Spectral characterization of sums of commutators II

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\textbf{Abstract.} For countably generated ideals, $\mathcal{J}$, of $B(\mathcal{H})$, geometric stability is necessary for the canonical spectral characterization of sums of $(\mathcal{J}, B(\mathcal{H}))$-commutators to hold. This answers a question raised by Dykema, Figiel, Weiss and Wodzicki. There are some ideals, $\mathcal{J}$, having quasi-nilpotent elements that are not sums of $(\mathcal{J}, B(\mathcal{H}))$-commutators. Also, every trace on every geometrically stable ideal is a spectral trace.

\textbf{Introduction}

Let $\mathcal{H}$ be a separable infinite-dimensional Hilbert space, and let $\mathcal{J}$ be a (two-sided) ideal contained in the ideal of compact operators $K(\mathcal{H})$ on $\mathcal{H}$. We define the \textit{commutator subspace} $\text{Com}\, \mathcal{J}$ to be the set of all finite linear combinations of commutators $[A, B] = AB - BA$ where $A \in \mathcal{J}$ and $B \in \mathcal{B}(\mathcal{H})$.

In the immediately preceding paper, [5] N.J. Kalton showed that, for a wide class of ideals including quasi-Banach ideals (i.e. ideals equipped with a complete ideal quasi-norm) $\text{Com}\, \mathcal{J}$ has the following spectral characterization. For $T \in \mathcal{J}$, let $\lambda_k = \lambda_k(T)$ be the eigenvalues of $T$ listed according to algebraic multiplicity and such that $|\lambda_1| \geq |\lambda_2| \geq \cdots$. Then $T \in \text{Com}\, \mathcal{J}$ if and only if $\text{diag}\left\{ \frac{1}{n} \sum_{k=1}^{n} \lambda_k \right\} \in \mathcal{J}$. The sufficient condition from [5] for $\text{Com}\, \mathcal{J}$ to have a spectral characterization in the above sense is \textit{geometric stability}, i.e. the condition that if $(s_n)$ is a monotone decreasing real sequence then $\text{diag}\{s_n\} \in \mathcal{J}$ if and only if $\text{diag}\{(s_1 \ldots s_n)^{1/n}\} \in \mathcal{J}$. (See the introduction of [5] for background.)

In §1 of this paper, we show that when $\mathcal{J}$ is a countably generated ideal, then geometric stability is a necessary condition for this spectral characterization of $\text{Com}\, \mathcal{J}$. In particular, for every countably generated ideal, $\mathcal{J}$ which is not geometrically stable, we

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find an element $T$, of $\text{Com} \mathcal{J}$ for which $\text{diag}\left\{ \frac{1}{n} \left( \lambda_1(T) + \cdots + \lambda_n(T) \right) \right\} \notin \mathcal{J}$. We also give examples of ideals for which spectral characterization of $\text{Com} \mathcal{J}$ fails in the opposite direction, and in an extreme way, in that $\mathcal{J}$ has a quasi-nilpotent element that is not in $\text{Com} \mathcal{J}$.

In §2, we show, based on an elementary decomposition result for compact operators, that every trace, $\tau$, on an arbitrary geometrically stable ideal, $\mathcal{J}$, is a spectral trace, i.e. for every $T \in \mathcal{J}$, $\tau(T)$ depends only on the eigenvalues of $T$ and their multiplicities.

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1. Countably generated ideals

**Proposition 1.1.** Suppose $s_1 \geq s_2 \geq \cdots \geq 0$. Suppose that $(\lambda_n)_{n=1}^\infty$ is any sequence of complex numbers with $|\lambda_1| \geq |\lambda_2| \geq \cdots$ and such that $|\lambda_1 \cdots \lambda_n| \leq s_1 \cdots s_n$ for all $n$. Then there is an upper triangular operator $A = (a_{jk})_{j,k}$ such that $s_n(A) \leq s_n$ for all $n$ and $a_{jj} = \lambda_j$ for all $j$.

**Proof.** This is the infinite dimensional extension of a result of Horn [3] (see Gohberg-Krein [2], Remark II. 3.1). By Horn’s result we can find for each $n$ an upper-triangular matrix $A^{(n)} = (a^{(n)}_{jk})$ such that $a^{(n)}_{jk} = 0$ if $\max(j,k) > n$, $a^{(n)}_{jj} = \lambda_j$ for $1 \leq j \leq n$ and $s_j(A^{(n)}) \leq s_j$ for $1 \leq j < \infty$.

Now we can pass to a subsequence $(B^{(n)})$ of $(A^{(n)})$ which is convergent to the weak operator topology to some operator $A = (a_{ij})$ which is upper-triangular and has $a_{jj} = \lambda_j$ for all $j$. It remains to show that $s_j(A) \leq s_j$ for all $j$.

Fix $j > 1$ (the case $j = 1$ is well known). For each $n$ there is a subspace $E_n$ of dimension $j-1$ so that if $P_n$ is the orthogonal projection with kernel $E_n$ then $\|B^{(n)} P_n\| \leq s_j$. Let $(f_k)_{k=1}^{n-1}$ be an orthonormal basis of $E_n$. By passing to a further subsequence we can suppose that $\lim_{n \to \infty} f_k^n = f_k$ exists weakly for each $1 \leq k \leq j-1$. Let $E$ be the span of $(f_k)_{k=1}^{n-1}$, so that $\dim E \leq j-1$. Suppose $x \in E$. Then $\lim_{n \to \infty} (x, f_k^n) = 0$ for $1 \leq k \leq j-1$. Hence $\lim_{n \to \infty} \|x - P_n x\| = 0$. Thus $\|A x\| \leq \liminf_{n \to \infty} \|B^{(n)} x\| \leq \liminf_{n \to \infty} \|B^{(n)} P_n x\| \leq s_j \|x\|$. This shows that $s_j(A) \leq s_j$. \qed

Let us introduce a bit of notation. Suppose $u = (u_j)_{j=1}^\infty$ and $t = (t_j)_{j=1}^\infty$ are decreasing sequences of positive numbers. We write $t \preceq u$ to mean that $t_j \leq u_j$ for every $j$. For $c > 0$ we let $cu$ denote the sequence $(cu_j)_{j=1}^\infty$. We let $u \oplus u$ denote the sequence

$$(u_1, u_1, u_2, u_2, u_3, u_3, \ldots).$$
Lemma 1.2. Let \( u = (u_j)_{j=1}^{\infty} \) and \( t = (t_j)_{j=1}^{\infty} \) be sequences of positive numbers decreasing to zero such that \( \text{diag} \{ t_j \} \) is in the ideal of compact operators generated by \( \text{diag} \{ u_j \} \). Then
\[
\text{diag} \{ t_1 t_2 \ldots t_k \}^{1/k}
\]
is in the ideal of compact operators generated by \( \text{diag} \{ (u_1 u_2 \ldots u_k)^{1/k} \} \).

Proof. The hypotheses imply that there is a positive integer \( n \) and a positive number \( c \) such that \( t_{nj} \leq cu_j \) for every \( j \), and we may suppose \( n \) is a power of 2. Hence, in order to prove the lemma it will suffice to prove it in each of the following special cases, in turn:

(i) \( t \leq u \),

(ii) \( t = cu \) for \( c > 1 \),

(iii) \( t = u \oplus u \).

The cases (i) and (ii) are clear. If (iii) holds then \( (t_1 \ldots t_n)^{1/n} = (u_1 \ldots u_n)^{1/n} \), which shows that \( \text{diag} \{ t_1 \ldots t_k \} \) is in the ideal generated by \( \text{diag} \{ (u_1 \ldots u_n)^{1/n} \} \), and the lemma is proved. \( \square \)

Theorem 1.3. Let \( \mathcal{J} \) be a countably generated ideal in \( \mathcal{K}(\mathcal{H}) \). Then the following conditions are equivalent:

(i) \( \mathcal{J} \) is geometrically stable.

(ii) If \( T \in \mathcal{J} \) then \( \text{diag} \{ \lambda_n \} \in \mathcal{J} \) where \( \lambda_n = \lambda_n(T) \).

(iii) If \( T \in \text{Com } \mathcal{J} \) then \( \text{diag} \left\{ \frac{1}{n} (\lambda_1 + \cdots + \lambda_n) \right\} \in \mathcal{J} \), for some ordering of \( \lambda_n \) so that \( (\lambda_n) \) is decreasing.

Proof. (i) implies (iii) is proved for any ideals in [5]. That (i) implies (ii) is also noted in [5]; this follows from the inequalities \( |\lambda_n(T)| \leq (s_1(T) \ldots s_n(T))^{1/n} \).

Since \( \mathcal{J} \) is countably generated there is a countable family, \( \{ u^{(k)} \} \) of decreasing sequences of positive numbers such that \( T \in \mathcal{J} \) if and only if for some \( k \) we have \( s_n(T) \leq u^{(k)}_n \) for all \( n \geq 1 \). Indeed, let \( \{ u^{(k)} \} = \{ w^{(i,j)} | i, j \in \mathbb{N} \} \) where \( w^{(i,j)}_n = j w^{(i)}_{n+j} \) and where \( w^{(i)}_n \) is a countable generating set of the ideal \( \mathcal{J} \).

We will prove that each of (ii) and (iii) individually implies (i) by supposing that (i) fails and showing that both (ii) and (iii) fail. If (i) fails then there is a sequence \( (u^{(k)})_{k=1}^{\infty} \) such that \( \text{diag} \{ u^{(k)} \} \in \mathcal{J} \) but \( \text{diag} \{ (t_1 t_2 \ldots t_n)^{1/n} \} \notin \mathcal{J} \). Now for every \( k \in \mathbb{N} \), since \( \text{diag} \{ (t_{n+k})^{1/n} \}_{n=1}^{\infty} \) generates the same ideal as \( \text{diag} \{ t_n \} \), it follows from Lemma 1.2 that \( \text{diag} \{ (t_{k+1} t_{k+2} \ldots t_{k+n})^{1/n} \}_{n=1}^{\infty} \notin \mathcal{J} \).
It follows that we can find a sequence \( m_n \uparrow \infty \) such that if \( m_0 = 0 \) and \( p_n = m_n - m_{n-1} \) for \( n \geq 1 \) then \( p_n/m_n \to 1 \) and

\[
\hat{w}_n \overset{\text{def}}{=} \left( \prod_{k=m_{n-1}+1}^{m_n} t_k \right)^{1/p_n} > u_{m_n}^{(n)}.
\]

Let \( w_k = \hat{w}_n \) for every \( m_{n-1} + 1 \leq k \leq m_n \). We may assume without loss of generality that \( \hat{w}_n > \hat{w}_{n+1} \) for every \( n \in \mathbb{N} \). By Proposition 1.1 there is an upper triangular operator \( A \in \mathcal{J} \) with diagonal entries \( d_k = w_k \) for \( k \in \mathbb{N} \). Now \( \text{diag}(w_k) \notin \mathcal{J} \) by construction and hence (ii) fails. Therefore (ii) implies (i).

We now show that also (iii) fails, which will finish the proof of the theorem. We can choose \( \bar{\sigma}_1 > \bar{\sigma}_2 > \cdots > 0 \) decreasing quickly enough so that \( \hat{w}_n > \bar{\sigma}_n > \hat{w}_{n+1} \) for every \( n \) and such that, letting \( \sigma_k = \bar{\sigma}_n \) for every \( m_{n-1} + 1 \leq k \leq m_n \), we have \( \text{diag}(\sigma_k) \in \mathcal{J} \). Define for each \( n \in \mathbb{N} \)

\[
\varepsilon_{2n-1} = \varepsilon_{2n} = \frac{1}{2} \min \{ \bar{\sigma}_{2n-1} - \bar{\sigma}_{2n} \}.
\]

Then for every \( n \in \mathbb{N} \) we have that \( \varepsilon_n > 0 \), that \( \bar{\sigma}_n - \varepsilon_n > \bar{\sigma}_{n-1} \) and that

\[
\left| \sum_{j=1}^{n} (-1)^{j-1} \varepsilon_j \right| \leq \begin{cases} \bar{\sigma}_n & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases}
\]

Let \((\mu_k)_{k=1}^{\infty}\) be the sequence defined by

\[
\mu_k = \begin{cases} \bar{\sigma}_n & \text{if } m_{n-1} + 1 \leq k \leq m_n \text{ and } n \text{ is odd,} \\ \bar{\sigma}_n - (\varepsilon_n/p_n) & \text{if } m_{n-1} + 1 \leq k \leq m_n \text{ and } n \text{ is even,} \end{cases}
\]

and let \((v_k)_{k=1}^{\infty}\) be the sequence defined by

\[
v_k = \begin{cases} \bar{\sigma}_n - (\varepsilon_n/p_n) & \text{if } m_{n-1} + 1 \leq k \leq m_n \text{ and } n \text{ is odd,} \\ \bar{\sigma}_n & \text{if } m_{n-1} + 1 \leq k \leq m_n \text{ and } n \text{ is even.} \end{cases}
\]

Note that \((u_k)_{k=1}^{\infty}\) and \((v_k)_{k=1}^{\infty}\) are decreasing sequences. Let \( D_1 = \text{diag} \{ \mu_k \} \) and \( D_2 = \text{diag} \{ v_k \} \). We will show that \( D_1 \oplus (-D_2) \in \text{Com} \mathcal{J} \). Let \((\lambda_k)_{k=1}^{\infty}\) be the eigenvalue sequence, arranged in some order of decreasing absolute value, of the operator \( D_1 \oplus (-D_2) \).

We will show that \( \text{diag} \left\{ \frac{1}{k} (\lambda_1' + \cdots + \lambda_k') \right\} \in \mathcal{J} \), which by [1] shows \( D_1 \oplus (-D_2) \in \text{Com} \mathcal{J} \). The sequence \((\lambda_k')_{k=1}^{\infty}\) is
\[
\frac{\bar{\sigma}_1, \ldots, \bar{\sigma}_1}{p_1 \text{ times}}, \quad \frac{\bar{\sigma}_1 + \frac{e_1}{p_1}, \ldots, \bar{\sigma}_1 + \frac{e_1}{p_1}}{p_1 \text{ times}}, \quad \frac{\bar{\sigma}_2, \ldots, \bar{\sigma}_2}{p_2 \text{ times}}, \quad \frac{\bar{\sigma}_2 + \frac{e_2}{p_2}, \ldots, \bar{\sigma}_2 + \frac{e_2}{p_2}}{p_2 \text{ times}}, \quad \ldots, \frac{\bar{\sigma}_{2n-1}, \ldots, \bar{\sigma}_{2n-1}}{p_{2n-1} \text{ times}}, \quad \frac{\bar{\sigma}_{2n-1} + \frac{e_{2n-1}}{p_{2n-1}}, \ldots, \bar{\sigma}_{2n-1} + \frac{e_{2n-1}}{p_{2n-1}}}{p_{2n-1} \text{ times}}, \quad \frac{-\bar{\sigma}_{2n}, \ldots, -\bar{\sigma}_{2n}}{p_{2n} \text{ times}}, \quad \frac{\bar{\sigma}_{2n} + \frac{e_{2n}}{p_{2n}}, \ldots, \bar{\sigma}_{2n} + \frac{e_{2n}}{p_{2n}}}{p_{2n} \text{ times}}.
\]

Let \( \eta'_k = \sum_{j=1}^k \lambda'_k \). Then for every \( n \in \mathbb{N} \),

\[
\begin{align*}
\eta'_{2m_{2n-2} + 2k} &= k \bar{\sigma}_{2n-1} \\
(1 \leq k \leq p_{2n-1}), \\
\eta'_{2m_{2n-2} + p_{2n-1} + k} &= (p_{2n-1} - k) \bar{\sigma}_{2n-1} + \frac{k e_{2n-1}}{p_{2n-1}} \\
(1 \leq k \leq p_{2n-1}), \\
\eta'_{2m_{2n-1} + k} &= e_{2n-1} - k \bar{\sigma}_{2n} \\
(1 \leq k \leq p_{2n}), \\
\eta'_{2m_{2n-1} + p_{2n} + k} &= e_{2n-1} - (p_{2n} - k) \bar{\sigma}_{2n} - \frac{k e_{2n}}{p_{2n}} \\
(1 \leq k \leq p_{2n}).
\end{align*}
\]

Making crude (but sufficient) estimates, we get

\[
\begin{align*}
\left| \frac{\eta'_{2m_{2n-2} + 2k}}{2m_{2n-2} + 2k} \right| &\leq \bar{\sigma}_{2n-1} \\
(1 \leq k \leq p_{2n-1}), \\
\left| \frac{\eta'_{2m_{2n-2} + p_{2n-1} + k}}{2m_{2n-2} + p_{2n-1} + k} \right| &\leq 2 \bar{\sigma}_{2n-1} \\
(1 \leq k \leq p_{2n-1}), \\
\left| \frac{\eta'_{2m_{2n-1} + k}}{2m_{2n-1} + k} \right| &\leq \bar{\sigma}_{2n-1} + \bar{\sigma}_{2n} \\
(1 \leq k \leq p_{2n}), \\
\left| \frac{\eta'_{2m_{2n-1} + p_{2n} + k}}{2m_{2n-1} + p_{2n} + k} \right| &\leq \bar{\sigma}_{2n-1} + \bar{\sigma}_{2n} \\
(1 \leq k \leq p_{2n}).
\end{align*}
\]

Using \( \text{diag}\{\lambda_k\} \in \mathcal{J} \) we see that \( \text{diag}\left\{ \frac{1}{k} (\lambda'_1 + \cdots + \lambda'_k) \right\} \in \mathcal{J} \). Hence by [1]

\[
D_1 \oplus (-D_2) \in \text{Com} \mathcal{J}.
\]

Let \( A \in \mathcal{J} \) be the upper triangular operator with diagonal entries \( a_{kk} = w_k \) as constructed above. Consider the operator

\[
\begin{align*}
D_1 \oplus (-D_2) \in \text{Com} \mathcal{J}.
\end{align*}
\]
Then $T \in \text{Com} \mathcal{J}$ because $A \oplus (-A) \in \text{Com} \mathcal{J}$ and $D_1 \oplus (-D_2) \in \text{Com} \mathcal{J}$. Let $\lambda_k = \lambda_k(T)$ be the eigenvalues of $T$ listed according to algebraic multiplicity and in order of decreasing absolute value. Then

$$
\begin{align*}
\hat{\lambda}_{2m_{2n-1} + k} & = \lambda'_{2m_{2n-1} + k} + \bar{w}_{2n-1} & (1 \leqslant k \leqslant p_{2n-1}), \\
\hat{\lambda}_{2m_{2n-2} + p_{2n-1} + k} & = \lambda'_{2m_{2n-2} + p_{2n-1} + k} - \bar{w}_{2n-1} & (1 \leqslant k \leqslant p_{2n-1}), \\
\hat{\lambda}_{2m_{2n-1} + k} & = \lambda'_{2m_{2n-1} + k} - \bar{w}_{2n} & (1 \leqslant k \leqslant p_{2n}), \\
\hat{\lambda}_{2m_{2n-1} + p_{2n} + k} & = \lambda'_{2m_{2n-1} + p_{2n} + k} + \bar{w}_{2n} & (1 \leqslant k \leqslant p_{2n}).
\end{align*}
$$

Let $\eta_k = \sum_{j=1}^k \lambda_j$. Then

$$
\begin{align*}
\eta_{2m_{2n-2} + k} & = \eta'_{2m_{2n-2} + k} + k \bar{w}_{2n-1} & (1 \leqslant k \leqslant p_{2n-1}), \\
\eta_{2m_{2n-2} + p_{2n-1} + k} & = \eta'_{2m_{2n-2} + p_{2n-1} + k} + (p_{2n-1} - k) \bar{w}_{2n-1} & (1 \leqslant k \leqslant p_{2n-1}), \\
\eta_{2m_{2n-1} + k} & = \eta'_{2m_{2n-1} + k} - k \bar{w}_{2n} & (1 \leqslant k \leqslant p_{2n}), \\
\eta_{2m_{2n-1} + p_{2n} + k} & = \eta'_{2m_{2n-1} + p_{2n} + k} - (p_{2n} - k) \bar{w}_{2n} & (1 \leqslant k \leqslant p_{2n}).
\end{align*}
$$

We will show that \( \text{diag} \left\{ \frac{1}{k} \hat{\lambda}_1 + \cdots + \hat{\lambda}_k \right\} \notin \mathcal{J} \). Since \( \text{diag} \left\{ \frac{1}{k} \eta_k \right\} \notin \mathcal{J} \), it will suffice to show that \( \text{diag} \left\{ \frac{1}{k} (\eta_k - \eta'_k) \right\} \notin \mathcal{J} \). However, taking the absolute value of a subsequence of \( \left\{ \frac{1}{k} (\eta_k - \eta'_k) \right\}_{k=1}^\infty \) and using \( \lim_{n \to \infty} (p_n/m_n) = 1 \), we see that \( \text{diag} \left\{ \frac{1}{k} (\eta_k - \eta'_k) \right\} \notin \mathcal{J} \) would imply \( \text{diag} \left\{ w_k \right\} \notin \mathcal{J} \), which would be a contradiction. Therefore

$$
\text{diag} \left\{ \frac{1}{k} \hat{\lambda}_1 + \cdots + \hat{\lambda}_k \right\} \notin \mathcal{J}
$$

and hence (iii) fails. \( \square \)

We will conclude by showing that it is in general impossible to characterize membership in \( \text{Com} \mathcal{J} \) by considering only the eigenvalues. To this end we will construct a quasinilpotent operator $T$ so that $T \notin \text{Com} \mathcal{J}_T$, where $\mathcal{J}_T$ is the ideal generated by $T$.

We will start by considering any singly generated ideal, $\mathcal{J}$. Then there is a decreasing sequence of positive numbers $(u_n)$ such that $T \in \mathcal{J}$ if and only if for some $C > 0$ and $0 < z \leqslant 1$ we have $s_z(T) \leq C u_{zn}$.

**Lemma 1.4.** Let $\mathcal{J}$ and $(u_n)$ be as above and suppose $(v_n)_{n=1}^\infty$ and $(w_n)_{n=1}^\infty$ are a pair of real sequences such that
(i) we have \( v_1 \geq v_2 \geq \cdots \geq 0 \);

(ii) for each \( k \geq 1 \) the sequence \((v_n)\) is constant on the dyadic block

\[
\{2^{k-1}, 2^{k-1} + 1, \ldots, 2^k - 1\};
\]

(iii) for each \( n \in \mathbb{N} \) we have

\[
\prod_{k=1}^{n} v_k \leq \prod_{k=1}^{n} u_k;
\]

(iv) for each \( n \) we have \(|w_n| \leq |u_n|\);

(v) for each \( n \) we have

\[
\left| \sum_{k=1}^{n} w_k \right| \leq n v_n.
\]

Then there is an upper-triangular operator \( A \in \text{Com } \mathcal{J} \) with \( a_{nn} = w_n \) for all \( n \).

**Proof.** The proof is essentially the same as for the corresponding result for diagonal operators in [4] and [1]. We start by setting

\[
\eta_n = 2^{1-k} \sum_{j=2^{k-1}}^{2^{k-1}} w_j
\]

when \( 1 \leq k < \infty \) and \( 2^{k-1} \leq n \leq 2^k - 1 \). Notice that

\[
|\eta_n - w_n| \leq u_{n/2}
\]

and that, if \( 2^{k-1} \leq n \leq 2^k - 1 \),

\[
\left| \sum_{j=1}^{n} (\eta_j - w_j) \right| = \left| \sum_{j=2^{k-1}}^{n} (\eta_j - w_j) \right| \leq 2^k u_{n/2}
\]

from which it follows that \( \text{diag}\{w_n - \eta_n\} \in \text{Com } \mathcal{J} \).

It therefore will suffice to show that there is an upper-triangular operator \( A \) in \( \text{Com } \mathcal{J} \) with \( a_{nn} = \eta_n \) for all \( n \). To this end we define a sequence \( \xi_n \) by setting

\[
\xi_n = 2^{1-k} \sum_{j=1}^{2^{k-1}} \eta_j = 2^{1-k} \sum_{j=1}^{2^{k-1}} w_j
\]

for \( 2^{k-1} \leq n < 2^k - 1 \). By hypothesis we have \( |\xi_n| \leq 2v_n \) for every \( n \in \mathbb{N} \). Now we note that by Proposition 1.1, there is an upper-triangular matrix \( C \in \mathcal{J} \) with \( c_{nn} = v_n \) for every \( n \); hence there is an upper-triangular matrix \( B \in \mathcal{J} \) with \( b_{nn} = \xi_n \) for \( n \in \mathbb{N} \).
Now consider the isometries $U_1, U_2$ defined by $U_1 e_n = e_{2n+1}$ and $U_2 e_n = e_{2n}$. Consider the commutator $[U^*_1 B, U^*_1]$. We have

$$[U^*_1, U_1 B](e_{2n+1}) = Be_{2n+1} - U_1 Be_n,$$
$$[U^*_1, U_1 B](e_{2n}) = Be_{2n}$$

and

$$[U^*_1, U_1 B]e_1 = Be_1.$$

Similarly

$$[U^*_2, U_2 B]e_{2n} = Be_{2n} - U_2 Be_n,$$
$$[U^*_2, U_2 B]e_{2n+1} = Be_{2n+1}$$

and

$$[U^*_2, U_2 B]e_1 = Be_1.$$

Thus if $A = \frac{1}{2} ([U^*_1, U_1 B] + [U^*_2, U_2 B])$ then $A e_1 = Be_1$ and

$$(A e_{2n+1}, e_{2m+1}) = (Be_{2n+1}, e_{2m+1}) - \frac{1}{2} (Be_n, e_m),$$

$$(A e_{2n+1}, e_{2m}) = (Be_{2n+1}, e_{2m+1}),$$

$$(A e_{2n+1}, e_1) = (Be_{2n+1}, e_1) = 0,$$

$$(A e_{2n}, e_{2m+1}) = (Be_{2n}, e_{2m+1}),$$

$$(A e_{2n}, e_{2m}) = (Be_{2n}, e_{2m}) - \frac{1}{2} (Be_n, e_m),$$

$$(A e_{2n}, e_1) = (Be_{2n}, e_1) = 0.$$

Thus $(A e_n, e_m) = 0$ if $m < n$ and $a_{11} = \xi_1 = \eta_1$ while if $2^{k-1} \leq n \leq 2^k - 1$ with $k \geq 2$ we have $a_{nn} = \xi_{2k-1} - \frac{1}{2} \xi_{2k-2} = \eta_n$. Since $A \in \text{Com} \mathcal{J}$, this completes the proof. \qed

We now turn to the construction of some examples of ideals, $\mathcal{J}$, having quasi-nilpotents that are not sums of commutators.

**Examples 1.5.** Let $0 = p_0 < p_1 < p_2 < \cdots$ be integers such that $p_{n+1} > p_n + 2n 2^{2n}$ for every $n \geq 0$. Let

$$u_k = 2^{-p_n} \quad \text{if} \quad 2^{p_n-1} \leq k \leq 2^{p_n} - 1 \quad (n \geq 1),$$

and let $\mathcal{J}$ be the ideal of $B(\mathcal{H})$ generated by diag $\{ u_k \}_{k=1}^{\infty}$. Then there is a quasi-nilpotent operator, $T \in \mathcal{J}$, such that $T \notin \text{Com} \mathcal{J}$. 
Proof. Let \( q_n = 2n + p_n \) and
\[
v_k = 2^n 2^{-p_{n+1}} \quad \text{if} \quad 2^{q_n - 1} \leq k \leq 2^{q_n} - 1 \quad (n \geq 1).
\]
We claim that
\[
\prod_{j=1}^{k} v_j \leq \prod_{j=1}^{k} u_j
\]
for every \( k \in \mathbb{N} \). Let \( \log \) denote the base 2 logarithm. For every \( n \in \mathbb{N} \), we have
\[
\log \left( \prod_{j=1}^{2^{q_n} - 1} v_j u_j^{-1} \right) = 2^{p_n} ((1 - 2^{-p_n + p_{n+1} + 2n - 2}) (p_{n+1} + p_n) + (1 - 2^{-p_n + p_{n-1} - 2}) n 2^{2n}).
\]
But
\[
1 - 2^{-p_n + p_{n-1} - 2} < 2(1 - 2^{-p_n + p_{n-1} + 2n - 2})
\]
so
\[
\log \left( \prod_{j=1}^{2^{q_n} - 1} v_j u_j^{-1} \right) < 2^{p_n} (1 - 2^{-p_n + p_{n-1} + 2n - 2}) (p_{n+1} + p_n + 2n 2^{2n}) < 0.
\]
Therefore, by induction on \( n \), (1) holds whenever \( k = 2^{q_n} - 1 \). But since \( v_j < u_j \) when \( 2^{q_n} \leq j \leq 2^{q_n+1} - 1 \) and \( v_j > u_j \) when \( 2^{q_n+1} \leq j \leq 2^{q_n+1} - 1 \), it follows that (1) holds for all \( k \in \mathbb{N} \).

Let \( \sigma_k = \sum_{j=1}^{k} u_j \) and
\[
\theta_k = \inf_{j \in \mathbb{N}} (j v_j + |\sigma_k - \sigma_j|).
\]
Then \( \theta_1 \leq v_1 \leq u_1 \) and for \( k \geq 2 \), \( |\theta_k - \theta_{k-1}| \leq u_k \). Let \( w_1 = \theta_1 \) and \( w_k = \theta_k - \theta_{k-1} \) \((k \geq 2)\). Then for every \( k \in \mathbb{N} \) we have \( |w_k| \leq u_k \) and \( 0 \leq \sum_{j=1}^{k} w_j = \theta_k \leq k v_k \). Therefore, by Lemma 1.4 there is an upper triangular \( A \in \text{Com} \mathcal{J} \) with diagonal elements \( a_{kk} = w_k \). However, also \( W \equiv \text{diag}(w_k) \in \mathcal{J} \). Let \( T = A - W \). Then \( T \in \mathcal{J} \) is quasi-nilpotent. We will show that \( T \notin \text{Com} \mathcal{J} \) by showing \( W \notin \mathcal{J} \).

Suppose for contradiction that \( W \in \text{Com} \mathcal{J} \). Let \( \{\lambda_k\}_{k=1}^{\infty} \) be a rearrangement of \( \{w_k\}_{k=1}^{\infty} \) such that \( |\lambda_1| \geq |\lambda_2| \geq \cdots \). Then by [1], \( \text{diag} \left( \frac{1}{k} (\lambda_1 + \cdots + \lambda_k) \right) \in \mathcal{J} \). For every \( k \in \mathbb{N} \) we have
\[
\left| \sum_{j=1}^{k} \lambda_j - \sum_{j=1}^{k} w_j \right| \leq 2 k u_k
\]
because every \( w_j \) having absolute value strictly greater than \( u_k \) appears in both summations above. Hence \( \text{diag} \left\{ \frac{1}{k} (w_1 + \cdots + w_k) \right\} \in \mathcal{F} \), thus by [5], 3.1 (3) there are \( \alpha > 0 \) and \( c > 0 \) such that

\[
\forall k \in \mathbb{N} \quad \frac{1}{k} \theta_k \leq cu_k.
\]

Let \( k = 2^{p_0 + n} \). We will find a lower bound for \( \theta_k \) which will contradict (2). Routine estimation reveals that

\[
jw_j + |\sigma_k - \sigma_j| \geq \begin{cases} \frac{1}{k} & \text{if } j < 2^{p_0 - 1}, \\ 1 & \text{if } 2^{p_0 - 1} \leq j \leq 2^{p_0 - 1} - 1, \\ 2^{-p_0 + 1 + p_0 + 2n} & \text{if } 2^{p_0 - 1} \leq j \leq 2^{p_0} - 1, \\ 2^{-p_0 + 1 + p_0 + 2n} & \text{if } 2^{p_0} \leq j \leq 2^{p_0 + n} - 1, \\ 2^{-p_0 + 1 + p_0 + 3n} & \text{if } 2^{p_0} \leq j \leq 2^{p_0 + 2n} - 1, \\ 2^{-p_0 + 1 + p_0 + 2n - 1} & \text{if } 2^{p_0} \leq j \leq 2^{p_0 + 1} - 1, \\ \frac{1}{k} & \text{if } 2^{p_0 + 1} \leq j. \end{cases}
\]

Considering all cases, we find \( \theta_k \geq 2^{-p_0 + 1 + p_0 + 2n - 1} \), so \( \frac{1}{k} \theta_k \geq 2^{-p_0 + 1 + n - 1} \). Let \( \alpha > 0 \) be arbitrary. If \( n \) is so large that \( 2^n > \alpha^{-1} \) then \( u_k = 2^{-p_0 + 1} \) and \( \frac{1}{k} \theta_k / u_k \geq 2^{n-1} \), which grows without bound as \( n \to \infty \), contradicting (2). \( \square \)

2. Spectral traces

Let \( \mathcal{F} \) be an ideal of compact operators. A trace on \( \mathcal{F} \) is a linear functional, \( \tau : \mathcal{F} \to \mathbb{C} \) that is unitarily invariant, or, equivalently, that vanishes on \( \text{Com} \mathcal{F} \). In this section, we show that, as a consequence of [5], Theorem 3.3, every trace on a geometrically stable ideal \( \mathcal{F} \) is a spectral trace, i.e. for every \( T \in \mathcal{F} \), \( \tau(T) \) depends only on the eigenvalues of \( T \), listed according to algebraic multiplicity.

We shall use the following decomposition result, which is certainly well-known.

**Proposition 2.1.** Let \( T \) be a compact operator on infinite dimensional Hilbert space \( \mathcal{H} \). Then \( T = D + Q \), where \( D \) is a normal operator whose eigenvalues and multiplicities are equal to those of \( T \), and where \( Q \) is a quasi-nilpotent operator.

**Proof.** Let \( E_T \) denote the closed linear span of all the root vectors for nonzero eigenvalues of \( T \), and let \( P \) be the orthogonal projection of \( \mathcal{H} \) onto \( E_T \). Note that \( E_T \) is invariant under \( T \), so \( PTP = TP \). By Schur’s Lemma (see [2], I. 4.1), there is an orthonormal basis for \( E_T \), with respect to which \( PTP \) is upper triangular and has diagonal entries form the list \( \lambda_1(T), \lambda_2(T), \ldots \). Let \( D \) be the operator on \( \mathcal{H} \) which on \( P\mathcal{H} \) is the diagonal part of \( PTP \) and on \( (1 - P)\mathcal{H} \) is zero. Then \( P(T - D)P \) is strictly upper triangular, hence is quasi-nilpotent. Moreover, by [2], I. 4.2, \((1 - P)T(1 - P) \) is quasi-nilpotent. Now, using the following elementary lemma, we are done. \( \square \)
Lemma 2.2. Let $A \in B(\mathcal{H})$. Suppose that for a projection $P$, $PAP$ and $(1 - P)A(1 - P)$ are quasi-nilpotent and $(1 - P)AP = 0$. Then $A$ is quasi-nilpotent.

Proof. For convenience write $P_1 = P$ and $P_2 = 1 - P$. Then

$$A^n = (P_1AP_1)^n + \sum_{k=0}^{n-1} (P_1AP_1)^{n-k-1}(P_1AP_2)(P_2AP_2)^k + (P_2AP_2)^n.$$ 

Now some elementary norm estimates and the quasi-nilpotence of $P_1AP_1$ and $P_2AP_2$ show that $A$ is quasi-nilpotent. □

Corollary 2.3. Let $\mathcal{J}$ be a geometrically stable ideal of compact operators and let $T \in \mathcal{J}$. Then $T = D + Q$ where $D \in \mathcal{J}$ is normal and $Q \in \mathcal{J}$ is quasi-nilpotent.

Proof. If $\lambda_k = \lambda_k(T)$ are the eigenvalues of $T$ listed according to algebraic multiplicity, then the inequality $|\lambda_1 \ldots \lambda_k| \leq |s_1(T) \ldots s_k(T)|$ and the geometric stability of $\mathcal{J}$ imply that $D \in \mathcal{J}$. □

Corollary 2.4. Let $\mathcal{J}$ be a geometrically stable ideal and suppose $\tau$ is a trace on $\mathcal{J}$. For every given $T \in \mathcal{J}$, the value $\tau(T)$ depends only on the eigenvalues of $T$ and their algebraic multiplicities.

Proof. Using the decomposition $T = D + Q$ from Corollary 2.3 and the spectral characterization of $\text{Com}\mathcal{J}$ in [5], 3.3, it follows that $Q \in \text{Com}\mathcal{J} \subseteq \ker\tau$. Hence $\tau(T) = \tau(D)$. □

References


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