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Spectral characterization of sums of commutators II

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Abstract. For countably generated ideals, $\mathscr{J}$, of $B(\mathscr{H})$, geometric stability is necessary for the canonical spectral characterization of sums of $(\mathscr{J}, B(\mathscr{H}))$-commutators to hold. This answers a question raised by Dykema, Figiel, Weiss and Wodzicki. There are some ideals, $\mathscr{J}$, having quasi-nilpotent elements that are not sums of $(\mathscr{J}, B(\mathscr{H}))$-commutators. Also, every trace on every geometrically stable ideal is a spectral trace.

Introduction

Let $\mathscr{H}$ be a separable infinite-dimensional Hilbert space, and let $\mathscr{J}$ be a (two-sided) ideal contained in the ideal of compact operators $\mathscr{K}(\mathscr{H})$ on $\mathscr{H}$. We define the *commutator subspace* $\operatorname{Com}\mathscr{J}$ to be the set of all finite linear combinations of commutators $[A, B] = AB - BA$ where $A \in \mathscr{J}$ and $B \in \mathscr{B}(\mathscr{H})$.

In the immediately preceding paper, [5] N.J. Kalton showed that, for a wide class of ideals including quasi-Banach ideals (i.e. ideals equipped with a complete ideal quasi-norm) $\operatorname{Com}\mathscr{J}$ has the following spectral characterization. For $T \in \mathscr{J}$, let $\lambda_k = \lambda_k(T)$ be the eigenvalues of T listed according to algebraic multiplicity and such that $|\lambda_1| \geqq |\lambda_2| \geqq \cdots$. Then $T \in \operatorname{Com}\mathscr{J}$ if and only if $\operatorname{diag}\left\{\frac{1}{n}\sum_{k=1}^{n}\lambda_k\right\} \in \mathscr{J}$. The sufficient condition from [5] for $\operatorname{Com}\mathscr{J}$ to have a spectral characterization in the above sense is *geometric stability*, i.e. the condition that if (s_n) is a monotone decreasing real sequence then $\operatorname{diag}\{s_n\} \in \mathscr{J}$ if and only if $\operatorname{diag}\{(s_1 \dots s_n)^{1/n}\} \in \mathscr{J}$. (See the introduction of [5] for background.)

In §1 of this paper, we show that when $\mathscr{J}$ is a countably generated ideal, then geometric stability is a necessary condition for this spectral characterization of $\operatorname{Com}\mathscr{J}$. In particular, for every countably generated ideal, $\mathscr{J}$, which is not geometrically stable, we

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find an element T, of $\operatorname{Com}\mathscr{J}$ for which $\operatorname{diag}\left\{\frac{1}{n}(\lambda_1(T)+\cdots+\lambda_n(T))\right\}\notin\mathscr{J}$. We also give examples of ideals for which spectral characterization of $\operatorname{Com}\mathscr{J}$ fails in the opposite direction, and in an extreme way, in that $\mathscr{J}$ has a quasi-nilpotent element that is not in $\operatorname{Com}\mathscr{J}$.

In § 2, we show, based on an elementary decomposition result for compact operators, that every trace, τ, on an arbitrary geometrically stable ideal, $\mathscr{J}$, is a spectral trace, i.e. for every $T\in\mathscr{J}$, $\tau(T)$ depends only on the eigenvalues of T and their multiplicities.

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1. Countably generated ideals

Proposition 1.1. *Suppose* $s_1\geqq s_2\geqq\cdots\geqq 0$. *Suppose that* $(\lambda_n)_{n=1}^{\infty}$ *is any sequence of complex numbers with* $|\lambda_1|\geqq|\lambda_2|\geqq\cdots$ *and such that* $|\lambda_1\ldots\lambda_n|\leqq s_1\ldots s_n$ *for all n. Then there is an upper triangular operator* $A=(a_{jk})_{j,k}$ *such that* $s_n(A)\leqq s_n$ *for all n and* $a_{jj}=\lambda_j$ *for all j.*

Proof. This is the infinite dimensional extension of a result of Horn [3] (see Gohberg-Krein [2], Remark II. 3.1). By Horn's result we can find for each n an upper-triangular matrix $A^{(n)}=(a_{jk}^{(n)})$ such that $a_{jk}^{(n)}=0$ if $\max(j,k)>n$, $a_{jj}^{n}=\lambda_j$ for $1\leqq j\leqq n$ and $s_j(A^{(n)})\leqq s_j$ for $1\leqq j<\infty$.

Now we can pass to a subsequence $(B^{(n)})$ of $(A^{(n)})$ which is convergent to the weak operator topology to some operator $A=(a_{ij})$ which is upper-triangular and has $a_{jj}=\lambda_j$ for all j. It remains to show that $s_j(A)\leqq s_j$ for all j.

Fix $j>1$ (the case $j=1$ is well known). For each n there is a subspace E_n of dimension $j-1$ so that if P_n is the orthogonal projection with kernel E_n then $\|B^{(n)}P_n\|\leqq s_j$. Let $\{f_k^n\}_{k=1}^{j-1}$ be an orthonormal basis of E_n. By passing to a further subsequence we can suppose that $\lim_{n\to\infty} f_k^n=f_k$ exists weakly for each $1\leqq k\leqq j-1$. Let E be the span of $(f_k)_{k=1}^{j-1}$, so that $\dim E\leqq j-1$. Suppose $x\in E^{\perp}$. Then $\lim_{n\to\infty}(x,f_k^n)=0$ for $1\leqq k\leqq j-1$. Hence $\lim_{n\to\infty}\|x-P_nx\|=0$. Thus $\|Ax\|\leqq\liminf_{n\to\infty}\|B^{(n)}x\|\leqq\liminf_{n\to\infty}\|B^{(n)}P_nx\|\leqq s_j\|x\|$. This shows that $s_j(A)\leqq s_j$. □

Let us introduce a bit of notation. Suppose $u=(u_j)_{j=1}^{\infty}$ and $t=(t_j)_{j=1}^{\infty}$ are decreasing sequences of positive numbers. We write $t\leqq u$ to mean that $t_j\leqq u_j$ for every j. For $c>0$ we let cu denote the sequence $(cu_j)_{j=1}^{\infty}$. We let $u\oplus u$ denote the sequence

$$(u_1,u_1,u_2,u_2,u_3,u_3,\ldots).$$

Lemma 1.2. *Let $u=(u_j)_{j=1}^{\infty}$ and $t=(t_j)_{j=1}^{\infty}$ be sequences of positive numbers decreasing to zero such that* $\operatorname{diag}\{t_j\}$ *is in the ideal of compact operators generated by* $\operatorname{diag}\{u_j\}$. *Then*

$$\operatorname{diag}\{(t_1 t_2 \dots t_k)^{1/k}\}$$

is in the ideal of compact operators generated by $\operatorname{diag}\{(u_1 u_2 \dots u_k)^{1/k}\}$.

Proof. The hypotheses imply that there is a positive integer n and a positive number c such that $t_{nj} \leqq c u_j$ for every j, and we may suppose n is a power of 2. Hence, in order to prove the lemma it will suffice to prove it in each of the following special cases, in turn:

(i) $t \leqq u$,

(ii) $t = cu$ for $c > 1$,

(iii) $t = u \oplus u$.

The cases (i) and (ii) are clear. If (iii) holds then $(t_1 \dots t_{2n})^{1/2n} = (u_1 \dots u_n)^{1/n}$, which shows that $\operatorname{diag}\{(t_1 \dots t_k)^{1/k}\}$ is in the ideal generated by $\operatorname{diag}\{(u_1 \dots u_n)^{1/n}\}$, and the lemma is proved. □

Theorem 1.3. *Let $\mathscr{J}$ be a countably generated ideal in $\mathscr{K}(\mathscr{H})$. Then the following conditions are equivalent*:

(i) *$\mathscr{J}$ is geometrically stable.*

(ii) *If $T \in \mathscr{J}$ then* $\operatorname{diag}\{\lambda_n\} \in \mathscr{J}$ *where $\lambda_n = \lambda_n(T)$.*

(iii) *If $T \in \operatorname{Com}\mathscr{J}$ then* $\operatorname{diag}\left\{\frac{1}{n}(\lambda_1 + \dots + \lambda_n)\right\} \in \mathscr{J}$, *for some ordering of λ_n so that $(|\lambda_n|)$ is decreasing.*

Proof. (i) implies (iii) is proved for any ideals in [5]. That (i) implies (ii) is also noted in [5]; this follows from the inequalities $|\lambda_n(T)| \leqq \big(s_1(T) \dots s_n(T)\big)^{1/n}$.

Since $\mathscr{J}$ is countably generated there is a countable family, $\{u^{(k)} \mid k \in \mathbb{N}\}$, of decreasing sequences of positive numbers such that $T \in \mathscr{J}$ if and only if for some k we have $s_n(T) \leqq u_n^{(k)}$ for all $n \geqq 1$. Indeed, let $\{u^{(k)} \mid k \in \mathbb{N}\} = \{w^{(i,j)} \mid i, j \in \mathbb{N}\}$ where $w_n^{(i,j)} = j w_{n/j}^{(i)}$ and where $w_n^{(i)} = s_n(T_1) + s_n(T_2) + \dots + s_n(T_i)$, with $\{T_i \mid i \in \mathbb{N}\}$ a countable generating set of the ideal $\mathscr{J}$.

We will prove that each of (ii) and (iii) individually implies (i) by supposing that (i) fails and showing that both (ii) and (iii) fail. If (i) fails then there is a sequence $(t_n)_{n=1}^{\infty}$ such that $\operatorname{diag}\{t_n\} \in \mathscr{J}$ but $\operatorname{diag}\{(t_1 t_2 \dots t_n)^{1/n}\} \notin \mathscr{J}$. Now for every $k \in \mathbb{N}$, since $\operatorname{diag}\{(t_{n+k})_{n=1}^{\infty}\}$ generates the same ideal as $\operatorname{diag}\{t_n\}$, it follows from Lemma 1.2 that $\operatorname{diag}\{\big((t_{k+1} t_{k+2} \dots t_{k+n})^{1/n}\big)_{n=1}^{\infty}\} \notin \mathscr{J}$.

It follows that we can find a sequence $m_n \uparrow \infty$ such that if $m_0 = 0$ and $p_n = m_n - m_{n-1}$ for $n \geqq 1$ then $p_n / m_n \to 1$ and

$$\bar{w}_n \stackrel{\text{def}}{=} \left(\prod_{k=m_{n-1}+1}^{m_n} t_k \right)^{1/p_n} > u_{m_n}^{(n)} .$$

Let $w_k = \bar{w}_n$ for every $m_{n-1} + 1 \leqq k \leqq m_n$. We may assume without loss of generality that $\bar{w}_n > \bar{w}_{n+1}$ for every $n \in \mathbb{N}$. By Proposition 1.1 there is an upper triangular operator $A \in \mathcal{J}$ with diagonal entries $a_{kk} = w_k$ for $k \in \mathbb{N}$. Now $\operatorname{diag}(w_k) \notin \mathcal{J}$ by construction and hence (ii) fails. Therefore (ii) implies (i).

We now show that also (iii) fails, which will finish the proof of the theorem. We can choose $\bar{\sigma}_1 > \bar{\sigma}_2 > \cdots > 0$ decreasing quickly enough so that $\bar{w}_n - \bar{\sigma}_n > \bar{w}_{n+1}$ for every n and such that, letting $\sigma_k = \bar{\sigma}_n$ for every $m_{m-1} + 1 \leqq k \leqq m_n$, we have $\operatorname{diag}(\sigma_k) \in \mathcal{J}$. Define for each $n \in \mathbb{N}$

$$\varepsilon_{2n-1} = \varepsilon_{2n} = \frac{1}{2} \min(\bar{\sigma}_{2n-1} - \bar{\sigma}_{2n}, \bar{\sigma}_{2n} - \bar{\sigma}_{2n+1}) .$$

Then for every $n \in \mathbb{N}$ we have that $\varepsilon_n > 0$, that $\bar{\sigma}_n - \varepsilon_n > \bar{\sigma}_{n-1}$ and that

$$\left| \sum_{j=1}^{n} (-1)^{j-1} \varepsilon_j \right| \leqq \begin{cases} \bar{\sigma}_n & \text{if } n \text{ is odd}, \\ 0 & \text{if } n \text{ is even}. \end{cases}$$

Let $(\mu_k)_{k=1}^{\infty}$ be the sequence defined by

$$\mu_k = \begin{cases} \bar{\sigma}_n & \text{if } m_{n-1} + 1 \leqq k \leqq m_n \text{ and } n \text{ is odd}, \\ \bar{\sigma}_n - (\varepsilon_n / p_n) & \text{if } m_{n-1} + 1 \leqq k \leqq m_n \text{ and } n \text{ is even}, \end{cases}$$

and let $(v_k)_{k=1}^{\infty}$ be the sequence defined by

$$v_k = \begin{cases} \bar{\sigma}_n - (\varepsilon_n / p_n) & \text{if } m_{n-1} + 1 \leqq k \leqq m_n \text{ and } n \text{ is odd}, \\ \bar{\sigma}_n & \text{if } m_{n-1} + 1 \leqq k \leqq m_n \text{ and } n \text{ is even}. \end{cases}$$

Note that $(u_k)_{k=1}^{\infty}$ and $(v_k)_{k=1}^{\infty}$ are decreasing sequences. Let $D_1 = \operatorname{diag}\{\mu_k\}$ and $D_2 = \operatorname{diag}\{v_k\}$. We will show that $D_1 \oplus (-D_2) \in \operatorname{Com} \mathcal{J}$. Let $(\lambda'_k)_{k=1}^{\infty}$ be the eigenvalue sequence, arranged in some order of decreasing absolute value, of the operator $D_1 \oplus (-D_2)$. We will show that $\operatorname{diag}\left\{\frac{1}{k}(\lambda'_1 + \cdots + \lambda'_k)\right\} \in \mathcal{J}$, which by [1] shows $D_1 \oplus (-D_2) \in \operatorname{Com} \mathcal{J}$. The sequence $(\lambda'_k)_{k=1}^{\infty}$ is

$$\underbrace{\bar{\sigma}_1, \ldots, \bar{\sigma}_1}_{p_1 \text{ times}}, \underbrace{-\bar{\sigma}_1 + \frac{\varepsilon_1}{p_1}, \ldots, -\bar{\sigma}_1 + \frac{\varepsilon_1}{p_1}}_{p_1 \text{ times}}, \underbrace{-\bar{\sigma}_2, \ldots, -\bar{\sigma}_2}_{p_2 \text{ times}}, \underbrace{\bar{\sigma}_2 - \frac{\varepsilon_2}{p_2}, \ldots, \bar{\sigma}_2 - \frac{\varepsilon_2}{p_2}}_{p_2 \text{ times}}, \ldots$$

$$\ldots, \underbrace{\bar{\sigma}_{2n-1}, \ldots, \bar{\sigma}_{2n-1}}_{p_{2n-1} \text{ times}}, \underbrace{-\bar{\sigma}_{2n-1} + \frac{\varepsilon_{2n-1}}{p_{2n-1}}, \ldots, -\bar{\sigma}_{2n-1} + \frac{\varepsilon_{2n-1}}{p_{2n-1}}}_{p_{2n-1} \text{ times}},$$

$$\underbrace{-\bar{\sigma}_{2n}, \ldots, -\bar{\sigma}_{2n}}_{p_{2n} \text{ times}}, \underbrace{\bar{\sigma}_{2n} - \frac{\varepsilon_{2n}}{p_{2n}}, \ldots, \bar{\sigma}_{2n} - \frac{\varepsilon_{2n}}{p_{2n}}}_{p_{2n} \text{ times}}, \ldots .$$

Let $\eta'_k = \sum_{j=1}^{k} \lambda'_k$. Then for every $n \in \mathbb{N}$,

$$\eta'_{2m_{2n-2}+k} = k\bar{\sigma}_{2n-1} \qquad (1 \leqq k \leqq p_{2n-1}),$$

$$\eta'_{2m_{2n-2}+p_{2n-1}+k} = (p_{2n-1} - k)\,\bar{\sigma}_{2n-1} + \frac{k\varepsilon_{2n-1}}{p_{2n-1}} \qquad (1 \leqq k \leqq p_{2n-1}),$$

$$\eta'_{2m_{2n-1}+k} = \varepsilon_{2n-1} - k\bar{\sigma}_{2n} \qquad (1 \leqq k \leqq p_{2n}),$$

$$\eta'_{2m_{2n-1}+p_{2n}+k} = \varepsilon_{2n-1} - (p_{2n} - k)\bar{\sigma}_{2n} - \frac{k\varepsilon_{2n}}{p_{2n}} \qquad (1 \leqq k \leqq p_{2n}).$$

Making crude (but sufficient) estimates, we get

$$\left|\frac{\eta'_{2m_{2n-2}+k}}{2m_{2n-2}+k}\right| \leqq \bar{\sigma}_{2n-1} \qquad (1 \leqq k \leqq p_{2n-1}),$$

$$\left|\frac{\eta'_{2m_{2n-2}+p_{2n-1}+k}}{2m_{2n-2}+p_{2n-1}+k}\right| \leqq 2\bar{\sigma}_{2n-1} \qquad (1 \leqq k \leqq p_{2n-1}),$$

$$\left|\frac{\eta'_{2m_{2n-1}+k}}{2m_{2n-1}+k}\right| \leqq \bar{\sigma}_{2n-1} + \bar{\sigma}_{2n} \qquad (1 \leqq k \leqq p_{2n}),$$

$$\left|\frac{\eta'_{2m_{2n-1}+p_{2n}+k}}{2m_{2n-1}+p_{2n}+k}\right| \leqq \bar{\sigma}_{2n-1} + \bar{\sigma}_{2n} \qquad (1 \leqq k \leqq p_{2n}).$$

Using $\operatorname{diag}\{\sigma_k\} \in \mathscr{J}$ we see that $\operatorname{diag}\left\{\frac{1}{k}(\lambda'_1 + \cdots + \lambda'_k)\right\} \in \mathscr{J}$. Hence by [1]

$$D_1 \oplus (-D_2) \in \operatorname{Com} \mathscr{J}.$$

Let $A \in \mathscr{J}$ be the upper triangular operator with diagonal entries $a_{kk} = w_k$ as constructed above. Consider the operator

$$T=(A+D_1)\oplus(-A-D_2)\,.$$

Then $T\in\operatorname{Com}\mathscr{J}$ because $A\oplus(-A)\in\operatorname{Com}\mathscr{J}$ and $D_1\oplus(-D_2)\in\operatorname{Com}\mathscr{J}$. Let $\lambda_k=\lambda_k(T)$ be the eigenvalues of T listed according to algebraic multiplicity and in order of decreasing absolute value. Then

$$\begin{aligned}
\lambda_{2m_{2n-2}+k} &= \lambda'_{2m_{2n-2}+k}+\bar{w}_{2n-1} && (1\leqq k\leqq p_{2n-1}),\\
\lambda_{2m_{2n-2}+p_{2n-1}+k} &= \lambda'_{2m_{2n-2}+p_{2n-1}+k}-\bar{w}_{2n-1} && (1\leqq k\leqq p_{2n-1}),\\
\lambda_{2m_{2n-1}+k} &= \lambda'_{2m_{2n-1}+k}-\bar{w}_{2n} && (1\leqq k\leqq p_{2n}),\\
\lambda_{2m_{2n-1}+p_{2n}+k} &= \lambda'_{2m_{2n-1}+p_{2n}+k}+\bar{w}_{2n} && (1\leqq k\leqq p_{2n}).
\end{aligned}$$

Let $\eta_k=\sum_{j=1}^{k}\lambda_j$. Then

$$\begin{aligned}
\eta_{2m_{2n-2}+k} &= \eta'_{2m_{2n-2}+k}+k\bar{w}_{2n-1} && (1\leqq k\leqq p_{2n-1}),\\
\eta_{2m_{2n-2}+p_{2n-1}+k} &= \eta'_{2m_{2n-2}+p_{2n-1}+k}+(p_{2n-1}-k)\,\bar{w}_{2n-1} && (1\leqq k\leqq p_{2n-1}),\\
\eta_{2m_{2n-1}+k} &= \eta'_{2m_{2n-1}+k}-k\bar{w}_{2n} && (1\leqq k\leqq p_{2n}),\\
\eta_{2m_{2n-1}+p_{2n}+k} &= \eta'_{2m_{2n-1}+p_{2n}+k}-(p_{2n}-k)\,\bar{w}_{2n} && (1\leqq k\leqq p_{2n}).
\end{aligned}$$

We will show that $\operatorname{diag}\left\{\frac{1}{k}\lambda_1+\cdots+\lambda_k)\right\}\notin\mathscr{J}$. Since $\operatorname{diag}\left\{\frac{1}{k}\eta'_k\right\}\in\mathscr{J}$, it will suffice to show that $\operatorname{diag}\left\{\frac{1}{k}(\eta_k-\eta'_k)\right\}\notin\mathscr{J}$. However, taking the absolute value of a subsequence of $\left(\frac{1}{k}(\eta_k-\eta'_k)\right)_{k=1}^{\infty}$ and using $\lim_{n\to\infty}(p_n/m_n)=1$, we see that $\operatorname{diag}\left\{\frac{1}{k}(\eta_k-\eta'_k)\right\}\in\mathscr{J}$ would imply $\operatorname{diag}\{w_k\}\in\mathscr{J}$, which would be a contradiction. Therefore

$$\operatorname{diag}\left\{\frac{1}{k}(\lambda_1+\cdots+\lambda_k)\right\}\notin\mathscr{J}$$

and hence (iii) fails. □

We will conclude by showing that it is in general impossible to characterize membership in $\operatorname{Com}\mathscr{J}$ by considering only the eigenvalues. To this end we will construct a quasinilpotent operator T so that $T\notin\operatorname{Com}\mathscr{J}_T$, where $\mathscr{J}_T$ is the ideal generated by T.

We will start by considering any singly generated ideal, $\mathscr{J}$. Then there is a decreasing sequence of positive numbers (u_n) such that $T\in\mathscr{J}$ if and only if for some $C>0$ and $0<\alpha\leqq 1$ we have $s_n(T)\leqq Cu_{\alpha n}$.

Lemma 1.4. *Let $\mathscr{J}$ and (u_n) be as above and suppose $(v_n)_{n=1}^{\infty}$ and $(w_n)_{n=1}^{\infty}$ are a pair of real sequences such that*

(i) *we have* $v_1 \geqq v_2 \geqq \cdots \geqq 0$;

(ii) *for each* $k \geqq 1$ *the sequence* (v_n) *is constant on the dyadic block*

$$\{2^{k-1}, 2^{k-1}+1, \ldots, 2^k - 1\};$$

(iii) *for each* $n \in \mathbb{N}$ *we have* $\prod_{k=1}^{n} v_k \leqq \prod_{k=1}^{n} u_k$;

(iv) *for each n we have* $|w_n| \leqq |u_n|$;

(v) *for each n we have* $\left| \sum_{k=1}^{n} w_k \right| \leqq n v_n$.

Then there is an upper-triangular operator $A \in \operatorname{Com} \mathcal{J}$ *with* $a_{nn} = w_n$ *for all n.*

Proof. The proof is essentially the same as for the corresponding result for diagonal operators in [4] and [1]. We start by setting

$$\eta_n = 2^{1-k} \sum_{j=2^{k-1}}^{2^k-1} w_j$$

when $1 \leqq k < \infty$ and $2^{k-1} \leqq n \leqq 2^k - 1$. Notice that

$$|\eta_n - w_n| \leqq u_{n/2}$$

and that, if $2^{k-1} \leqq n \leqq 2^k - 1$,

$$\left| \sum_{j=1}^{n} (\eta_j - w_j) \right| = \left| \sum_{j=2^{k-1}}^{n} (\eta_j - w_j) \right| \leqq 2^k u_{n/2}$$

from which it follows that $\operatorname{diag}\{(\eta_n - w_n)\} \in \operatorname{Com} \mathcal{J}$.

It therefore will suffice to show that there is an upper-triangular operator A in $\operatorname{Com} \mathcal{J}$ with $a_{nn} = \eta_n$ for all n. To this end we define a sequence ξ_n by setting

$$\xi_n = 2^{1-k} \sum_{j=1}^{2^k-1} \eta_j = 2^{1-k} \sum_{j=1}^{2^k-1} w_j$$

for $2^{k-1} \leqq n < 2^{k-1}$. By hypothesis we have $|\xi_n| \leqq 2 v_n$ for ever $n \in \mathbb{N}$. Now we note that by Proposition 1.1, there is an upper-triangular matrix $C \in \mathcal{J}$ with $c_{nn} = v_n$ for every n; hence there is an upper-triangular matrix $B \in \mathcal{J}$ with $b_{nn} = \xi_n$ for $n \in \mathbb{N}$.

Now consider the isometries U_1, U_2 defined by $U_1 e_n = e_{2n+1}$ and $U_2 e_n = e_{2n}$. Consider the commutator $[U_1 B, U_1^*]$. We have

$$[U_1^*, U_1 B](e_{2n+1}) = Be_{2n+1} - U_1 Be_n ,$$
$$[U_1^*, U_1 B](e_{2n}) = Be_{2n}$$

and

$$[U_1^*, U_1 B] e_1 = Be_1 .$$

Similarly

$$[U_2^*, U_2 B] e_{2n} = Be_{2n} - U_2 Be_n ,$$
$$[U_2^*, U_2 B] e_{2n+1} = Be_{2n+1}$$

and

$$[U_2^*, U_2 B] e_1 = Be_1 .$$

Thus if $A = \frac{1}{2}([U_1^*, U_1 B] + [U_2^*, U_2 B])$ then $Ae_1 = Be_1$ and

$$(Ae_{2n+1}, e_{2m+1}) = (Be_{2n+1}, e_{2m+1}) - \frac{1}{2}(Be_n, e_m) ,$$
$$(Ae_{2n+1}, e_{2m}) = (Be_{2n+1}, e_{2m+1}) ,$$
$$(Ae_{2n+1}, e_1) = (Be_{2n+1}, e_1) = 0 ,$$
$$(Ae_{2n}, e_{2m+1}) = (Be_{2n}, e_{2m+1}) ,$$
$$(Ae_{2n}, e_{2m}) = (Be_{2n}, e_{2m}) - \frac{1}{2}(Be_n, e_m) ,$$
$$(Ae_{2n}, e_1) = (Be_{2n}, e_1) = 0 .$$

Thus $(Ae_n, e_m) = 0$ if $m < n$ and $a_{11} = \xi_1 = \eta_1$ while if $2^{k-1} \leqq n \leqq 2^k - 1$ with $k \geqq 2$ we have $a_{nn} = \xi_{2^{k-1}} - \frac{1}{2}\xi_{2^{k-2}} = \eta_n$. Since $A \in \operatorname{Com} \mathscr{J}$, this completes the proof. □

We now turn to the construction of some examples of ideals, $\mathscr{J}$, having quasi-nilpotents that are not sums of commutators.

Examples 1.5. Let $0 = p_0 < p_1 < p_2 < \cdots$ be integers such that $p_{n+1} > p_n + 2n2^{2n}$ for every $n \geqq 0$. Let

$$u_k = 2^{-p_n} \quad \text{if} \quad 2^{p_{n-1}} \leqq k \leqq 2^{p_n} - 1 \quad (n \geqq 1) ,$$

and let $\mathscr{J}$ be the ideal of $B(\mathscr{H})$ generated by $\operatorname{diag}\{u_k\}_{k=1}^{\infty}$. Then there is a quasi-nilpotent operator, $T \in \mathscr{J}$ such that $T \notin \operatorname{Com} \mathscr{J}$.

Proof. Let $q_n = 2n + p_n$ and

$$v_k = 2^n 2^{-p_{n+1}} \quad \text{if} \quad 2^{q_{n-1}} \leqq k \leqq 2^{q_n} - 1 \quad (n \geqq 1).$$

We claim that

$$\prod_{j=1}^{k} v_j \leqq \prod_{j=1}^{k} u_j \tag{1}$$

for every $k \in \mathbb{N}$. Let log denote the base 2 logarithm. For every $n \in \mathbb{N}$, we have

$$\log\left(\prod_{j=2^{q(n-1)}}^{(2^{q_n})-1} v_j u_j^{-1}\right) = 2^{p_n}\big((1 - 2^{-p_n + p_{n+1} + 2n - 2})(-p_{n+1} + p_n) + (1 - 2^{-p_n + p_{n-1} - 2}) n 2^{2n}\big).$$

But

$$1 - 2^{-p_n + p_{n-1} - 2} < 2(1 - 2^{-p_n + p_{n-1} + 2n - 2})$$

so

$$\log\left(\prod_{j=2^{q(n-1)}}^{(2^{q_n})-1} v_j u_j^{-1}\right) < 2^{p_n}(1 - 2^{-p_n + p_{n-1} + 2n - 2})(-p_{n+1} + p_n + 2n 2^{2n}) < 0.$$

Therefore, by induction on n, (1) holds whenever $k = 2^{q_n} - 1$. But since $v_j < u_j$ when $2^{q_n} \leqq j \leqq 2^{p_{n+1}} - 1$ and $v_j > u_j$ when $2^{p_{n+1}} \leqq j \leqq 2^{q_{n+1}} - 1$, it follows that (1) holds for all $k \in \mathbb{N}$.

Let $\sigma_k = \sum_{j=1}^{k} u_j$ and

$$\theta_k = \inf_{j \in \mathbb{N}} (j v_j + |\sigma_k - \sigma_j|).$$

Then $\theta_1 \leqq v_1 \leqq u_1$ and for $k \geqq 2$, $|\theta_k - \theta_{k-1}| \leqq u_k$. Let $w_1 = \theta_1$ and $w_k = \theta_k - \theta_{k-1}$ $(k \geqq 2)$. Then for every $k \in \mathbb{N}$ we have $|w_k| \leqq u_k$ and $0 \leqq \sum_{j=1}^{k} w_j = \theta_k \leqq k v_k$. Therefore, by Lemma 1.4 there is an upper triangular $A \in \operatorname{Com} \mathcal{J}$ with diagonal elements $a_{kk} = w_k$. However, also $W \overset{\text{def}}{=} \operatorname{diag}\{w_k\} \in \mathcal{J}$. Let $T = A - W$. Then $T \in \mathcal{J}$ is quasi-nilpotent. We will show that $T \notin \operatorname{Com} \mathcal{J}$ by showing $W \notin \operatorname{Com} \mathcal{J}$.

Suppose for contradiction that $W \in \operatorname{Com} \mathcal{J}$. Let $\{\lambda_k\}_{k=1}^{\infty}$ be a rearrangement of $\{w_k\}_{k=1}^{\infty}$ such that $|\lambda_1| \geqq |\lambda_2| \geqq \cdots$. Then by [1], $\operatorname{diag}\left\{\frac{1}{k}(\lambda_1 + \cdots + \lambda_k)\right\} \in \mathcal{J}$. For every $k \in \mathbb{N}$ we have

$$\left|\sum_{j=1}^{k} \lambda_j - \sum_{j=1}^{k} w_j\right| \leqq 2k u_k$$

because every w_j having absolute value strictly greater than u_k appears in both summations above. Hence $\operatorname{diag}\left\{\frac{1}{k}(w_1 + \cdots + w_k)\right\} \in \mathscr{J}$, thus by [5], 3.1 (3) there are $\alpha > 0$ and $c > 0$ such that

$$(2) \qquad \forall k \in \mathbb{N} \quad \frac{1}{k}\theta_k \leqq c u_{\alpha k}.$$

Let $k = 2^{p_n + n}$. We will find a lower bound for θ_k which will contradict (2). Routine estimation reveals that

$$j v_j + |\sigma_k - \sigma_j| \geqq \begin{cases} \frac{1}{2} & \text{if } j < 2^{p_{n-1}}, \\ 1 & \text{if } 2^{p_{n-1}} \leqq j \leqq 2^{q_{n-1}} - 1, \\ 2^{-p_{n+1} + p_n + 2n} & \text{if } 2^{q_{n-1}} \leqq j \leqq 2^{p_n} - 1, \\ 2^{-p_{n+1} + p_n + 2n} & \text{if } 2^{p_n} \leqq j \leqq 2^{p_n + n} - 1, \\ 2^{-p_{n+1} + p_n + 3n} & \text{if } 2^{p_n + n} \leqq j \leqq 2^{p_n + 2n} - 1, \\ 2^{-p_{n+1} + p_n + 2n - 1} & \text{if } 2^{q_n} \leqq j \leqq 2^{p_{n+1}} - 1, \\ \frac{1}{2} & \text{if } 2^{p_{n+1}} \leqq j. \end{cases}$$

Considering all cases, we find $\theta_k \geqq 2^{-p_{n+1} + p_n + 2n - 1}$, so $\frac{1}{k}\theta_k \geqq 2^{-p_{n+1} + n - 1}$. Let $\alpha > 0$ be arbitrary. If n is so large that $2^n > \alpha^{-1}$ then $u_{\alpha k} = 2^{-p_{n+1}}$ and $\frac{1}{k}\theta_k / u_{\alpha k} \geqq 2^{n-1}$, which grows without bound as $n \to \infty$, contradicting (2). □

2. Spectral traces

Let $\mathscr{J}$ be an ideal of compact operators. A *trace* on $\mathscr{J}$ is a linear functional, $\tau : \mathscr{J} \to \mathbb{C}$ that is unitarily invariant, or, equivalently, that vanishes on $\operatorname{Com} \mathscr{J}$. In this section, we show that, as a consequence of [5], Theorem 3.3, every trace on a geometrically stable ideal $\mathscr{J}$ is a *spectral trace*, i.e. for every $T \in \mathscr{J}$, $\tau(T)$ depends only on the eigenvalues of T, listed according to algebraic multiplicity.

We shall use the following decomposition result, which is certainly well-known.

Proposition 2.1. *Let T be a compact operator on infinite dimensional Hilbert space $\mathscr{H}$. Then $T = D + Q$, where D is a normal operator whose eigenvalues and multiplicities are equal to those of T, and where Q is a quasi-nilpotent operator.*

Proof. Let E_T denote the closed linear span of all the root vectors for nonzero eigenvalues of T, and let P be the orthogonal projection of $\mathscr{H}$ onto E_T. Note that E_T is invariant under T, so $PTP = TP$. By Schur's Lemma (see [2], I. 4.1), there is an orthonormal basis for E_T, with respect to which PTP is upper triangular and has diagonal entries form the list $\lambda_1(T), \lambda_2(T), \ldots$. Let D be the operator on $\mathscr{H}$ which on $P\mathscr{H}$ is the diagonal part of PTP and on $(1-P)\mathscr{H}$ is zero. Then $P(T-D)P$ is strictly upper triangular, hence is quasi-nilpotent. Moreover, by [2], I. 4.2, $(1-P)T(1-P)$ is quasi-nilpotent. Now, using the following elementary lemma, we are done. □

Lemma 2.2. *Let $A \in B(\mathscr{H})$. Suppose that for a projection P, PAP and $(1-P)A(1-P)$ are quasi-nilpotent and $(1-P)AP = 0$. Then A is quasi-nilpotent.*

Proof. For convenience write $P_1 = P$ and $P_2 = 1 - P$. Then

$$A^n = (P_1 A P_1)^n + \sum_{k=0}^{n-1} (P_1 A P_1)^{n-k-1} (P_1 A P_2)(P_2 A P_2)^k + (P_2 A P_2)^n.$$

Now some elementary norm estimates and the quasi-nilpotence of $P_1 A P_1$ and $P_2 A P_2$ show that A is quasi-nilpotent. □

Corollary 2.3. *Let $\mathscr{J}$ be a geometrically stable ideal of compact operators and let $T \in \mathscr{J}$. Then $T = D + Q$ where $D \in \mathscr{J}$ is normal and $Q \in \mathscr{J}$ is quasi-nilpotent.*

Proof. If $\lambda_k = \lambda_k(T)$ are the eigenvalues of T listed according to algebraic multiplicity, then the inequality $|\lambda_1 \dots \lambda_k| \leqq |s_1(T) \dots s_k(T)|$ and the geometric stability of $\mathscr{J}$ imply that $D \in \mathscr{J}$. □

Corollary 2.4. *Let $\mathscr{J}$ be a geometrically stable ideal and suppose τ is a trace on $\mathscr{J}$. For every given $T \in \mathscr{J}$, the value $\tau(T)$ depends only on the eigenvalues of T and their algebraic multiplicities.*

Proof. Using the decomposition $T = D + Q$ from Corollary 2.3 and the spectral characterization of $\mathrm{Com}\,\mathscr{J}$ in [5], 3.3, it follows that $Q \in \mathrm{Com}\,\mathscr{J} \subseteqq \ker \tau$. Hence $\tau(T) = \tau(D)$. □

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