J. reine angew. Math. **621** (2008), 81—121
DOI 10.1515/CRELLE.2008.059

Journal für die reine und angewandte Mathematik

Symmetric norms and spaces of operators

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Abstract. We show that if $(E, \|\cdot\|_E)$ is a symmetric Banach sequence space then the corresponding space $\mathscr{S}_E$ of operators on a separable Hilbert space, defined by $T \in \mathscr{S}_E$ if and only if $\big(s_n(T)\big)_{n=1}^{\infty} \in E$, is a Banach space under the norm $\|T\|_{\mathscr{S}_E} = \big\|\big(s_n(T)\big)_{n=1}^{\infty}\big\|_E$. Although this was proved for finite-dimensional spaces by von Neumann in 1937, it has never been established in complete generality in infinite-dimensional spaces; previous proofs have used the stronger hypothesis of full symmetry on E. The proof that $\|\cdot\|_{\mathscr{S}_E}$ is a norm requires the apparently new concept of uniform Hardy-Littlewood majorization; completeness also requires a new proof. We also give the analogous results for operator spaces modelled on a semifinite von Neumann algebra with a normal faithful semi-finite trace.

1. Introduction

In 1937, von Neumann [23] or [24], pp. 205–218, showed that if $\|\cdot\|_E$ is a symmetric norm on $\mathbb{R}^n$ then one can define a norm on the space of $n \times n$ matrices by

$$\|A\|_E = \big\|\big(s_1(A), \dots, s_n(A)\big)\big\|_E$$

where $s_1(A), \dots, s_n(A)$ are the singular values of A (i.e. the eigenvalues of $(A^*A)^{1/2}$) in decreasing order. Surprisingly, the infinite-dimensional analogue of this result, although well-known in special cases, has never been established in complete generality. This is the aim of the current paper.

Consider a (real or complex) symmetric Banach sequence space E; i.e., E is a sequence space invariant under permutations equipped with a norm $\|\cdot\|_E$ such that $(E, \|\cdot\|_E)$ is complete and has the property that

$$f \in E, \quad g^* \leqq f^* \quad \Rightarrow \quad g \in E, \quad \|g\|_E \leqq \|f\|_E.$$

(Here by f^* we denote the usual decreasing rearrangment of the sequence $|f|$.) Then if $\mathscr{H}$ is a separable Hilbert space we can define an associated *Schatten ideal* $\mathscr{S}_E \subset \mathscr{L}(\mathscr{H})$ by

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The first author acknowledges support from NSF grant DMS-0555670; the second author acknowledges support from the ARC.

$$T \in \mathscr{S}_E \quad \Leftrightarrow \quad \big(s_n(T)\big)_{n=1}^{\infty} \in E$$

with a quasi-norm

$$\|T\|_{\mathscr{S}_E} = \big\|\big(s_n(T)\big)_{n=1}^{\infty}\big\|_E.$$

In this paper we address the following natural problem. Is $(\mathscr{S}_E, \|\cdot\|_E)$ a Banach space? It is curious that this question is up to now unresolved in this generality; see for example [28], [29], [14] and [31]. In fact it is well-known that there is a positive answer if E is *fully symmetric*. For this we need the definition of Hardy-Littlewood majorization:

$$f \prec\prec g \quad \Leftrightarrow \quad \sum_{j=1}^{n} f^*(j) \leqq \sum_{j=1}^{n} g^*(j), \quad n = 1, 2, \ldots .$$

Then E is fully symmetric provided

$$f \in E, \quad g^* \prec\prec f^* \quad \Rightarrow \quad g \in E, \quad \|g\|_E \leqq \|f\|_E.$$

This is equivalent to the requirement that E is a 1-interpolation space between ℓ_1 and ℓ_∞ (cf. [3]). More generally a positive answer is known if E is *relatively fully symmetric* i.e.

$$f, g \in E, \quad g^* \prec\prec f^* \quad \Rightarrow \quad \|g\|_E \leqq \|f\|_E.$$

See [5], [10] and [34]. We may note that E is relatively fully symmetric if and only if it is a closed subspace of a fully symmetric space, for we may define F by $f \in F$ if and only if there exists $g \in E$ such that $f \prec\prec g$ and define

$$\|f\|_F = \inf\{\|g\|_E : g \in E, f \prec\prec g\}$$

and it can be verified that F is then a fully symmetric Banach sequence space containing E as a closed subspace.

The same problem can be formulated in a continuous form. If E is a symmetric Banach function space on $(0, \infty)$ (in the terminology of [18]) and $\mathscr{M}$ is a von Neumann algebra with a faithful normal semi-finite trace τ then we can define the space $E(\mathscr{M}, \tau)$ as a subset of the space $\tilde{\mathscr{M}}$ of measurable operators affiliated with $\mathscr{M}$ by

$$x \in E(\mathscr{M}, \tau) \quad \Leftrightarrow \quad \mu(x) \in E$$

where $\mu(x)(t) = \mu_t(x)$ are the generalized singular values of x (see e.g. [13]). The associated quasinorm is

$$\|x\|_E = \|\mu(x)\|_E.$$

As in the discrete case it is known that $E(\mathscr{M}, \tau)$ is a Banach space under $\|\cdot\|_E$ if E is relatively fully symmetric. In this case

$$f \prec\prec g \quad \Leftrightarrow \quad \int_0^t f^*(s)\, ds \leqq \int_0^t g^*(s)\, ds, \quad 0 < t < \infty.$$

We note that some authors (e.g. [19], [9] or [11]) define symmetric or rearrangement-invariant spaces to have additional properties which imply full symmetry or relative full symmetry. However, according to our definition above, there are examples of symmetric Banach function spaces or sequence spaces which are not (relatively) fully symmetric. In our earlier paper [16] we showed that for example on the Marcinkiewicz sequence space M (sometimes denoted $\mathscr{L}^{1,\infty}$) of all sequences $\big(f(n)\big)_{n=1}^{\infty}$ such that

$$\|f\|_M = \sup_{n \geqq 1} \frac{1}{\log(n+1)} \sum_{j=1}^{n} f^*(j) < \infty$$

one can define a bounded positive symmetric linear functional $\varphi \neq 0$ such that

$$\varphi\big((1/n)_{n=1}^{\infty}\big) = 0.$$

Thus with the equivalent norm on M given by

$$\|f\|' = \|f\|_M + \varphi(|f|)$$

M is not fully symmetric and further the smallest closed symmetric subspace M_0 containing $(1/n)_{n=1}^{\infty}$ is not stable under Hardy-Littlewood majorization and therefore not fully symmetric. This last remark is only a special case of a result of Russu [26] and Mekler [20] (see also [2]) who proved a similar result for a wide class of Marcinkiewicz spaces. A more extreme example was given by Sedaev [30], who showed the existence of a symmetric Banach function space which fails to have any equivalent relatively fully symmetric norm. Thus the known results in the literature are far from a complete solution of the problem of the extension of von Neumann's result to infinite dimensions.

Our question divides into two parts. The first is the *convexity problem*: is the induced quasi-norm $\|\cdot\|_{\mathscr{S}_E}$ on $\mathscr{S}_E$ or $E(\mathscr{M},\tau)$ a norm? The second is the *completeness problem*: is space $\mathscr{S}_E$ or $E(\mathscr{M},\tau)$ complete for this (quasi-)norm? We show in this paper that both questions have a positive answer and so indeed the spaces $\mathscr{S}_E$ and $E(\mathscr{M},\tau)$ are Banach spaces.

In order to answer the convexity problem we consider a more general situation. To simplify the discussion let us concentrate on the continuous case. Let $L_0(0,\infty)$ denote the space of measurable functions whose supports are of finite measure and let $\tilde{L}_\infty = L_0 + L_\infty$. Suppose $g \in \tilde{L}_\infty$ and let $\mathscr{Q}(g)$ denote the convex hull of the set of functions $\mathscr{E}(g) = \{f : f^* \leqq g^*\}$. Then define

$$\langle f; g\rangle = \inf\{\lambda > 0 : f \in \lambda\mathscr{Q}(g)\}$$

(and $\langle f; g\rangle = \infty$ if f is not in the linear span of $\mathscr{Q}(g)$).

We also introduce the concept of *uniform Hardy-Littlewood majorization*, which plays a fundamental role in this paper. We write $f \trianglelefteq g$ (for $f, g \in \tilde{L}_\infty$) if there exists $\lambda > 1$ so that if $0 < \lambda a < b$ we have

$$\int_{\lambda a}^{b} f^*(s)\,ds \leqq \int_{a}^{b} g^*(s)\,ds.$$

If $g \in L_1 + L_\infty$ then $f \trianglelefteq g$ implies $f \prec\prec g$ but not conversely. We show in Theorem 6.3 that $f \trianglelefteq g$ implies that $\langle f; g\rangle \leqq 1$ (but not necessarily $f \in \mathscr{Q}(g)$) and this gives an explicit formula for $\langle f; g\rangle$ i.e. (Theorem 6.4):

$$\langle f; g\rangle = \lim_{\lambda\to\infty} \sup_{0<\lambda a<b} \frac{\int_{\lambda a}^{b} f^*(s)\,ds}{\int_{a}^{b} g^*(s)\,ds}.$$

The analogous discrete results are Theorem 5.4 and Theorem 5.5. Let us note that $\langle f, g\rangle < 1$ implies $f \trianglelefteq g$ but in general $\langle f; g\rangle \leqq 1$ does not imply $f \trianglelefteq g$ (see Lemma 4.4 and the examples following Theorem 5.5).

To emphasize the role of uniform Hardy-Littlewood majorization let us point out that $f \trianglelefteq g$ implies $\|f\|_E \leqq \|g\|_E$ for every symmetric norm $\|\cdot\|_E$ while $f \prec\prec g$ implies $\|f\|_E \leqq \|g\|_E$ only if $\|\cdot\|_E$ is fully symmetric.

These results are related to calculations of Banach envelopes of some weak type spaces, and in a special case when $g(x) = 1/x$ this formula can be shown to be equivalent to a result of Cwikel and Fefferman [6], [7] on the envelope of weak L_1.

Now suppose as before that $\mathscr{M}$ is von Neumann algebra with a faithful semi-finite normal trace τ. Then if $x_1, \dots, x_k$ are measurable affiliated with $\mathscr{M}$ we show (Proposition 8.6):

$$\mu(x_1 + \cdots + x_k) \trianglelefteq \mu(x_1) + \cdots + \mu(x_k).$$

Here $\mu(x)$ denotes the functions $t \to \mu_t(x)$. This can be regarded as answering (in an infinite-dimensional setting) a question raised by von Neumann in [23], p. 298 or [24], p. 218. From this and our previously stated results it is easy to show that any symmetric seminorm defined on a symmetric subspace of $\tilde{L}_\infty$ induces a seminorm on the corresponding subspace of $\tilde{\mathscr{M}}$; thus the convexity problem has a positive answer. The completeness problem also uses the same approach (Theorem 8.11) and we prove some generalizations to p-convex quasi-Banach function spaces.

Acknowledgements. We would like to thank Peter Dodds for many helpful comments on the content of this paper. We also thank Aleksandr Sedaev and Yves Raynaud for their comments on earlier drafts.

2. Functionals on spaces of measurable functions

Let $\mathbb{J}$ denote either $\mathbb{N}$ with counting measure (discrete case) or $(0, \infty)$ with Lebesgue measure (continuous case). The measure of a measurable set E is denoted in either case by $|E|$. We let $L_0 = L_0(\mathbb{J})$ be the linear space of all measurable functions with support of finite measure, i.e. $|\{f \neq 0\}| < \infty$. We then consider the space $\tilde{L}_\infty = L_0 + L_\infty$; in the discrete case this reduces to $\ell_\infty(\mathbb{N})$. If $f \in \tilde{L}_\infty$ we define the decreasing rearrangement by

$$f^*(t) = \inf\{\lambda : |\{|f| > \lambda\}| < t\}, \quad t \in \mathbb{J}.$$

We will then write $f \sim g$ if $f^* = g^*$ and $f \prec g$ if $f^* \leqq g^*$. Note that for example in the discrete case $(2, 1, 2, 1, \ldots) \sim (2, 0, 2, 0, \ldots)$ according to our definition.

A subset U of $\tilde{L}_\infty$ will be called *symmetric* if $f \in U$, $g \prec f \Rightarrow g \in U$ and *solid* if $f \in U$, $|g| \leqq |f| \Rightarrow g \in U$.

If $f, g \in L_1 + L_\infty$ (note that $L_1 + L_\infty = \ell_\infty$ in the discrete case) we write $f \prec\prec g$ if

$$\sum_{k=1}^{n} f^*(k) \leqq \sum_{k=1}^{n} g^*(k), \quad n = 1, 2, \ldots,$$

for the discrete case and

$$\int_0^t f^*(s)\, ds \leqq \int_0^t g^*(s)\, ds, \quad 0 < t < \infty,$$

for the continuous case. A subset U of $L_0 + L_\infty$ is called *fully symmetric* if $f \in U$, $g \prec\prec f \Rightarrow g \in U$. A fully symmetric set is automatically absolutely convex and symmetric.

Consider a functional $\Phi : \tilde{L}_\infty \to [0, \infty]$. We define the domain of Φ as the set $\mathrm{Dom}(\Phi) = \{f : \Phi(f) < \infty\}$. We say that Φ is *solid* if

$$\Phi(f) \leqq \Phi(g), \quad |f| \leqq |g|,\ f, g \in \tilde{L}_\infty.$$

Φ is *homogeneous* if

$$\Phi(\alpha f) = |\alpha| \Phi(f), \quad \alpha \in \mathbb{C},\ f \in \tilde{L}_\infty,$$

and *subadditive* if

$$\Phi(f + g) \leqq \Phi(f) + \Phi(g), \quad f, g \in \tilde{L}_\infty.$$

A homogeneous and subadditive functional $\|\cdot\|$ is called an *extended-valued seminorm*; in this case $\|\cdot\|$ is a seminorm on its domain $\{f : \|f\| < \infty\}$. Conversely if E is a linear subspace of $\tilde{L}_\infty$ and $\|\cdot\|_E$ is a seminorm on E we can extend the definition of $\|\cdot\|_E$ by putting $\|f\|_E = \infty$ when $f \notin E$ and then $E = \mathrm{Dom}(\|\cdot\|)$. $\|\cdot\|_E$ is an extended-valued seminorm.

A solid functional $\Phi : \tilde{L}_\infty \to [0, \infty]$ is called *symmetric* if

$$\Phi(f) = \Phi(g), \quad f \sim g,$$

or, equivalently,

$$\Phi(f) \leqq \Phi(g), \quad f \prec g,\ f, g \in \tilde{L}_\infty.$$

Φ is called *fully symmetric* if $\mathrm{Dom}(\Phi) \subset L_1 + L_\infty$ and

$$\Phi(f) \leqq \Phi(g), \quad f \prec\prec g,\ f, g \in L_1 + L_\infty.$$

Let us note at this point that the distinction between symmetry and full symmetry can be understood in the context of interpolation. Let $\mathscr{B}_{1,\infty}$ be the set of all linear maps $T : L_1 + L_\infty \to L_1 + L_\infty$ such that $\|T\|_{1,\infty} = \max(\|T\|_{L_1 \to L_1}, \|T\|_{L_\infty \to L_\infty}) \leqq 1$. Then it follows from the Caldéron-Mityagin Theorem [3] that if $g \in L_1 + L_\infty$,

$$\Omega(g) := \{f \in L_1 + L_\infty : f \prec\prec g\} = \{Tg : T \in \mathscr{B}_{1,\infty}\}.$$

Thus a functional Φ with $\mathrm{Dom}(\Phi) \subset L_1 + L_\infty$ is fully symmetric if and only if

$$\Phi(Tf) \leqq \Phi(f), \quad T \in \mathscr{B}_{1,\infty},\ f \in L_1 + L_\infty.$$

It is then natural to consider the orbit of a fixed $g \in L_1 + L_\infty$. We refer to [3], [25] and [18] for the method of orbits in interpolation theory. For fixed g the *orbit* of g for this couple is $\mathrm{Orb}(g; L_1, L_\infty)$ is the linear span of $\Omega(g)$ under the norm

$$\|f\| = \inf\{\|T\|_{1,\infty} : Tf = g\}.$$

This is simply the *Marcinkiewicz space* $M(\psi_g)$ of all f such that

$$\|f\|_{M(\psi_g)} = \sup_{t>0} \frac{\int_0^t f^*(s)\,ds}{\psi_g(t)} < \infty$$

where $\psi_g(t) = \int_0^t g^*(s)\,ds$ or in the discrete case

$$\|f\|_{M(\psi_g)} = \sup_{n \geqq 1} \frac{\sum_{j=1}^n f^*(j)}{\psi_g(n)} < \infty$$

where $\psi_g(n) = \sum_{j=1}^n g^*(j)$.

In [1] the author considers $L_0(0, \infty)$ as with the subadditive symmetric functional $\|f\|_0 = |\mathrm{supp}\, f|$. We will refer to this functional as a *G-norm* since with the induced metric L_0 becomes a metric abelian group. He then considers (L_0, L_∞) as a couple. In this case if $T : \tilde{L}_\infty \to \tilde{L}_\infty$ is a linear operator we define $\|T\|_{0,\infty} = \max(\|T\|_{L_0 \to L_0}, \|T\|_{L_\infty \to L_\infty})$ where

$$\|T\|_0 = \sup\left\{\frac{\|Tf\|_0}{\|f\|_0} : \|f\|_0 > 0\right\}.$$

We then set $\mathscr{B}_{0,\infty} = \{T : \|T\|_{0,\infty} \leqq 1\}$. Now if $g \in \tilde{L}_\infty$ we have

$$\mathscr{E}_g := \{f \in \tilde{L}_\infty : f \prec g\} = \{Tg : T \in \mathscr{B}_{0,\infty}\}$$

and so a functional Φ on $\tilde{L}_\infty$ is symmetric if

$$\Phi(Tf) \leqq \Phi(f), \quad T \in \mathscr{B}_{0,\infty}, \, f \in \tilde{L}_\infty.$$

The orbit of g which we denote $\mathscr{X}_g$ is the smallest symmetric subspace of $\tilde{L}_\infty$ and is a complete metric abelian group under the orbit G-norm defined by

$$|f|_g = \inf\{\|T\|_{0,\infty} : Tg = f\}.$$

In fact ([1], Theorem 1),

$$|f|_g = \inf\{\lambda > 0 : f^*(t) \leqq \lambda g^*(t/\lambda), 0 < t < \infty\}.$$

If g satisfies the condition

$$g^*(t/2) \leqq Cg^*(t), \quad t > 0,$$

for a suitable constant C, this space coincides with the symmetric quasi-Banach function space of all f such that

$$\|f\|_{\mathscr{X}_g} = \sup_{t>0} \frac{f^*(t)}{g^*(t)} < \infty.$$

Furthermore the quasinorm $\|\cdot\|_{\mathscr{X}_g}$ defines an equivalent topology. For example if $g(x) = 1/x$ we have $\mathscr{X}_g$ coincides with weak L_1.

Suppose $0 < p < \infty$. Then a *quasi-Banach function space* or *quasi-Banach sequence space* in the discrete case is a linear subspace E of $\tilde{L}_\infty$ together with a solid homogeneous map $\|\cdot\|_E : \tilde{L}_\infty \to [0,\infty]$ such that $\mathrm{Dom}(\|\cdot\|_E) = E$, $\|\cdot\|_E$ is a quasi-norm on E and $(E, \|\cdot\|_E)$ is a quasi-Banach space. If $\|\cdot\|_E$ is an extended-valued norm then E is a *Banach function space*. E is said to be *p-convex* where $0 < p < \infty$ if there is a constant C so that

$$\|(|f_1|^p + \cdots + |f_n|^p)^{1/p}\|_E \leqq C(\|f_1\|_E^p + \cdots + \|f_n\|_E^p)^{1/p}, \quad f_1, \ldots, f_n \in E.$$

If $C = 1$ we say that E is *exactly p-convex*. Every p-convex quasi-Banach function space can be given an equivalent quasi-norm so that it is exactly p-convex; if E is exactly p-convex for any $p \geqq 1$ then E is a Banach function space (i.e., $\|\cdot\|_E$ is a norm).

3. Noncommutative functionals and function spaces

We now discuss the non-commutative setting. First, for simplicity we treat the case of the von Neumann algebra $\mathscr{L}(\mathscr{H})$ where $\mathscr{H}$ is a separable Hilbert space. We will denote by $\mathscr{P}$ the collection of all orthogonal projections P on $\mathscr{H}$. If $\Phi : \ell_\infty(\mathbb{N}) \to [0,\infty]$ is any symmetric functional then we can unambiguously define an associated map (which we also denote by Φ) $\Phi : \mathscr{L}(\mathscr{H}) \to [0,\infty]$ by setting

$$\Phi(T) = \Phi\big(s_1(T), s_2(T), \ldots\big), \quad T \in \mathscr{L}(\mathscr{H}).$$

Here the sequence $(s_n(T))_{n=1}^{\infty}$ is the sequence of singular values of T which we define by

$$s_n(T) = \inf\{\|T(I-P)\| : P \in \mathscr{P},\ \mathrm{rank}(P) < n\}.$$

If T is compact $(s_n(T))_{n=1}^{\infty}$ is the sequence of eigenvalues of $|T| = (T^*T)^{1/2}$ in decreasing order, repeated according to multiplicity. Note that Φ obeys the condition

$$\Phi(RST) \leqq \Phi(S), \quad \|R\|, \|T\| \leqq 1. \tag{3.1}$$

Conversely if $\Phi : \mathscr{L}(\mathscr{H}) \to [0, \infty]$ satisfies (3.1) then Φ can be induced from a symmetric map $\Phi : \ell_\infty \to [0, \infty]$.

In particular if E is a symmetric quasi-Banach sequence space we can define the associated ideal $\mathscr{S}_E$ by setting

$$\|T\|_{\mathscr{S}_E} = \left\|(s_n(T))_{n=1}^{\infty}\right\|_E$$

and then $S_E = \{T : \|T\|_E < \infty\}$. It may be shown that $(\mathscr{S}_E, \|\cdot\|_{\mathscr{S}_E})$ is then a quasi-normed space. This follows from the fact that

$$s_{2n-1}(S+T) \leqq s_n(S) + s_n(T), \quad n = 1, 2 \ldots .$$

Thus if $\|\cdot\|_E$ satisfies

$$\|f+g\|_E \leqq C(\|f\|_E + \|g\|_E), \quad f, g \in \ell_\infty,$$

we have

$$\|S+T\|_{\mathscr{S}_E} \leqq 2C\left\|(s_{2n-1}(S+T))_{n=1}^{\infty}\right\|_E \leqq 2C^2(\|S\|_{\mathscr{S}_E} + \|T\|_{\mathscr{S}_E}).$$

In the case $C = 1$ this crude calculation only gives that $\|S+T\|_{\mathscr{S}_E} \leqq 2(\|S\|_{\mathscr{S}_E} + \|T\|_{\mathscr{S}_E})$. It is one of the results of this paper, but not immediately clear, that if E is a symmetric Banach sequence space then indeed $\|\cdot\|_{\mathscr{S}_E}$ is actually a norm on $\mathscr{S}_E$ and $(\mathscr{S}_E, \|\cdot\|_E)$ is complete (i.e. a Banach space). If $\|\cdot\|_E$ is fully symmetric it is well-known that the ideal $\mathscr{S}_E$ is a Banach space (see e.g. [14]).

We next turn to a more general situation. We assume that $\mathscr{M}$ is a semi-finite von Neumann algebra on a Hilbert space $\mathscr{H}$ with a fixed faithful normal semi-finite trace τ and unit element $\mathbf{1}$ (our references for the theory of von Neumann algebras are [8], [33], [35], [36]). $\|\cdot\|_{\mathscr{M}}$ stands for the uniform operator norm on $\mathscr{M}$. In this case we denote by $\mathscr{P} = \mathscr{P}_{\mathscr{M}}$ the set of orthogonal projections, i.e. self-adjoint p such that $p^2 = p$.

Given a self-adjoint operator $a : \mathrm{Dom}(a) \to \mathscr{H}$ in the Hilbert space $\mathscr{H}$, the *spectral measure* of a is denoted by e^a. We write $e^a_\lambda = e^a((-\infty, \lambda])$ for all $\lambda \in \mathbb{R}$. A linear operator $x : \mathrm{Dom}(x) \to \mathscr{H}$, with domain $\mathrm{Dom}(x) \subseteqq \mathscr{H}$, is said to be *affiliated* with $\mathscr{M}$ if $ux = xu$ for all unitary operators u in the commutant $\mathscr{M}'$ of $\mathscr{M}$. A self-adjoint operator a in $\mathscr{H}$ is affiliated with $\mathscr{M}$ if and only if $e^a(B) \in \mathscr{M}$ for all Borel sets $B \subseteqq \mathbb{R}$, or equivalently, $e^a_\lambda \in \mathscr{M}$ for all $\lambda \in \mathbb{R}$ (see e.g. [33], E.9.10, E.9.25). If x is a closed and densely defined linear operator in $\mathscr{H}$ with polar decomposition $x = v|x|$, then x is affiliated with $\mathscr{M}$ if and only if $v \in \mathscr{M}$

and $|x|$ is affiliated with $\mathcal{M}$ (see e.g. [33], 9.29; [36], IX.2). We have $v^*v = s(|x|)$, where $s(|x|) = \mathbf{1} - e_0^{|x|}$ is the support projection of $|x|$ (see e.g. [33], 9.4).

A closed and densely defined linear operator x, affiliated with $\mathcal{M}$, is called *τ-measurable* if there exists $\lambda > 0$ such that $\tau\big(e^{|x|}(\lambda, \infty)\big) < \infty$. The set $\tilde{\mathcal{M}}$ of all τ-measurable operators is a $*$-algebra with the sum and product defined as the closure of the algebraic sum and product, respectively. For $\varepsilon, \delta > 0$ we denote by $N(\varepsilon, \delta)$ the set of all $x \in \tilde{\mathcal{M}}$ for which there exists an orthogonal projection $p \in \mathcal{M}$ such that $p(\mathcal{H}) \subseteq \mathrm{Dom}(x)$, $\|xp\| \leqq \varepsilon$ and $\tau(\mathbf{1} - p) \leqq \delta$. The sets $\{N(\varepsilon, \delta) : \varepsilon, \delta > 0\}$ are a base at 0 for a metrizable Hausdorff vector space topology in $\tilde{\mathcal{M}}$, which is called the *measure topology*. Convergence with respect to this topology is referred to as *convergence in measure*. Equipped with the measure topology, $\tilde{\mathcal{M}}$ is a complete topological $*$-algebra in which $\mathcal{M}$ is dense. In fact, if $0 \leqq x \in \tilde{\mathcal{M}}$, then $\{xe_\lambda^x\}_{\lambda \geqq 0}$, which is contained in $\mathcal{M}$, converges in measure to x as $\lambda \to \infty$. Furthermore, if $\{x_n\}_{n=1}^\infty$ is a sequence in $\tilde{\mathcal{M}}$, then

$$(3.2) \qquad x_n \to 0 \text{ in measure} \quad \Leftrightarrow \quad \tau\big(e^{|x_n|}(\lambda, \infty)\big) \to 0 \text{ as } n \to \infty \ \forall \lambda > 0.$$

The proofs of these facts can be found in [13] and [22] (see also [36]).

Let $x \in \tilde{\mathcal{M}}$; the generalized singular value function of x is $\mu(x) : t \to \mu_t(x)$, where, for $0 < t < \tau(\mathbf{1})$,

$$\mu_t(x) = \inf\big\{s \geqq 0 : \tau\big(e^{|x|}(s, \infty)\big) < t\big\} = \inf\{\|x(\mathbf{1} - e)\|_{\mathcal{M}} : e \in \mathcal{P}, \tau(e) < t\}.$$

Consider $\mathcal{M} = L^\infty\big([0, \infty)\big)$ as an abelian von Neumann algebra acting via multiplication on the Hilbert space $\mathcal{H} = L^2(0, \infty)$, with the trace given by integration with respect to m. It is easy to see that the set of all τ-measurable operators affiliated with $\mathcal{M}$ consists of all measurable functions on $[0, \infty)$ which are bounded except on a set of finite measure, that is $\tilde{\mathcal{M}} = \tilde{L}_\infty$ and that the generalized singular value function $\mu(f)$ is precisely the decreasing rearrangement f^*.

If $\mathcal{M} = \mathcal{L}(\mathcal{H})$ (respectively, $\ell_\infty(\mathbb{N})$) and τ is the standard trace tr (respectively, the counting measure on $\mathbb{N}$), then it is not difficult to see that $\tilde{\mathcal{M}} = \mathcal{M}$. In this case, for $x \in \tilde{\mathcal{M}}$ we have

$$\mu_n(x) = \mu_t(x), \quad t \in (n-1, n],\ n = 0, 1, 2, \ldots .$$

For $\mathcal{M} = \mathcal{L}(\mathcal{H})$ the sequence $\{\mu_n(T)\}_{n=1}^\infty$ is just the sequence of singular values $\big(s_n(T)\big)_{n=1}^\infty$.

The trace τ on $\mathcal{M}^+$ extends uniquely to an additive, positively homogeneous, unitarily invariant and normal functional $\tilde{\tau} : \tilde{\mathcal{M}}^+ \to [0, \infty]$, which is given by $\tilde{\tau}(a) = \int_0^\infty \mu_t(a)\, dt$ for all $a \in \tilde{\mathcal{M}}^+$ (for the details see e.g. [11], Section 3). For convenience, we denote this extension $\tilde{\tau}$ again by τ. An operator $x \in \tilde{\mathcal{M}}$ belongs to $L_1(\mathcal{M}) = L_1(\mathcal{M}, \tau)$ if and only if $\|x\|_1 := \tau(|x|) < \infty$. If $\mathcal{M} = \mathcal{L}(\mathcal{H})$ as above, then $L_1(\mathcal{M})$ is precisely the *trace class*. We denote by $L_0(\mathcal{M})$ the set of all $x \in \tilde{\mathcal{M}}$ for which $\|x\|_0 := \tau\big(\mathrm{supp}(x)\big) < \infty$. Thus, it follows from the definitions that $\tilde{\mathcal{M}} = L_0(\mathcal{M}) + \mathcal{M}$.

Let us note that it is always possible to replace $\mathcal{M}$ by $\mathcal{M} \otimes L_\infty(0, \infty)$ and thus to ensure that $\mathcal{M}$ has no minimal orthogonal projections, i.e. is *atomless* and that $\tau(\mathbf{1}) = \infty$.

Convention. Throughout the remainder of the paper $\mathcal{M}$ will denote an atomless semi-finite von Neumann algebra with a faithful normal infinite trace τ.

Now suppose $\Phi : \tilde{L}_\infty(0, \infty) \to [0, \infty]$ is a symmetric functional. Then we can as before define $\Phi : \tilde{\mathcal{M}} \to [0, \infty]$ by the formula:

$$\Phi(x) = \Phi\big(\mu(x)\big)$$

where $\mu(x)$ denotes the function $\mu(x)(t) = \mu_t(x)$ for $0 < t < \infty$. As before if E is a symmetric quasi-Banach function space we can define the associated *noncommutative* version of E, $E(\mathcal{M}, \tau)$ by defining

$$\|x\|_{E(\mathcal{M},\tau)} = \|\mu(x)\|_E$$

and then setting $E(\mathcal{M}, \tau) = \{x \in \tilde{M} : \|x\|_{E(\mathcal{M},\tau)} < \infty\}$. The same questions arise as in the discrete case: if E is a symmetric Banach function space, is $E(\mathcal{M}, \tau)$ a Banach space under $\|\cdot\|_E$?

4. Uniform Hardy-Littlewood majorization

In this section we will introduce the fundamental notion of *uniform Hardy-Littlewood majorization*. If $f, g \in \ell_\infty(\mathbb{N})$ we will write $f \trianglelefteq g$ if there exists $\lambda \in \mathbb{N}$ so that

$$\sum_{j=\lambda r+1}^{n} f^*(j) \leqq \sum_{j=r+1}^{n} g^*(j), \quad 0 \leqq \lambda r < n.$$

If $f, g \in \tilde{L}_\infty(0, \infty)$ we write $f \trianglelefteq g$ if there exists $\lambda > 0$ so that

$$\int_{\lambda a}^{b} f^*(s)\, ds \leqq \int_{a}^{b} g^*(s)\, ds, \quad 0 < \lambda a \leqq b.$$

(In the continuous case we can also restrict λ to be an integer of course.)

Let us start with the simple observation that if $f \in \tilde{L}_\infty(0, \infty)$ we have

$$\int_a^b f^*(s)\, ds = \sup_{|E|=b} \inf_{\substack{F\subset E\\ |F|=b-a}} \int_F |f(s)|\, ds. \tag{4.1}$$

Similarly we have

$$\int_a^b f^*(s)\, ds = \inf_{|E|\leqq a} \sup_{\substack{E\cap F=\emptyset\\ |F|=b-a}} \int_F |f(s)|\, ds. \tag{4.2}$$

Similarly if $f \in \ell_\infty(\mathbb{N})$,

$$\sum_{j=r+1}^{n} f^*(j) = \sup_{|E|=n} \inf_{\substack{F\subset E\\ |F|=n-r}} \sum_{j\in F} |f(j)| \tag{4.3}$$

and

$$(4.4)\qquad \sum_{j=r+1}^{n} f^*(j) = \inf_{|E|\leqq r} \sup_{\substack{E\cap F=\emptyset\\ |F|=n-r}} \sum_{j\in F} |f(j)|.$$

Lemma 4.1. (a) *Suppose* $f_1, \dots, f_k \in \ell_\infty(\mathbb{N})$. *Then*

$$(4.5)\qquad \sum_{j=kr+1}^{n} \big(f_1(j) + \cdots + f_k(j)\big)^* \leqq \sum_{j=r+1}^{n} \big(f_1^*(j) + \cdots + f_k^*(j)\big), \quad 0 \leqq kr < n.$$

Thus

$$(f_1 + \cdots + f_k) \trianglelefteq (f_1^* + \cdots + f_k^*).$$

(b) *Suppose* $f_1, \dots, f_k \in \tilde{L}_\infty(0, \infty)$. *Then*

$$(4.6)\qquad \int_{ka}^{b} (f_1 + \cdots + f_k)^*(s)\, ds \leqq \int_{a}^{b} \big(f_1^*(s) + \cdots + f_k^*(s)\big)\, ds, \quad 0 < ka < b < \infty.$$

Thus

$$(f_1 + \cdots + f_k) \trianglelefteq (f_1^* + \cdots + f_k^*).$$

Proof. The proofs of (a) and (b) are very similar so we indicate the proof of (b). For fixed $a, b > 0$ with $ka < b$ we can find F_j be measurable subsets of $(0, \infty)$ with $|F_j| = a$ and such that $|f_j(t)| \geqq f_j^*(a)$ for $t \in E_j$; indeed we let $F_j = \{t : f^*(t) > f^*(a)\} \cup H_j$ where $H_j \subset \{t : f(t) = f^*(a)\}$ is a measurable set with $|H_j| + |\{t : f^*(t) > f^*(a)\}| = a$. If $F = \bigcup_{j=1}^{k} F_j$ then $|F| \leqq ka$. Now if E is any measurable set with $|E| = b$, let G be a measurable subset of E with $|G| = b - ka$ and $G \cap F = \emptyset$. We have, using (4.2),

$$\int_G |f_1(s) + \cdots + f_k(s)|\, ds \leqq \int_G \big(|f_1(s)| + \cdots + |f_k(s)|\big)\, ds$$
$$\leqq \sum_{j=1}^{k} \int_a^b f_j^*(s)\, ds$$

since $G \cap F_j = \emptyset$ and $|G| = b - ka \leqq b - a$. The lemma follows by (4.1). □

Let $\mathbb{J} = \mathbb{N}$ or $(0, \infty)$ as before. For $g \in \tilde{L}_\infty(\mathbb{J})$ we define $\mathcal{Q}(g)$ to be the absolutely convex hull of the set $\mathcal{E}_g = \{f : f \prec g\}$. It is clear that if $\|\cdot\|$ is a symmetric extended valued seminorm on $\tilde{L}_\infty(\mathbb{J})$ we have

$$\|f\| \leqq \|g\|, \quad f \in \mathcal{Q}(g).$$

We therefore introduce the Minkowski functional of $\mathcal{Q}(g)$:

$$\langle f; g \rangle = \inf\{\lambda > 0 : f \in \lambda \mathcal{Q}(g)\},$$

with $\langle f; g \rangle = \infty$ if $f \notin \bigcup_{\lambda > 0} \lambda \mathcal{Q}(g)$.

Proposition 4.2. *$f \to \langle f; g \rangle$ is a symmetric extended-valued seminorm. Furthermore*

$$\langle f; g_1 \rangle \geqq \langle f; g_2 \rangle \quad \text{if } g_1^* \leqq g_2^*.$$

In particular, $\langle f; g \rangle = \langle f^*; g^* \rangle$.

Proof. It is clear that $f \to \langle f; g \rangle$ is a solid extended valued seminorm. To prove that it is symmetric requires us to show that if $f_1, f_2 \geqq 0$ and $f_1 \sim f_2$ then $\langle f_1; g \rangle = \langle f_2; g \rangle$. We can assume f_1, f_2 are Borel maps.

Suppose $\theta > 1$ and let $\rho_\theta : [0, \infty) \to [0, \infty)$ be the function defined by $\rho_\theta(0) = 0$ and otherwise

$$\rho_\theta(t) = \theta^n, \quad \theta^n \leqq t < \theta^{n+1}, \, n \in \mathbb{Z}.$$

Assume first that $|\{f_1 > t\}| < \infty$ for all $t > 0$. Then there is an invertible measure-preserving Borel map $\sigma : \mathbb{J} \to \mathbb{J}$ so that $\rho_\theta \circ f_1 = \rho_\theta \circ f_2 \circ \sigma$. It follows that $\langle \rho_\theta \circ f_1; g \rangle = \langle \rho_\theta \circ f_2; g \rangle$. Thus

$$\langle f_1; g \rangle \leqq \theta \langle \rho_\theta \circ f_1; g \rangle = \theta \langle \rho_\theta \circ f_2; g \rangle \leqq \langle \theta f_2, g \rangle.$$

Since $\theta > 1$ is arbitrary and the roles of f_1, f_2 can be interchanged it follows that $\langle f_1; g \rangle = \langle f_2; g \rangle$.

In the case when $|\{f_1 > t\}| = \infty$ for some $t > 0$ we can find a measure preserving Borel map $\sigma : \mathbb{J} \to \mathbb{J}$ which is not necessarily surjective so that $\rho_\theta \circ f_1 \leqq \rho_\theta \circ f_2 \circ \sigma$. The argument then proceeds similarly. □

Remark. It does not seem clear whether $\mathcal{Q}(g)$ is itself a symmetric set according to our definition. However the above proposition shows that its Minkowski functional is a symmetric functional.

We also have the following elementary properties:

Proposition 4.3. *If $\|\cdot\|$ is a symmetric extended-valued seminorm on $\tilde{L}_\infty$, then*

$$\|f\| \leqq \langle f; g \rangle \|g\|, \quad f, g \in \tilde{L}_\infty,$$

and in particular

$$\langle f; h \rangle \leqq \langle f; g \rangle \langle g; h \rangle, \quad f, g, h \in \tilde{L}_\infty.$$

Thus $\langle f; f \rangle = 0$ or 1 for all $f \in \tilde{L}_\infty$.

An immediate conclusion from Lemma 4.1 is that:

Lemma 4.4. (a) *If* $f, g \in \ell_\infty(\mathbb{N})$ *and* $f \in \mathcal{Q}(g)$ *then* $f \trianglelefteq g$.

(b) *If* $f, g \in \tilde{L}_\infty(0, \infty)$ *and* $f \in \mathcal{Q}(g)$ *then* $f \trianglelefteq g$.

Hence in either case if $\langle f; g \rangle < 1$ *then* $f \trianglelefteq g$.

We now introduce an auxiliary functional. Assume that either $0 \leqq f, g \in \ell_\infty(\mathbb{N})$ or $0 \leqq f, g \in \tilde{L}_\infty(0, \infty)$. We define:

$$[[f; g]] = \inf\left\{N : f \leqq \sum_{j=1}^{N} g_j, g_j^* = g^*\right\}. \tag{4.7}$$

$[[f; g]]$ is taken to be ∞ if no such representation as above exists. It is clear that

$$[[f_1, g]] \leqq [[f_2, g]] \quad \text{if } f_1 \leqq f_2$$

and

$$[[f, g_1]] \geqq [[f, g_2]] \quad \text{if } g_1 \leqq g_2.$$

Lemma 4.5.

$$\langle f; g \rangle = \lim_{m \to \infty} \frac{1}{m} [[mf, g]]. \tag{4.8}$$

Proof. For an arbitrary $\epsilon > 0$, let $N \in \mathbb{N}$, $c_j \geqq 0$, $1 \leqq j \leqq N$, and $g_j \in \tilde{L}_\infty$, $1 \leqq j \leqq N$ be such that $g_j^* = g$ for $1 \leqq j \leqq N$ and

$$f \leqq \sum_{j=1}^{N} c_j g_j, \quad \sum_{j=1}^{N} c_j < \langle f; g \rangle + \epsilon.$$

Then, for every positive integer M, we have

$$Mf \leqq \sum_{j=1}^{N} ([Mc_j] + 1) g_j$$

where $[a]$ is the integral part of a. Consequently,

$$[[Mf, g]] \leqq \sum_{j=1}^{N} ([Mc_j] + 1) < M(\langle f; g \rangle + \epsilon) + N.$$

Conversely, for any given $M \in \mathbb{N}$, let $K = [[Mf, g]]$ and let $h_1, \ldots, h_K \geqq 0$ be such that $h_j^* = g^*$, $j = 1, 2, \ldots, K$ and

$$Mf \leqq \sum_{j=1}^{K} h_j.$$

Then we have

$$f \leqq \sum_{j=1}^{K} \frac{1}{M} h_j,$$

and, consequently, $\langle f; g \rangle \leqq \sum_{j=1}^{K} \frac{1}{M} = \frac{1}{M}[[Mf, g]]$. Hence,

$$\langle f; g \rangle \leqq \frac{1}{M}[[Mf, g]] \leqq \langle f; g \rangle + \epsilon + \frac{N}{M}.$$

Tending M to ∞ and keeping in mind that N is fixed and $\epsilon > 0$ is arbitrary, we conclude that equality (4.8) holds. □

In the next lemma we use the notation g_E for the function $g\chi_E$.

Lemma 4.6. *If $(E_n)_{n=1}^{\infty}$ is a sequence of disjoint measurable sets such that $\bigcup_n E_n = \mathbb{J}$, then*

$$[[f; g]] \leqq \sup_{n \in \mathbb{N}} [[f_{E_n}, g_{E_n}]].$$

Proof. Setting, $M_n := [[f_{E_n}, g_{E_n}]]$, $n \geqq 1$ and $M := \sup_n M_n < \infty$, we have $f_{E_n} \leqq \sum_{j=1}^{M_n} g_n^j$ where $g_n^j \sim g_{E_n}$. Trivially,

$$f_{E_n} \leqq \sum_{j=1}^{M_n} g_n^j \leqq \sum_{j=1}^{M_n} g_n^j + \sum_{j=M_n+1}^{M} g_{E_n} = \sum_{k=1}^{M} g_n^k, \quad g_n^k \sim g_{E_n}.$$

Hence, $g^k := \sum_{n=1}^{\infty} g_n^k \chi_{E_n} \sim g$. Consequently, $f \leqq \sum_{k=1}^{M} g^k$, $g^k \sim g$, that is $[[f, g]] \leqq M$. □

5. The discrete case

We first prove our main result in the discrete case. We start with an elementary deduction from Carathéodory's theorem.

Lemma 5.1. *Suppose $0 \leqq f \in c_{00}(\mathbb{N})$ and $g \in \ell_\infty(\mathbb{N})$ with $|\operatorname{supp} f| \leqq N$. Suppose*

$$\sum_{k=0}^{n} f^*(k) \leqq M \sum_{k=0}^{n} g^*(k), \quad n = 0, 1, 2, \dots,$$

where $M \in \mathbb{N}$. Then

$$[[f; g]] \leqq M + N^2.$$

Proof. We can suppose both f, g are supported on $\{1, 2, \dots, N\}$. Since the set of all extreme points of $\Omega(g)$ coincides with $\mathscr{E}(g)$ (cf. e.g. [4], [27]), it follows from the classical Carathéodory's theorem, that there are sequences $g_j \in \mathscr{E}(g)$, $1 \leqq j \leqq N^2$, such that

$$f \leqq \sum_{j=1}^{N^2} c_j g_j$$

where

$$\sum_{j=1}^{N^2} c_j = M.$$

Thus

$$f \leqq \sum_{j=1}^{N^2} ([c_j] + 1) g_j.$$

Hence

$$[[f; g]] \leqq \sum_{j=1}^{N^2} [c_j] + 1 \leqq M + N^2. \quad \square$$

Let us define the translation $\mathcal{T}_r : \ell_\infty(\mathbb{N}) \to \ell_\infty(\mathbb{N})$ by $\mathcal{T}_r(f)(j) = f(j + r)$ for $r \geqq 0$.

Lemma 5.2. *Suppose $g = g^* \in \ell_\infty(\mathbb{N})$. Suppose $f \in c_{00}$ and $|\operatorname{supp} f| = L$. Suppose further that $0 \leqq r \leqq L$ and $M \in \mathbb{N}$ are such that $f \prec\prec M\mathcal{T}_r g$, i.e.*

$$\sum_{k=1}^{s} f^*(k) \leqq M \sum_{k=r+1}^{s+r} g(k), \quad 0 \leqq s \leqq L. \tag{5.1}$$

Then

$$[[\mathcal{T}_r f; g]] \leqq M + (L/r)^2. \tag{5.2}$$

Proof. Without loss of generality we may assume that $f = f^*$. Define $\xi, \eta \in \ell_\infty(\mathbb{N})$ by

$$\xi(n) = f(nr + 1), \quad \eta(n) = g(nr + 1), \quad n \in \mathbb{N}.$$

We have $|\operatorname{supp} \xi| \leqq L/r$ and

$$r\xi(n) \leqq \sum_{i=(n-1)r+1}^{nr} f(i), \quad r\eta(n) \geqq \sum_{i=nr+1}^{(n+1)r} g(i), \quad n \in \mathbb{N}.$$

Now, using the estimates above and assumption (5.1), we obtain the estimate

$$\begin{aligned} r\sum_{k=1}^{n} \xi(k) &\leqq \sum_{k=1}^{n} \sum_{i=(k-1)r+1}^{kr} f(i) = \sum_{i=1}^{nr} f(i) \\ &\leqq M\sum_{i=1}^{nr} g(i + r) = M\sum_{k=1}^{n} \sum_{i=kr+1}^{(k+1)r} g(i) \\ &\leqq rM\sum_{k=1}^{n} \eta(k), \end{aligned}$$

for $n \in \mathbb{N}$. Thus, by Lemma 5.1, we have

$$[[\xi;\eta]] \leqq M + (L/r)^2.$$

Now,

$$\mathcal{T}_r f(n) \leqq \xi([n/r]+1), \quad n \in \mathbb{N},$$

and

$$g(n) \geqq \eta([n/r]+1), \quad n \in \mathbb{N}.$$

Thus

$$[[\mathcal{T}_r f, g]] \leqq [[\xi,\eta]] \leqq M + (L/r)^2. \quad \square$$

Lemma 5.3. *Suppose $f = f^*$, $g = g^* \in \ell_\infty(\mathbb{N})$ are such that for some $\lambda, M \in \mathbb{N}$, we have*

$$\sum_{k=\lambda m+1}^{n} f(k) \leqq M \sum_{k=m+1}^{n} g(k), \quad 0 \leqq \lambda m < n < \infty.$$

Suppose $p \in \mathbb{N}$ be such that $p-1$ is a factor of M and set $\gamma = \gamma(p,\lambda) := 1 + 2\lambda p$. If $u, v \in \mathbb{N}$ satisfy $\gamma u < v$, then we have

$$[[f_{[\gamma u+1,v]}; g_{[u+1,v]}]] \leqq Mp(p-1)^{-1} + (v/u)^2. \tag{5.3}$$

Proof. Let $N = Mp(p-1)^{-1}$, so that N is an integer and $N > M$.

For fixed u, v as in the statement of the lemma, let us introduce the function

$$H(r) := M\frac{r-2u}{r-2\lambda u}, \quad r > 2\lambda u.$$

Note that $H(r)$ is decreasing and

$$\lim_{r\to\infty} H(r) = M.$$

If $r > 2\lambda u$ we have

$$\sum_{k=2\lambda u+1}^{r} f(k) \leqq M \sum_{k=2u+1}^{r} g(k) \leqq M\frac{r-2u}{r-2\lambda u}\sum_{k=2u+1}^{r-2(\lambda-1)u} g(k)$$

so that

$$\sum_{k=2\lambda u+1}^{r} f(k) \leqq H(r) \sum_{k=2u+1}^{r-2(\lambda-1)u} g(k). \tag{5.4}$$

We shall define the scalar $w \geqq 2\lambda u$ according to the following rules.

If for all $r \in \mathbb{N}$, $r \geqq 2\lambda u + 1$, we have

$$\sum_{k=2\lambda u+1}^{r} f(k) \leqq N \sum_{k=2u+1}^{r-2(\lambda-1)u} g(k),$$

then we set $w := 2\lambda u$.

Now consider the case when the inequality above fails for some $r \geqq 2\lambda u$. By (5.4) and the properties of $H(k)$, the inequality holds for sufficiently large $r \geqq 2\lambda u$. In this case, we define w by the condition that $w \geqq 2\lambda u + 1$ is the greatest integer such that

$$\sum_{k=2\lambda u+1}^{w} f(k) > N \sum_{k=2u+1}^{w-2(\lambda-1)u} g(k). \tag{5.5}$$

In particular, by (5.4) we then have

$$H(w) = M \frac{w - 2u}{w - 2\lambda u} > N,$$

i.e.

$$Mw - 2Mu > Nw - 2\lambda Nu,$$

or

$$w < \frac{(2\lambda N - 2M)}{N - M} u \leqq 2\lambda pu \leqq (\gamma - 1)u.$$

Thus in either case we have

$$w < (\gamma - 1)u < v. \tag{5.6}$$

Now it follows that for $r \geqq w$

$$\sum_{k=2\lambda u+1}^{r} f(k) \leqq N \sum_{k=2u+1}^{r-2(\lambda-1)u} g(k).$$

Now if $w \leqq r \leqq v$ we have using (5.5) and defining empty sums to be zero,

$$\begin{aligned}
\sum_{k=w+1}^{r} f(k) &= \sum_{k=2\lambda u+1}^{r} f(k) - \sum_{k=2\lambda u+1}^{w} f(k) \\
&\leqq N \sum_{k=2u+1}^{r-2(\lambda-1)u} g(k) - \sum_{k=2\lambda u+1}^{w} f(k) \\
&\leqq N \sum_{k=2u+1}^{r-2(\lambda-1)u} g(k) - N \sum_{k=2u+1}^{w-2(\lambda-1)u} g(k) \\
&= N \sum_{k=w-2(\lambda-1)u+1}^{r-2(\lambda-1)u} g(k).
\end{aligned}$$

Thus

$$f_{[w+1,v]} \prec\prec N\mathcal{T}_{w+2u-2\lambda u}g.$$

We rewrite this as

$$f_{[w+1,v]} \prec\prec N\mathcal{T}_u\mathcal{T}_{w+u-2\lambda u}g.$$

Now Lemma 5.2 applies

$$[[\mathcal{T}_u f_{[w+1,v]}; \mathcal{T}_{w+u-2\lambda u}g]] \leqq N + \left(\frac{v-w}{u}\right)^2.$$

Since $v > w \geqq 2\lambda u$ this implies

$$[[f_{[u+w+1,u+v]}; \mathcal{T}_u g]] \leqq N + (v/u)^2$$

which gives

$$[[f_{[\gamma u+1,v)}; g_{[u+1,\infty)}]] \leqq N + (v/u)^2.$$

However it is clear that since $|\mathrm{supp}\, f_{[\gamma u+1,v)}| = v - \gamma u < v - u$ this also implies

$$[[f_{[\gamma u+1,v)}; g_{[u+1,v)}]] \leqq N + (v/u)^2. \quad \square$$

Theorem 5.4. *Suppose $f, g \in \ell_\infty(\mathbb{N})$. Then $f \trianglelefteq g$ implies $\langle f; g\rangle \leqq 1$.*

Proof. We may suppose $f = f^*$ and $g = g^*$ by Proposition 4.2. Since $f \trianglelefteq g$ there exists $\lambda \in \mathbb{N}$ so that

$$\sum_{k=\lambda m+1}^{n} f(k) \leqq \sum_{k=m+1}^{n} g(k), \quad 0 \leqq \lambda m \leqq n.$$

We now fix $p, q, r \in \mathbb{N}$ with $p > 1$ and $r > 1$. Let $u = 2\lambda p + 1$. Let $A(j,k)$ denote the set $[u^j + 1, u^k]$ for $0 \leqq j < k$.

Now we let $M = (p-1)q$ and note that we can now appeal to Lemma 5.3 to deduce that

$$[[(p-1)qf_{A(j+1,k)}; g_{A(j,k)}]] \leqq pq + u^{2(k-j)}, \quad 0 \leqq j < k-1.$$

In particular, if $n \in \mathbb{N}$,

$$[[(p-1)qf_{A(k+1+(n-1)r,k+nr)}; g_{A(k+(n-1)r,k+nr)}]] \leqq pq + u^{2r}, \quad 1 \leqq k \leqq r.$$

On the other hand, by Lemma 5.1,

$$[[(p-1)qf_{[1,u^k]}; g_{[1,u^k]}]] \leqq (p-1)q + u^{2r}, \quad 1 \leqq k \leqq r.$$

Let

$$f_k = f_{[1,u^k]} + \sum_{n=1}^{\infty} f_{A(k+1+(n-1)r,k+nr)}.$$

Then, by Lemma 4.6, we obtain

$$[[(p-1)qf_k;g]] \leqq pq + u^{2r}.$$

This implies that

$$\langle f_k;g\rangle \leqq \frac{pq+u^{2r}}{(p-1)q}.$$

Now

$$f - f_k = \sum_{n=1}^{\infty} f_{A(k+(n-1)r,k+1+(n-1)r)}$$

so that

$$\sum_{k=1}^{r} (f - f_k) \leqq f$$

or

$$\sum_{k=1}^{r} f_k \geqq (r-1)f.$$

Thus

$$\langle (r-1)f;g\rangle \leqq r\frac{pq+u^{2r}}{(p-1)q}$$

or

$$\langle f;g\rangle \leqq \frac{r\big(pq+(1+2\lambda p)^{2r}\big)}{(p-1)q(r-1)}.$$

Letting $q \to \infty$ we have

$$\langle f;g\rangle \leqq \frac{rp}{(p-1)(r-1)},$$

and then letting $r \to \infty$, $p \to \infty$ we have $\langle f;g\rangle \leqq 1$. $\square$

The following theorem is now almost immediate:

Theorem 5.5. *If $f, g \in \ell_\infty(\mathbb{N})$ we have*

$$\langle f; g \rangle = \lim_{\lambda \to \infty} \sup_{0 \leqq \lambda m < n} \frac{\sum_{k=\lambda m+1}^{n} f^*(k)}{\sum_{k=m+1}^{n} g^*(k)}.$$

Remark. Here we interpret 0/0 as 0.

Proof. Suppose

$$\theta = \lim_{\lambda \to \infty} \sup_{0 \leqq \lambda m < n} \frac{\sum_{k=\lambda m+1}^{n} f^*(k)}{\sum_{k=m+1}^{n} g^*(k)}.$$

Assume $\theta < \infty$. Then for any $\alpha > \theta$ there exists λ so that

$$\sup_{0 \leqq \lambda m < n} \frac{\sum_{k=\lambda m+1}^{n} f^*(k)}{\sum_{k=m+1}^{n} g^*(k)} < \alpha$$

and so by Theorem 5.4 we have $\langle f; g \rangle < \alpha$. Hence $\langle f; g \rangle \leqq \theta$.

Conversely if $\alpha > \langle f; g \rangle$ we have $\langle \alpha^{-1} f; g \rangle < 1$ and so by Lemma 4.4 we conclude that $\alpha^{-1} f \trianglelefteq g$ and hence $\theta \leqq \alpha$. □

Examples. At this point we give two examples to illustrate the above results. Let us give a simple example to show that it is not true that $f \trianglelefteq g$ implies $f \in \mathcal{Q}(g)$, although it implies $f \in (1+\epsilon)\mathcal{Q}(g)$ for every $\epsilon > 0$. Define the sets $A_0 = \{1\}$ and then $A_r = [2^{r-1}+1, 2^r]$. Let g be any strictly decreasing positive sequence with the property that if $r \geqq 2$ then

$$\frac{1}{|A_r|} \sum_{k \in A_r} g(k) > g(2^{r-1}+2).$$

Let

$$f(k) = \frac{1}{|A_r|} \sum_{k \in A_r} g(k), \quad k \in A_r, \ r = 0, 1, \ldots .$$

Then it is easily verified that $f \trianglelefteq g$. Indeed $f \prec\prec g$ and if $m \geqq 1$ then

$$\sum_{k=2m+1}^{n} f(k) \leqq \sum_{k=m+1}^{n} g(k), \quad 2m \leqq n,$$

since if $m \in A_r$ then $2m \in A_{r+1}$.

On the other hand suppose

$$f \leqq \sum_{j=1}^{N} c_j g_j$$

where $c_j > 0$, $\sum c_j = 1$ and $g_j = g \circ \sigma_j$ where each σ_j is a permutation of $\mathbb{N}$. Since

$$\sum_{k=1}^{2^r} f(k) = \sum_{k=1}^{2^r} g(k), \quad r = 0, 1, \dots,$$

and g is *strictly* decreasing it is clear that each σ_j leaves the sets $[1, 2^r]$ invariant and hence also the sets A_r.

Pick r such that $2^{r-1} > N$. Then there exists $k \in A_r$ so that $\sigma_j(k) \neq 2^{r-1} + 1$ for all $1 \leqq j \leqq N$ and hence $g_j(k) < f(k)$ for all j so that

$$\sum_{j=1}^{N} c_j g_j(k) < f(k).$$

Hence $f \notin \mathcal{Q}(g)$ and the first example is concluded.

Next we answer a question raised by Peter Dodds. As he observed if $\|f\|_E \leqq \|g\|_E$ for all fully symmetric norms we have $f \prec\prec g$. However we show that if $\|f\|_E \leqq \|g\|_E$ for all symmetric norms it does not follow that we have $f \trianglelefteq g$. To do this we only need to show, by Propositions 4.2 and 4.3, that it is not true that $\langle f; g \rangle \leqq 1$ implies that $f \trianglelefteq g$. Define the sets A_r as above. Define a decreasing sequence g by $g(1) = 1$ and $g(k) = 2^{1-r}$ when $k \in A_r$. We next define a sequence $\alpha_r = \epsilon_r / 4(r+1)$ where $\epsilon_r = \pm 1$ are chosen so that we have the properties

$$\sup_{s \geqq 0} \sum_{r=0}^{s} \alpha_r = \limsup_{s \to \infty} \sum_{r=0}^{s} \alpha_r = 0$$

and

$$\liminf \sum_{r=0}^{s} \alpha_r = -\infty.$$

Then let

$$f(k) = (1 + \alpha_r) g(k), \quad k \in A_r.$$

f is also a decreasing sequence. For $r \geqq 0$ we have

$$\sum_{k=1}^{2^r} f(k) = \sum_{j=0}^{r} \alpha_j + r + 1 \leqq \sum_{k=1}^{2^r} g(k)$$

and hence by interpolation

$$\sum_{k=1}^{n} f(k) \leqq \sum_{k=1}^{n} g(k), \quad n = 1, 2, \ldots,$$

i.e. $f \prec\prec g$. If $\lambda \in \mathbb{N}$ and $m \geqq 1$, then for $n \geqq \lambda m + 1$,

$$\sum_{k=\lambda m+1}^{n} f(k) \leqq \left(1 + (4\lambda m)^{-1}\right) \sum_{k=\lambda m+1}^{n} g(k),$$

and so combined with the case $m = 0$ we have $\langle f; g\rangle \leqq \left(1 + (4\lambda m)^{-1}\right)$. Hence $\langle f; g\rangle \leqq 1$.

Finally let us suppose that $f \trianglelefteq g$. Then for some $\lambda \in \mathbb{N}$ we have

$$\sum_{k=\lambda m+1}^{n} f(k) \leqq \sum_{k=m+1}^{n} g(k), \quad 0 \leqq \lambda m < n.$$

We can suppose λ is a power of two, i.e. $\lambda = 2^s$ for some fixed $s \geqq 1$. It follows that

$$\sum_{r=m+s+1}^{t} \sum_{k \in A_r} f(k) \leqq \sum_{r=m+1} \sum_{k \in A_r} g(k), \quad m + s < t.$$

This implies that

$$t - m - s + \sum_{r=m+s+1}^{t} \alpha_r \leqq t - m$$

or

$$\sum_{r=m+s+1}^{t} \alpha_r \leqq s, \quad m + s < t.$$

Since m, t are arbitrary this contradicts the choice of the $(\alpha_r)_{r=0}^{\infty}$. This concludes the second example.

6. The continuous case

The details for the continuous case are quite similar but require a few modifications. We start with the analogue of Lemma 5.2.

We define the translation $\mathcal{T}_r : \tilde{L}_\infty(0, \infty) \to \tilde{L}_\infty(0, \infty)$ by $\mathcal{T}_r(f)(s) = f(r+s)$ for $r \geqq 0$.

Lemma 6.1. *Let $f = f^*$, $g = g^* \in \tilde{L}_\infty(0, \infty)$. Suppose $|\operatorname{supp} f| = L < \infty$. Suppose further that $0 \leqq r \leqq L$ and $M \in \mathbb{N}$ are such that $f \prec\prec M\mathcal{T}_r g$, i.e.*

$$\int_0^a f(s)\,ds \leqq M \int_r^{a+r} g(s)\,ds, \quad 0 \leqq s \leqq L. \tag{6.1}$$

Then

$$[[\mathcal{T}_r f; g]] \leqq M + (L/r)^2. \tag{6.2}$$

Proof. Define $\xi, \eta \in \ell_\infty(\mathbb{N})$ by

$$\xi(n) = f(nr), \quad \eta(n) = g(nr), \quad n \in \mathbb{N}.$$

Then $|\operatorname{supp} \xi| \leqq L/r$ and

$$r\xi(k) \leqq \int_{(k-1)r}^{kr} f(t)\,dt, \quad r\eta(k) \geqq \int_{kr}^{(k+1)r} g(t)\,dt, \quad k \in \mathbb{N}.$$

Hence

$$r\sum_{k=1}^{n} \xi(k) \leqq \int_0^{nr} f(t)\,dt \leqq M \int_r^{(n+1)r} g(t)\,dt \leqq M \sum_{k=1}^{n} \eta(k)$$

for $n \in \mathbb{N}$.

Thus, according to Lemma 5.1,

$$[[\xi; \eta]] \leqq M + (L/r)^2.$$

Now,

$$\mathcal{T}_r f \leqq \sum_{k=1}^{\infty} \xi(k) \chi_{[(k-1)r, kr)}$$

and

$$g \geqq \sum_{k=1}^{\infty} \eta(k) \chi_{[(k-1)r, kr)},$$

and it follows easily that

$$[[\mathcal{T}_r f; g]] \leqq M + (L/r)^2. \quad \square$$

Lemma 6.2. *Suppose $f = f^*$, $g = g^* \in \tilde{L}_\infty(0, \infty)$ are such that for some $\lambda, M \in \mathbb{N}$, we have*

$$\int_{\lambda a}^{b} f(s)\,ds \leqq M \int_a^b g(s)\,ds, \quad 0 < \lambda a < b < \infty.$$

Suppose $p \in \mathbb{N}$ is such that $(p-1)$ is a factor of M and set $\gamma = \gamma(p,\lambda) := 1 + 2\lambda p$. If $u, v \in (0, \infty)$ satisfy $\gamma u < v$, then we have

$$[[f_{[\gamma u, v)}; y_{[u,v)}]] \leqq Mp(p-1)^{-1} + (v/u)^2. \tag{6.3}$$

Proof. As in Lemma 5.3 let $N = Mp(p-1)^{-1}$.

For fixed u, v as before let

$$H(r) := M\frac{r-2u}{r-2\lambda u}, \quad r > 2\lambda u.$$

If $r > 2\lambda u$ we have (as in Lemma 5.3):

$$\int_{2\lambda u}^{r} f(s)\,ds \leqq H(r) \int_{2u}^{r-2(\lambda-1)u} g(s)\,ds.$$

Consider the equation in $r > 2\lambda u$:

$$\int_{2\lambda u}^{r} f(s)\,ds = N \int_{2u}^{r-2(\lambda-1)u} g(s)\,ds.$$

If this equation has a solution then it has a largest solution which we denote by w. If it has no solution let $w = 2\lambda u$.

Arguing as in Lemma 5.3 we obtain the analogue of (5.6):

$$w < \frac{(2\lambda N - 2M)}{N-M} u \leqq 2\lambda p u \leqq (\gamma - 1)u. \tag{6.4}$$

If $w \leqq r \leqq v$,

$$\begin{aligned} \int_{w}^{r} f(s)\,ds &= \int_{2\lambda u}^{r} f(s)\,ds - \int_{2\lambda u}^{w} f(s)\,ds \\ &\leqq N \int_{2u}^{r-2(\lambda-1)u} g(s)\,ds - N \int_{2u}^{w-2(\lambda-1)u} g(s)\,ds \\ &= N \int_{w-2(\lambda-1)u}^{r-2(\lambda-1)u} g(s)\,ds. \end{aligned}$$

Thus

$$f_{[w,v)} \prec\prec N g_{[w-2(\lambda-1)u, v-2(\lambda-1)u)}.$$

The proof is completed as in Lemma 5.3, using Lemma 6.1. □

Theorem 6.3. *Suppose $f, g \in \tilde{L}_\infty(0, \infty)$. Then $f \trianglelefteq g$ implies $\langle f; g\rangle \leqq 1$.*

Proof. We may suppose $f = f^*$ and $g = g^*$ by Proposition 4.2. Since $f \trianglelefteq g$ there exists $\lambda \in \mathbb{N}$ so that

$$\int_{\lambda a}^{b} f(s)\,ds \leqq \int_{a}^{b} g(s)\,ds, \quad 0 < \lambda a \leqq b.$$

As in Theorem 5.4 we fix $p, q, r \in \mathbb{N}$ with $p > 1$ and $r > 1$ and let $u = 2\lambda p + 1$. Let $A(j,k)$ denote the set $[u^j, u^k)$ for $j, k \in \mathbb{Z}$ with $j < k$. We appeal to Lemma 6.2 to deduce that

$$[[(p-1)qf_{A(j+1,k)}; g_{A(j,k)}]] \leqq pq + u^{2(k-j)}, \quad 0 \leqq j < k-1.$$

In particular, if $n \in \mathbb{N}$,

$$[[(p-1)qf_{A(k+1+(n-1)r,k+nr)}; g_{A(k+(n-1)r,k+nr)}]] \leqq pq + u^{2r}, \quad 1 \leqq k \leqq r.$$

Let

$$f_k = \sum_{n\in\mathbb{Z}} f_{A(k+1+(n-1)r,k+nr)} = f - \sum_{n\in\mathbb{Z}} f_{A(k+nr,k+1+nr)}.$$

Then, by Lemma 4.6, we obtain

$$[[(p-1)qf_k; g]] \leqq pq + u^{2r}.$$

This implies by Lemma 4.5 that

$$\langle f_k; g\rangle \leqq \frac{pq + u^{2r}}{(p-1)q}.$$

Now

$$(r-1)f = f_1 + \cdots + f_r$$

so that

$$\langle (r-1)f; g\rangle \leqq r\frac{pq + u^{2r}}{(p-1)q},$$

and this implies that $\langle f; g\rangle \leqq 1$ as before. □

Theorem 6.4. *If $f, g \in \tilde{L}_\infty(0, \infty)$ we have*

$$\langle f; g\rangle = \lim_{\lambda\to\infty} \sup_{0<\lambda a<b} \frac{\int_{\lambda a}^{b} f^*(s)\,ds}{\int_{a}^{b} g^*(s)\,ds}.$$

We remark at this point on one essential difference between the discrete and continuous cases. In the discrete case $\langle f; g\rangle = \langle g; f\rangle = 1$ implies $f \prec\prec g$ and $g \prec\prec f$ and hence $f^* = g^*$. In the continuous case, let $0 < \theta < 1/\sqrt{2}$ and consider

$$f(x) = 1/x, \quad g(x) = \big(1 + \theta \sin(\log x)\big)/x.$$

Then f and g are decreasing positive functions and $\langle f; g\rangle, \langle g; f\rangle \leqq 1$. However $\langle f; f\rangle \neq 0$ since by the results of Cwikel and Fefferman [6] and Sparr [32] the set $\mathcal{Q}(f)$ is a proper subset of weak L_1 (see the next section §7 for more details). Hence by Proposition 4.3, $\langle f; g\rangle = \langle g; f\rangle = 1$.

7. Some applications to envelopes

We start with a standard and well-known lemma:

Lemma 7.1. *Let g be any decreasing function on $(0, \infty)$. Then the following conditions are equivalent*:

(i) *There exist $\delta > 0$ and a constant C so that*

$$g(s) \leqq C(t/s)^{1+\delta} g(t), \quad s > t.$$

(ii)

$$\lim_{\lambda\to\infty} \sup_{t>0} \frac{\lambda g(\lambda t)}{g(t)} = 0.$$

Proof. It is clear that (i) implies (ii). Pick $\lambda > 1$ so that

$$\lambda g(\lambda t) \leqq \frac{1}{2} g(t), \quad t > 0.$$

Then if $\lambda^{n-1} t < s \leqq \lambda^n t$ where $n \in \mathbb{N}$, we have

$$g(s) \leqq (2\lambda)^{-(n-1)} g(t) \leqq 2\lambda (t/s)^{1+\delta} g(t)$$

where

$$\delta = \frac{\log 2}{\log \lambda}. \quad \square$$

The proof of the following lemma is quite similar and we omit it.

Lemma 7.2. *Let g be any decreasing function on $(0, \infty)$. Then the following conditions are equivalent*:

(i) *There exist $\delta > 0$ and a constant C so that*

$$g(s) \leqq C(t/s)^{1-\delta} g(t), \quad 0 < s < t.$$

(ii)

$$\lim_{\lambda\to\infty}\sup_{t>0}\frac{\lambda g(\lambda t)}{g(t)}=\infty.$$

Proposition 7.3. *Let $g \in \tilde{L}_\infty(0, \infty)$ be a decreasing non-negative function which is not identically zero. Then the following are equivalent*:

(7.1) $$\langle g; g\rangle = 0,$$

(7.2) $$\lim_{\lambda\to\infty}\sup_{t>0}\frac{\lambda g(\lambda t)}{g(t)}=0.$$

Remark. In (7.2) we take $0/0 = 0$ as usual.

Proof. If $\langle g; g\rangle = 0$ then there exists $\mu > 1$ so that

$$\int_{\mu a}^{b} g(s)\,ds \leqq \frac{1}{2}\int_a^b g(s)\,ds, \quad 0 < \mu a < b.$$

This implies

$$\int_a^{\mu a} g(s)\,ds \geqq \frac{1}{2}\int_a^b g(s)\,ds, \quad 0 < \mu a < b.$$

In particular g is integrable on (a, ∞) if $a > 0$. Let

$$H(t) = \int_t^\infty g(s)\,ds.$$

Then

$$tg(2t) \leqq H(t) \leqq Ctg(t), \quad 0 < t < \infty,$$

where $C = 2(\mu - 1)$. Now H is decreasing and

$$\frac{H'(t)}{H(t)} \leqq -\frac{1}{Ct} \quad \text{a.e. } 0 < t < \infty,$$

so that $t^{1/C}H(t)$ is decreasing. This implies (7.2), since

$$\frac{\lambda g(\lambda t)}{g(t)} \leqq 2C\frac{H(\lambda t/2)}{H(t)} \leqq 2C(\lambda/2)^{-1/C}, \quad \lambda > 2,\ t > 0.$$

If we assume (7.2) then by Lemma 7.1, for some $C, \delta > 0$ we have an estimate

$$g(s) \leqq C(t/s)^{1+\delta}g(t), \quad s > t.$$

Hence if $\lambda > 1$ and $0 < \lambda a < b$, we have

$$\int_{\lambda a}^{b} g(s)\,ds \leqq \frac{C}{\lambda^{1+\delta}} \int_{\lambda a}^{b} g(s/\lambda)\,ds$$

$$= C\lambda^{-\delta} \int_{a}^{\lambda^{-1}b} g(s)\,ds$$

$$\leqq C\lambda^{-\delta} \int_{a}^{b} g(s)\,ds$$

so that $\langle g; g\rangle = 0$, i.e. (7.1) holds. $\square$

Suppose $g \in \tilde{L}_\infty(0, \infty)$ is a decreasing non-negative function (which is not identically zero). We recall that the metric abelian group $\mathscr{X}_g$ is the space of functions $f \in \tilde{L}_\infty(0, \infty)$ such that

$$|f|_g = \inf\{\lambda > 0 : f^*(\lambda t) \leqq \lambda g(t)\} < \infty.$$

These spaces have been studied in detail in [20], [2] and [1].

If g satisfies the condition

$$\sup_{t>0} \frac{g(t/2)}{g(t)} < \infty$$

then $\mathscr{X}_g$ is a topological vector space, which is a quasi-Banach space under the quasi-norm

$$\|f\|_{\mathscr{X}_g} = \sup_{t>0} \frac{f^*(t)}{g(t)}.$$

Under these circumstances we say that g is *weakly regular* ([21]).

A linear functional φ on $\mathscr{X}_g$ is continuous if and only if

$$\|\varphi\| = \sup_{f \prec g} |\varphi(f)| < \infty.$$

Here we use the fact that, even in the non weakly regular case, if φ is bounded on any set $\{f : |f|_g < \epsilon\}$ then it is also bounded on $\{f : |f|_g \leqq 1\}$ since there is an integer N so that if $|f|_g \leqq 1$ then $f = f_1 + \cdots + f_N$ with $|f_j|_g < \epsilon$ for $j = 1, 2, \ldots, N$. We also use the fact that $\{f : |f|_g < 1\} \subset \{f : f \prec g\} \subset \{f : |f|_g \leqq 1\}$. Thus $\mathscr{X}_g^*$ is always identifiable as a Banach space. Furthermore

$$\|\varphi\| = \sup_{f \in \mathscr{Q}(g)} |\varphi(f)|.$$

If we define the seminorm

$$\|f\|_{\hat{\mathscr{X}}_g} = \sup_{\|\varphi\| \leqq 1} |\varphi(f)|, \quad f \in \mathscr{X}_g,$$

then $\mathscr{X}_g^*$ can be identified with the dual of the seminormed space $(\mathscr{X}_g, \|\cdot\|_{\hat{\mathscr{X}}_g})$. If g is weakly regular, the Hausdorff completion of this space is usually called the *Banach envelope* of $\mathscr{X}_g$ and denoted $\hat{\mathscr{X}}_g$. We refer to [15] for a fuller discussion of this concept. By the Hahn-Banach theorem it is clear that $\|\cdot\|_{\hat{\mathscr{X}}_g}$ is simply the Minkowski functional associated to the set $\mathscr{Q}(g)$. Hence

Theorem 7.4.

$$\|f\|_{\hat{\mathscr{X}}_g} = \langle f; g\rangle = \lim_{\lambda\to\infty} \sup_{0<\lambda a<b} \frac{\int_{\lambda a}^{b} f^*(s)\,ds}{\int_a^b g^*(s)\,ds}, \quad f \in \mathscr{X}_g. \tag{7.3}$$

Let us use this to recover two known results:

Theorem 7.5. $\mathscr{X}_g^* = \{0\}$ *if and only if*

$$\lim_{\lambda\to\infty} \sup_{t>0} \frac{\lambda g(\lambda t)}{g(t)} = 0. \tag{7.4}$$

Remark. This was first proved by A. Sparr in her thesis in 1971 [32].

Proof. The statement that $\mathscr{X}_g^* = \{0\}$ is, by the above remarks, equivalent to the fact that $\langle g; g\rangle = 0$. Thus the theorem is a restatement of Proposition 7.3. □

If $g \in L_1 + L_\infty$ is non-negative and decreasing, we will write

$$\psi(t) = \int_0^t g(s)\,ds$$

and $h(t) = \psi(t)/t$. Note that $g(t) \leqq h(t)$. The following theorem is essentially known; in the case of finite measure spaces it is due to Mekler [20] and [21]. Mekler's proof could be modified to work in infinite measure spaces, but for completeness we give an independent proof.

Theorem 7.6. *If* $g \in \tilde{L}_\infty(0, \infty)$ *is a non-negative decreasing function, the following conditions are equivalent*:

(i) $\mathscr{X}_g$ *is isomorphic to a Banach space.*

(ii) *We have* $g \in L_1 + L_\infty$ *and* $h \in \mathscr{X}_g$ *or equivalently for some constant* C *we have* $h(t) \leqq Cg(t)$ *for all* $t > 0$.

(iii) *We have*

$$\lim_{\lambda\to\infty} \inf_{t>0} \frac{\lambda g(\lambda t)}{g(t)} = \infty. \tag{7.5}$$

(iv) $g \in L_1 + L_\infty$ *and*

$$\langle f; g\rangle = \|f\|_{M(\psi)}, \quad f \in \tilde{L}_\infty(0, \infty). \tag{7.6}$$

Proof. (i) ⇒ (ii). In this case $\|\cdot\|_{\mathscr{X}_g}$ is equivalent to a norm and so there is a constant C so that if $\|f_j\|_{\mathscr{X}_g} \leqq 1$ for $j = 1, 2, \ldots, N$ then

$$\|f_1 + \cdots + f_N\|_{\mathscr{X}_g} \leqq CN.$$

For any fixed t we can find rearrangements $f_1, \ldots, f_N$ of g so that

$$f_1(t) + \cdots + f_N(t) \geqq \sum_{k=1}^{N} g(kt/N)$$

and hence

$$\frac{1}{N}\sum_{k=1}^{N} g(kt/N) \leqq Cg(t), \quad N = 1, 2, \ldots, \ t > 0.$$

Letting $N \to \infty$ we obtain that $g \in L_1 + L_\infty$ and $h(t) \leqq Cg(t)$.

(ii) ⇒ (iii). In this case we have $\psi(t) \leqq Ct\psi'(t)$ almost everywhere and hence $t^{-1/C}\psi(t)$ is increasing. Hence if $\lambda > 1$,

$$\begin{aligned}\frac{\lambda g(\lambda t)}{g(t)} &\geqq \frac{\lambda h(\lambda t)}{Ch(t)} = C^{-1}\frac{\psi(\lambda t)}{\psi(t)} \\ &\geqq C^{-1}\lambda^{1/C}.\end{aligned}$$

(iii) ⇒ (iv). By Lemma 7.2 it is trivial that (7.5) implies $g \in L_1 + L_\infty$. Notice also that $\langle f; g\rangle \geqq \|f\|_{M(\psi)}$ for all $f \in \tilde{L}_\infty(0, \infty)$.

Now, for any $\epsilon > 0$ we can choose λ_0 so that if $\lambda \geqq \lambda_0$ we have $\lambda g(\lambda t) \geqq \epsilon^{-1} g(t)$. Thus

$$\int_0^{\lambda t} g(s)\,ds = \int_0^{t} \lambda g(\lambda s)\,ds \geqq \epsilon^{-1}\int_0^t g(s)\,ds.$$

It follows that if $b > \lambda_0 a$ then

$$\int_a^b g(s)\,ds \geqq (1-\epsilon)\int_0^b g(s)\,ds$$

and so if $\lambda a < b$ then

$$\frac{\int_{\lambda a}^{b} f^*(s)\,ds}{\int_a^b g(s)\,ds} \leqq (1-\epsilon)^{-1}\sup_{t>0}\frac{\int_0^t f^*(s)\,ds}{\int_0^t g(s)\,ds}$$

so that (7.6) holds.

(iv) $\Rightarrow$ (i). In this case $\mathscr{X}_g = \{f : \langle f; g\rangle < \infty\}$ coincides as a set with M_ψ. First we show that g must be weakly regular so that $\mathscr{X}_g$ is a quasi-Banach space. Indeed if we set

$$F(t) = \frac{1}{t^2}\int_0^t sg(s)\,ds$$

then F is decreasing and

$$\int_0^t F(s)\,ds = \int_0^t \left(1 - \frac{s}{t}\right) g(s)\,ds \leqq \psi(t)$$

so that $F \in M(\psi)$. Hence for some choice of λ we have

$$F(t) \leqq \lambda g(\lambda t)$$

or

$$F(t/\lambda) \leqq \lambda g(t).$$

In particular

$$\frac{1}{2} g(t/\lambda) \leqq \lambda g(t)$$

whence g is weakly regular. Thus $\mathscr{X}_g$ is a quasi-Banach space and we can apply the Closed Graph Theorem to show that the quasi-norm topology agrees with the norm topology of $M(\psi)$. In particular we have (i). □

We also note that a special case of Theorem 7.4 was found by Cwikel and Fefferman [6] and [7]. They considered the case $g(x) = 1/x$ when $\mathscr{X}_g$ coincides with $L_{1,\infty}$ = weak L_1 and obtained the formula

$$\|f\|_{\hat{L}_{1,\infty}} = \lim_{\lambda\to\infty} \sup_{b/a \geqq \lambda} \frac{1}{\log(b/a)} \int_{a \leqq |f(t)| \leqq b} |f(t)|\,dt. \tag{7.7}$$

Let us show that the Cwikel-Fefferman formula (7.7) coincides with the formula (7.3) of Theorem 7.4. Assume, without loss of generality, that $\|f\|_{L_{1,\infty}} = 1$ so that $f^*(t) \leqq 1/t$. Then for $a < b$ let

$$I(a,b) = \int_a^b f^*(t)\,dt, \quad J(a,b) = \int_{1/b \leqq |f(t)| \leqq 1/a} |f(t)|\,dt.$$

Let $\alpha = \inf\{t : f^*(t) \leqq 1/a\}$ and $\beta = \sup\{t : f^*(t) \geqq 1/b\}$ so that

$$J(a,b) = \int_\alpha^\beta f^*(t)\,dt.$$

Note that $\alpha \leqq a$ and $\beta \leqq b$. We have

$$\int_{\alpha}^{a} f^*(t)\,dt \leqq a^{-1}(a-\alpha) \leqq 1$$

and

$$\int_{\beta}^{b} f^*(t)\,dt \leqq b^{-1}(b-\beta) \leqq 1.$$

Hence

$$|I(a,b) - J(a,b)| \leqq 1$$

and it is clear that (7.7) and (7.3) are equivalent.

8. Applications to non-commutative function spaces

We now turn to non-commutative applications. Throughout this section $\mathscr{M}$ will denote, as before, a semi-finite von Neumann algebra with a fixed faithful normal semi-finite trace τ such that $\mathscr{M}$ is atomless and $\tau(\mathbf{1}) = \infty$. However, we wish to emphasize that our results apply equally when $\mathscr{M} = \mathscr{L}(\mathscr{H})$ and so we will state the operator forms of our results, even though these are essentially special cases.

The first lemma is due to Wielandt [37]:

Lemma 8.1. *Suppose $T : \mathscr{H} \to \mathscr{H}$ is a positive operator. Then*

$$\sum_{k=m+1}^{n} s_k(T) = \sup_{\operatorname{tr} P = n} \inf_{\substack{Q \leqq P \\ \operatorname{tr} Q = n-m}} \operatorname{tr} QTQ,$$

where P, Q are orthogonal projections.

Now let us prove the corresponding result for $\mathscr{M}$:

Lemma 8.2. *Suppose $x \in \tilde{\mathscr{M}}_+$. Then*

$$\int_a^b \mu_t(x)\,dt = \sup_{\tau(e)=b} \inf_{\substack{f \leqq e \\ \tau(f)=b-a}} \tau(fx), \quad 0 < a < b < \infty,$$

where $e, f \in \mathscr{P}$.

Proof. Let us prove this under the additional assumption that $\lim_{t\to\infty} \mu_t(x) = 0$; the general case follows by an approximation argument. We first prove that if $\tau(e) = b$ then

$$\inf_{\substack{f \leqq e \\ \tau(f)=b-a}} \tau(fx) \leqq \int_a^b \mu_t(x)\,dt.$$

First assume $x \in \mathscr{M}_+$. Then if $\tau(e) = b$ we have using [13], Lemma 4.1,

$$\sup_{\substack{e \geqq e' \in \mathscr{P} \\ \tau(e')=a}} \tau(e'xe') = \int_0^a \mu_t(exe)\,dt$$

and so

$$\inf_{\substack{e \geqq f \in \mathscr{P} \\ \tau(f)=b-a}} \tau(fxf) = \int_a^b \mu_t(exe)\,dt \leqq \int_a^b \mu_t(x)\,dt.$$

For the general case we observe that if $\tau(e) = b$ and $\epsilon > 0$ there exists $e' \in \mathscr{P}$ with $e' \leqq e$ and $\tau(e') = b - \epsilon$ and $e'xe' \in \mathscr{M}_+$. Thus

$$\inf_{\substack{e \geqq f \in \mathscr{P} \\ \tau(f)=b-a}} \tau(fxf) \leqq \int_{a-\epsilon}^{b-\epsilon} \mu_t(e'xe')\,dt \leqq \int_{a-\epsilon}^{b-\epsilon} \mu_t(x)\,dt.$$

Letting $\epsilon \to 0$ gives the result.

To deduce equality we note that we can find $e \in \mathscr{P}$ so that $\tau(e) = b$ and $\mu(exe) = \mu(x)\chi_{[0,b)}$ by [9], Lemma 2.6. In this case

$$\int_a^b \mu_t(x)\,dt = \min_{\substack{f \leqq e \\ \tau(f)=b-a}} \tau(fx). \quad \square \tag{8.1}$$

Lemma 8.3. *Suppose $T_j : \mathscr{H} \to \mathscr{H}$ are positive operators for $j = 1, 2, \dots, k$. Then*

$$\sum_{j=kr+1}^{n} s_j(T_1 + \cdots + T_k) \leqq \sum_{j=r+1}^{n} \big(s_j(T_1) + \cdots + s_j(T_k)\big), \quad 0 \leqq kr < n.$$

Lemma 8.4. *Suppose $x_j \in \tilde{\mathscr{M}}_+$ for $j = 1, 2, \dots, k$. Then*

$$\int_{ka}^{b} \mu_t(x_1 + \cdots + x_k)\,dt \leqq \int_a^b \big(\mu_t(x_1) + \cdots + \mu_t(x_k)\big)\,dt, \quad 0 < ka < b.$$

We will actually need a stronger form of Lemma 8.4 which we prove below. This implies both Lemmas 8.3 and 8.4.

Lemma 8.5. *Suppose $x_j \in \tilde{\mathscr{M}}_+$ for $j = 1, 2, \dots, k$ and $\alpha_1, \dots, \alpha_k > 0$ with $\alpha_1 + \cdots + \alpha_k \leqq 1$. Then*

$$\int_a^b \mu_t(x_1 + \cdots + x_k)\,dt \leqq \sum_{j=1}^{k} \int_{\alpha_j a}^{b} \mu_t(x_j)\,dt, \quad 0 < a < b.$$

Proof. We prove this under the additional assumption that $\lim_{t\to\infty} \mu_t(x_j) = 0$ for $1 \leqq j \leqq k$. The case when this fails will then follow easily by an approximation argument. We pick $e \in \mathscr{P}$ with $\tau(e) = b$ and $\mu_t\big(e(x_1 + \cdots + x_k)e\big) = \mu_t(x_1 + \cdots + x_k)$ for $0 < t \leqq b$ (again using [9], Lemma 2.6). For each $j = 1, 2, \dots, k$ we can find $e \geqq e_j \in \mathscr{P}$ with $\tau(e_j) = \alpha_j a$ and $\mu_t(e_j x_j e_j) = \mu_t(e x_j e)$ for $0 \leqq t \leqq \alpha_j a$ and

$$\mu_t\big((e-e_j)x_j(e-e_j)\big) = \mu_{t+\alpha_j a}(ex_je).$$

Pick $f' = e_1 \vee \cdots \vee e_k$. Then $f' \leqq e$ and $\tau(f') \leqq a$. Since $\mathscr{M}$ is atomless there exists $f \in \mathscr{P}$ with $e \geqq f \geqq f'$ and $\tau(f) = a$. By (8.1)

$$\begin{aligned}\int_a^b \mu_t(x_1+\cdots+x_k)\,dt &\leqq \tau\big((e-f)(x_1+\cdots+x_k)\big)\\ &\leqq \sum_{j=1}^k \tau\big((e-f')x_j(e-f')\big) \leqq \sum_{j=1}^k \tau\big((e-e_j)x_j(e-e_j)\big)\\ &= \sum_{j=1}^k \int_{\alpha_j a}^b \mu_t(ex_je)\,dt \leqq \sum_{j=1}^k \int_{\alpha_j a}^b \mu_t(x_j)\,dt. \quad \Box\end{aligned}$$

Proposition 8.6. (i) *Suppose $T_j : \mathscr{H} \to \mathscr{H}$ are bounded operators for $j = 1, 2, \ldots, k$. Then*

$$\big(s_n(T_1+\cdots+T_k)\big)_{n=1}^\infty \trianglelefteq \left(\sum_{j=1}^k s_n(T_j)\right)_{n=1}^\infty.$$

(ii) *Suppose $x_1, \ldots, x_k \in \tilde{\mathscr{M}}$ for $j = 1, 2, \ldots, k$. Then*

$$\mu(x_1+\cdots+x_k) \trianglelefteq \mu(x_1)+\cdots+\mu(x_k).$$

Proof. We prove (ii). There exist partial isometries $u_1, \ldots, u_k \in \mathscr{M}$ such that

$$|x_1+\cdots+x_k| \leqq u_1|x_1|u_1^* + \cdots + u_k|x_k|u_k^*$$

([13], Lemma 4.3; see also [17]). This implies (ii) by Lemma 8.4. $\Box$

The following then follows immediately using Proposition 4.3 and Theorem 6.3, since as remarked in §3 we can always replace $\mathscr{M}$ by $\mathscr{M} \otimes L_\infty(0, \infty)$ (see for example [11], Proposition 4.6 (i) for a forerunner of this result):

Theorem 8.7. *Let $\mathscr{M}$ be any semi-finite von Neumann algebra equipped with a normal faithful semi-finite trace τ. Let $\|\cdot\|$ be a symmetric extended seminorm on $\tilde{L}_\infty(0, \infty)$. Then $x \to \|\mu(x)\|$ is an extended seminorm on $\tilde{\mathscr{M}}$.*

The following result can be deduced from Theorem 8.7 or proved directly using Lemma 8.3:

Corollary 8.8. *Let $\|\cdot\|$ be a symmetric extended seminorm on ℓ_∞. Then $T \to \|(s_n(T))_{n=1}^\infty\|$ is an extended seminorm on $\mathscr{L}(\mathscr{H})$.*

Proposition 8.9. *Suppose $\varphi : [0, \infty) \to [0, \infty)$ is a continuous increasing concave function. Suppose $x_1, \ldots, x_k \in \tilde{\mathscr{M}}_+$ and that $\alpha_1, \ldots, \alpha_k > 0$ with $\alpha_1 + \cdots + \alpha_k \leqq 1$. Then*

$$\int_a^b \varphi\big(\mu_t(x_1+\cdots+x_k)\big)\,dt \leqq \sum_{j=1}^k \int_{\alpha_j a}^b \varphi\big(\mu_t(x_j)\big)\,dt. \tag{8.2}$$

We also have:

$$\mu\big(\varphi(x_1+\cdots+x_k)\big) \trianglelefteq \mu\big(\varphi(x_1)\big)+\cdots+\mu\big(\varphi(x_k)\big). \tag{8.3}$$

Proof. We first show that if $0<a<b$ and $e\in\mathscr{P}$ satisfies $\tau(e)=b$ we have

$$\int_a^b \varphi\big(\mu_t(ex_1e+\cdots+ex_ke)\big)\,dt \leqq \sum_{j=1}^k \int_{\alpha_j a}^b \varphi\big(\mu_t(ex_je)\big)\,dt. \tag{8.4}$$

For $0<\sigma<\infty$ let

$$\psi_\sigma(t)=\min(t,\sigma), \quad t\geqq 0.$$

For $1\leqq j\leqq k$ let $y_j=ex_je$ and then $f_j=e^{y_j}(\sigma,\infty)$ so that $f_j\leqq e$.

Let us observe that if $s<\tau(f_j)$ we have

$$\int_{\tau(f_j)}^b \mu_t\big(\psi_\sigma(y_j)\big)\,dt = \int_s^b \mu_t\big(\psi_\sigma(y_j)\big)\,dt - \sigma\big(\tau(f_j)-s\big)$$

while if $\tau(f_j)<s\leqq b$ we have

$$\int_{\tau(f_j)}^b \mu_t\big(\psi_\sigma(y_j)\big)\,dt \leqq \int_s^b \mu_t\big(\psi_\sigma(y_j)\big)\,dt + \sigma\big(s-\tau(f_j)\big)$$

and we may combine these as

$$\int_{\tau(f_j)}^b \mu_t\big(\psi_\sigma(y_j)\big)\,dt \leqq \int_s^b \mu_t\big(\psi_\sigma(y_j)\big)\,dt + \sigma\big(s-\tau(f_j)\big), \quad 0<s\leqq b. \tag{8.5}$$

Let $f=f_1\vee\cdots\vee f_k$ and $f'=e-f$. Suppose $\tau(f)\leqq a$. Then if $g\in\mathscr{P}$ with $g\leqq f'$ and $\tau(g)=b-a$, we have, using (8.5),

$$\begin{aligned}\int_a^b \mu_t\big(\psi_\sigma(y_1+\cdots+y_k)\big)\,dt &\leqq \tau\big(g(y_1+\cdots+y_k)g\big) \leqq \tau\big(f'(y_1+\cdots+y_k)f'\big)\\ &\leqq \sum_{j=1}^k \tau\big((e-f_j)\psi_\sigma(y_j)(e-f_j)\big) = \sum_{j=1}^k \int_{\tau(f_j)}^b \mu_t\big(\psi_\sigma(y_j)\big)\,dt\\ &\leqq \sum_{j=1}^k \int_{\alpha_j a}^b \mu_t\big(\psi_\sigma(y_j)\big)\,dt + \sum_{j=1}^k \sigma\big(\alpha_j a-\tau(f_j)\big) \leqq \sum_{j=1}^k \int_{\alpha_j a}^b \mu_t\big(\psi_\sigma(y_j)\big)\,dt.\end{aligned}$$

If $\tau(f)>a$ then, again using (8.5),

$$\begin{aligned}\int_a^b \mu_t\big(\psi_\sigma(y_1+\cdots+y_k)\big)\,dt &\leqq \tau\big(f'(y_1+\cdots+y_k)f'\big) + \sigma\big(b-a-\tau(f')\big)\\ &\leqq \sum_{j=1}^k \tau\big((e-f_j)\psi_\sigma(y_j)(e-f_j)\big) + \sigma\big(\tau(f)-a\big)\end{aligned}$$

$$= \sum_{j=1}^{k} \int_{\tau(f_j)}^{b} \mu_t\big(\psi_\sigma(y_j)\big)\,dt + \sigma\big(\tau(f) - a\big)$$

$$\leqq \sum_{j=1}^{k} \int_{\alpha_j a}^{b} \mu_t\big(\psi_\sigma(y_j)\big)\,dt + \sigma\big(\tau(f) - a\big) + \sigma \sum_{j=1}^{k} \big(\alpha_j a - \tau(f_j)\big)$$

$$\leqq \sum_{j=1}^{k} \int_{\alpha_j a}^{b} \mu_t\big(\psi_\sigma(y_j)\big)\,dt.$$

Since φ is concave, it has left and right-derivatives $\varphi'(a-)$ and $\varphi'(a+)$ for each $a > 0$ and these coincide except on a countable set. Let ν be the nonnegative σ-finite measure on $(0, \infty)$ so that $\nu(a,b) = \varphi'(a+) - \varphi'(b-)$. Let $\kappa = \lim_{t\to\infty} \varphi'(t)$. Then

$$\varphi(t) - \varphi(0) = \int_0^t \varphi'(u)\,du = t\varphi'(t-) + \int_0^t \nu(u,t)\,du$$

$$= \kappa t + t\nu[t,\infty) + \int_{(0,t)} \sigma\,d\nu(\sigma) = \kappa t + \int_{(0,\infty)} \psi_\sigma(t)\,d\nu(\sigma).$$

Let us assume first that $\kappa = 0$. Thus

$$\int_a^b \mu_t\big(\varphi(y_1 + \cdots + y_k)\big)\,dt = \varphi(0) + \int_{(0,\infty)} \int_a^b \mu_t\big(\psi_\sigma(y_1 + \cdots + y_k)\big)\,dt\,d\nu(\sigma)$$

$$\leqq k\varphi(0) + \sum_{j=1}^{k} \int_{(0,\infty)} \int_{\alpha_j a}^{b} \mu_t\big(\psi_\sigma(y_j)\big)\,dt\,d\nu(\sigma)$$

$$= \sum_{j=1}^{k} \int_{\alpha_j a}^{b} \mu_t\big(\varphi(y_j)\big)\,dt.$$

This establishes (8.4) if $\kappa = 0$. The case $\kappa > 0$ is then easily handled by writing $\varphi(t) = \kappa t + \varphi_0(t)$ and using Lemma 8.5.

To prove (8.2), if $\lim_{t\to\infty} \mu_t(x_j) = 0$ for $1 \leqq j \leqq k$, fix e with $\tau(e) = b$ so that

$$\mu_t\big(e(x_1 + \cdots + x_k)e\big) = \mu_t(x_1 + \cdots + x_k), \quad 0 < t < b.$$

Then according to (8.4) we have

$$\int_a^b \mu_t\big(\varphi(x_1 + \cdots + x_k)\big)\,dt \leqq \sum_{j=1}^{k} \int_{\alpha_j a}^{b} \mu_t\big(\varphi(ex_je)\big)\,dt$$

$$\leqq \sum_{j=1}^{k} \int_{\alpha_j a}^{b} \mu_t\big(\varphi(x_j)\big)\,dt.$$

The general case follows easily by approximation.

Finally for (8.3) replace a by ka where $0 < ka < b$ and let $\alpha_j = 1/k$. $\square$

Theorem 8.10. *Suppose $\varphi : [0, \infty) \to [0, \infty)$ is a continuous increasing concave function.*

(i) *Let $\|\cdot\|$ be a symmetric extended seminorm on ℓ_∞. Then if $T_1, \dots, T_k \in \mathscr{L}(\mathscr{H})$ we have*

$$\|\varphi(|T_1 + \cdots + T_k|)\| \leqq \|\varphi(|T_1|)\| + \cdots + \|\varphi(|T_k|)\|.$$

(ii) *Let $\|\cdot\|$ be a symmetric extended seminorm on $\tilde{L}_\infty(0, \infty)$. Then if $x_1, \dots, x_k \in \tilde{\mathscr{M}}$ we have*

$$\|\varphi(|x_1 + \cdots + x_k|)\| \leqq \|\varphi(|x_1|)\| + \cdots + \|\varphi(|x_k|)\|.$$

Remark. In particular this theorem applies to $\varphi(t) = t^p$ when $0 < p \leqq 1$.

Proof. Using again [13], Lemma 4.3, we may find partial isometries $u_1, \dots, u_k$ so that $|x_1 + \cdots + x_k| \leqq \sum_{j=1}^{k} u_j x_j u_j^*$. Thus it suffices to establish the case when $x_j \geqq 0$. This now follows from Proposition 8.9 (8.3). □

Theorem 8.11. (i) *Let E be an r.i. p-convex quasi-Banach sequence space where $0 < p < \infty$. Then $\mathscr{C}_E$ is a p-convex quasi-Banach operator ideal.*

(ii) *Let E be an r.i. p-convex quasi-Banach function space on $(0, \infty)$ where $0 < p < \infty$. Then $E(\mathscr{M}, \tau)$ is a p-convex quasi-Banach operator space.*

Proof. We prove this in the case of (ii). To prove (i) it is simplest to embed E into a p-convex function space $\tilde{E}$ on $(0, \infty)$ e.g. by defining

$$\|f\|_{\tilde{E}} = \left\| \left(\left(\int_{n-1}^{n} f^*(t)^p \, dt \right)^{1/p} \right)_{n=1}^{\infty} \right\|_E$$

and $\mathscr{L}(\mathscr{H})$ into $\mathscr{L}(\mathscr{H}) \otimes L_\infty[0, 1]$.

(ii) Let us define $X = \{f \in \tilde{L}_\infty(0, \infty) : |f|^{1/p} \in E\}$ and $\|f\|_X = \|f^{1/p}\|_E^p$. Then X is a symmetric Banach function space on $(0, \infty)$. Now

$$\|x\|_{E(M,\tau)} = \|\mu(|x|)^p\|_X^{1/p}$$

and it follows that

$$\|x + y\|_{E(M,\tau)}^p \leqq \|x\|_{E(M,\tau)}^p + \|y\|_{E(M,\tau)}^p$$

when $0 < p < 1$ by Theorem 8.10; of course if $p \geqq 1$, $\|\cdot\|_{E(\mathscr{M},\tau)}$ is a norm by Theorem 8.7. The fact that $\|\cdot\|_{X(\mathscr{M},\tau)}$ is a norm implies that $E(\mathscr{M}, \tau)$ is p-convex.

It remains to show that $E(\mathscr{M}, \tau)$ is complete. We note that the injection $E(\mathscr{M}, \tau) \hookrightarrow \tilde{\mathscr{M}}$ is continuous. This follows from [11], Proposition 2.2 or the estimate

$$\mu_t(x) \leqq \|\chi_{(0,t)}\|_E^{-1} \|x\|_{E(\mathscr{M},\tau)}$$

(see [9], Lemma 4.4). Thus any Cauchy sequence in $E(\mathscr{M},\tau)$ converges in $\tilde{\mathscr{M}}$.

To establish completeness it suffices to show that for some $0 < \beta \leqq \min(p, 1)$ we have an estimate

$$\left\| \sum_{k=1}^{\infty} x_k \right\|_{E(\mathscr{M},\tau)}^{\beta} \leqq \sum_{n=1}^{\infty} \|x_n\|_{E(\mathscr{M},\tau)}^{\beta}, \quad x_k \geqq 0, \quad k = 1, 2, \dots, \tag{8.6}$$

whenever the right-hand side is finite. This is a non-commutative form of the socalled Riesz-Fischer Theorem (see [12] for more details).

By the above remarks $\sum_{n=1}^{\infty} x_k$ converges in $\tilde{\mathscr{M}}$ to some x. Note that for (8.6) to hold it is necessary that $x \in E(\mathscr{M},\tau)$. Once (8.6) is established it follows by considering the tail that $x = \sum_{n=1}^{\infty} x_n$ in $E(\mathscr{M},\tau)$ and then any series $\sum y_n$ in $E(\mathscr{M},\tau)$ with $\sum \|y_n\|_{E(\mathscr{M},\tau)}^{\beta} < \infty$ is convergent in $E(\mathscr{M},\tau)$, and completeness of $E(\mathscr{M},\tau)$ will follow.

First suppose $p \geqq 1$; in this case we take $\beta = 1/2$. If $g \in \tilde{L}_\infty(0,\infty)$ and $\alpha \leqq 1$ we define as before $g^{[\alpha]}(t) = \alpha g(\alpha t)$. Observe that if g is decreasing

$$\int_{\alpha^{-1}a}^{b} g^{[\alpha]}(t)\,dt = \int_{a}^{\alpha b} g(t)\,dt \leqq \int_{a}^{b} g(t)\,dt, \quad 0 < a < \alpha b.$$

Hence

$$\|g^{[\alpha]}\|_E \leqq \|g\|_E.$$

Let us assume $\sum_{k=1}^{\infty} \|x_k\|_{E(\mathscr{M},\tau)}^{1/2} = 1$ and that each x_k is non-zero. Let $\alpha_j = \|x_j\|_{E(\mathscr{M},\tau)}^{1/2}$. Let $f_k = \mu(x_k)$ and $g_k = \mu(x_1 + \cdots + x_k)$. Let $g = \mu(x)$. If $\lambda \in \mathbb{N}$ with $\lambda \geqq 2$, and $0 < \lambda a < b$ we have (using [13], Lemma 3.5)

$$\int_{\lambda a}^{b} g(t)\,dt = \lim_{n\to\infty} \int_{\lambda a}^{b} g_n(t)\,dt.$$

Now we use Lemma 8.5:

$$\begin{aligned}
\int_{\lambda a}^{b} g_n(t)\,dt &\leqq \sum_{k=1}^{n} \int_{\alpha_k \lambda a}^{b} f_k(t)\,dt = \sum_{k=1}^{n} \int_{\lambda a}^{\alpha_k^{-1} b} f_k^{[\alpha_k]}(t)\,dt \\
&\leqq \sum_{k=1}^{n} \frac{\alpha_k^{-1} b - \lambda a}{b - a} \int_{a}^{b} f_k^{[\alpha_k]}(t)\,dt \leqq \frac{\lambda}{\lambda - 1} \sum_{k=1}^{n} \alpha_k^{-1} \int_{a}^{b} f_k^{[\alpha_k]}(t)\,dt \\
&\leqq \frac{\lambda}{\lambda - 1} \int_{a}^{b} h(t)\,dt
\end{aligned}$$

where

$$h(t) = \sum_{k=1}^{\infty} \alpha_k^{-1} f_k^{[\alpha_k]}(t).$$

Note here that since $\|f_k^{[\alpha_k]}\|_E \leqq \alpha_k^2$ and $\sum \alpha_k = 1$ we have $h \in E$ and $\|h\|_E \leqq 1$. Hence

$$\langle g; h \rangle \leqq \frac{\lambda}{\lambda - 1}$$

and hence, as $\lambda \geqq 2$ is arbitrary,

$$\langle g; h \rangle \leqq 1.$$

Thus $g \in E$ and $\|g\|_E \leqq 1$. This implies (8.6)

In the case $p < 1$ we let $\beta = p/2$. Suppose $\sum_{k=1}^{\infty} \|x_k\|_{E(\mathcal{M},\tau)}^{p/2} = 1$ and that each x_k is non-zero. Let $\alpha_j = \|x_j\|_{E(\mathcal{M},\tau)}^{p/2}$. As before let $f_k = \mu(x_k)$ and $g_k = \mu(f_1 + \cdots + f_k)$ and $g = \mu(x)$. We now note that

$$\int_{\lambda a}^{b} g(t)^p \, dt = \lim_{n \to \infty} \int_{\lambda a}^{b} g_n(t)^p \, dt.$$

We now use Proposition 8.9:

$$\begin{aligned} \int_{\lambda a}^{b} g_n(t)^p \, dt &\leqq \sum_{k=1}^{n} \int_{\alpha_k \lambda a}^{b} f_k(t)^p \, dt \\ &\leqq \sum_{k=1}^{n} \int_{\lambda a}^{\alpha_k^{-1} b} (f_k^p)^{[\alpha_k]}(t) \, dt \leqq \frac{\lambda}{\lambda - 1} \int_a^b h(t) \, dt \end{aligned}$$

where

$$h(t) = \sum_{k=1}^{\infty} \alpha_k^{-1} (f_k^p)^{[\alpha_k]}(t).$$

We observe that

$$\|(f_k^p)^{[\alpha_k]}\|_X \leqq \|f_k^p\|_X = \alpha_k^2$$

so that $h \in X$ and $\|h\|_X \leqq 1$. Hence, since as before $\langle g^p; h \rangle \leqq 1$,

$$\|x\|_{E(\mathcal{M},\tau)} \leqq \|g^p\|_X^{1/p} \leqq \|h\|_X^{1/p} \leqq 1$$

and (8.6) is established. □

Let us remark here that if E is a fully symmetric Banach function space the completeness of $E(\mathcal{M}, \tau)$ was established first in [34], [9], [10] and [5]. In fact they prove completeness under the assumption of *relative full symmetry*:

$$0 \leqq f, g \in E, \quad f \prec\prec g \quad \Rightarrow \quad \|f\|_E \leqq \|g\|_E. \tag{8.7}$$

However the arguments in the above mentioned papers depend on the submajorization inequality

$$|\mu(x) - \mu(y)| \prec\prec \mu(x - y).$$

It is worth pointing out that there is no analogue to this inequality for uniform Hardy-Littlewood majorization even in the commutative case. Indeed if

$$f, g \in L_1(0, \infty) + L_\infty(0, \infty)$$

the inequality

$$|f^* - g^*| \prec\prec (f - g)^*$$

is well-known (see [18]). However set $f(t) = \sum_{n=0}^{\infty} 2^{-n}\chi_{(n,n+1]}$ and $g(t) = \sum_{n=1}^{\infty} 2^{-n}\chi_{(n,n+1]}$. Then $g^* = f^*/2$ and $f - g = \chi_{(0,1]}$. Thus $\langle |f^* - g^*|; (f - g)^* \rangle = \infty$.

In the case where $p < 1$ completeness is established under the Fatou condition by Xu [38].

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Eingegangen 31. März 2007