is immediate from Theorem 4.2.

Now consider an operator $T$ on a real Hilbert space. We may regard $T$ as an operator with a real matrix acting on a complex Hilbert space. It is easy to verify that if $T=\sum[A_k, B_k]$, with $A_k, B_k$ Hilbert–Schmidt, then $T=\sum[\mathfrak{R}A_k, \mathfrak{R}B_k]-\sum[\mathfrak{I}B_k, \mathfrak{I}A_k]$ where $\mathfrak{R}A$ denotes the operator whose matrix consists of the real parts of the matrix of $A$, etc. Thus Theorem 5.4 yields:

THEOREM 5.5 *In order that a trace-class operator T on a real Hilbert space be a finite linear combination of commutators of Hilbert–Schmidt operators it is necessary and sufficient that* (18) *holds.*

We conclude by giving yet another formulation of the condition $T \in \mathscr{C}h_1$, this time in terms of the "adjoint".

If $T$ is a trace-class operator on a complex Hilbert space $\mathscr{H}$ then we can define an entire function taking values in $\mathscr{B}(\mathscr{H})$ by

$$A(z) = \operatorname{adj}(I - zT) = D_T(z)(I - zT)^{-1}$$

where $A(z) = 0$ when $z^{-1}$ is an eigenvalue of $T$.

THEOREM 5.6 $T \in \mathscr{C}h_1$ *if and only if* $\operatorname{tr} T = 0$ *and*

$$\int_1^\infty \int_0^{2\pi} \log_+ \|A(z)\| \, d\theta \, \frac{dr}{r^2} < \infty. \tag{19}$$

*Proof* First if (19) holds then since $D_T(z)I = (I - zT)A(z)$ we obtain $D_T \in \mathscr{V}$. Thus if $\operatorname{tr} T = 0$ we have (17) and $T \in \mathscr{C}h_1$.

Conversely suppose $T \in \mathscr{C}h_1$. Then $\operatorname{tr} T = 0$ and $D_T \in \mathscr{V}$. We use an argument from Gohberg–Krein [4, p. 244]. We have

$$\begin{aligned} \|(I - zT)^{-1}\| &\leqslant \|I + zT\| \, \|(I - z^2T^2)^{-1}\| \\ &\leqslant \frac{(1 + |z| \, \|T\|)}{|D_{T^2}(z^2)|} \prod_{n=1}^\infty (1 + |z|^2 s_n) \end{aligned} \tag{20}$$

where $(s_n)$ are the singular values of $T^2$, which satisfy $\sum \sqrt{s_n} < \infty$. Now

$$D_{T^2}(z^2) = D_T(z) D_T(-z) = \prod_{n=1}^\infty (1 - \lambda_n^2 z^2)$$

where $(\lambda_n)$ are the eigenvalues of $T$. Hence (cf. Bernstein's theorem, Boas [2, p. 35])

$$\begin{aligned} \int_1^\infty \int_0^{2\pi} |\log|D_T(z)D_T(-z)|| \, d\theta \, \frac{dr}{r^2} &\leqslant \sum_{n=1}^\infty \int_0^{2\pi} |\log|1 - \lambda_n^2 z^2|| \, d\theta \, \frac{dr}{r^2} \\ &\leqslant M \sum_{n=1}^\infty |\lambda_n| < \infty \end{aligned}$$

where $M$ is a suitable constant. We also have

$$\int_0^\infty \log(1 + r^2 s_n) \frac{dr}{r^2} = \pi \sqrt{s_n}$$

so that (20) gives us

$$\int_0^\infty \int_0^{2\pi} \log_+ \|(I - zT)^{-1}\| \, d\theta \, \frac{dr}{r^2} < \infty \tag{21}$$

and hence since $D_T \in \mathscr{V}$ we obtain (19).



*Remark* Notice that the above proof shows that (21) holds for any trace-class operator $T$.

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