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Spectral characterization of sums of commutators I

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Abstract. Suppose $\mathscr{J}$ is a two-sided quasi-Banach ideal of compact operators an a separable infinite-dimensional Hilbert space $\mathscr{H}$. We show that an operator $T \in \mathscr{J}$ can be expressed as finite linear combination of commutators $[A, B]$ where $A \in \mathscr{J}$ and $B \in \mathscr{B}(\mathscr{H})$ if and only if its eigenvalues (λ_n) (arranged in decreasing order of absolute value, repeated according to algebraic multiplicity and augmented by zeros if necessary) satisfy the condition that the diagonal operator $\operatorname{diag}\left\{\frac{1}{n}(\lambda_1 + \cdots + \lambda_n)\right\}$ is a member of $\mathscr{J}$. This answers (for quasi-Banach ideals) a question raised by Dykema, Figiel, Weiss and Wodzicki.

1. Introduction

Let $\mathscr{H}$ be a separable infinite-dimensional Hilbert space, and let $\mathscr{J}$ be a (two-sided) ideal contained in the ideal of compact operators $\mathscr{K}(\mathscr{H})$ on $\mathscr{H}$. We define the *commutator subspace* $\operatorname{Com}\mathscr{J}$ to be the closed linear span of commutators $[\mathrm{A}, B] = AB - BA$ where $A \in \mathscr{J}$ and $B \in \mathscr{B}(\mathscr{H})$. It has been shown by Dykema, Figiel, Weiss and Wodzicki in [3] that if $\mathscr{I}_1$ and $\mathscr{I}_2$ are any two ideals then the linear span of commutators of the form $[A_1, A_2]$ where $A_j \in \mathscr{I}_j$ for $j = 1, 2$ coincides with the commutator subspace $\operatorname{Com}\mathscr{J}$ where $\mathscr{J} = \mathscr{I}_1 \mathscr{I}_2$.

Pearcy and Topping ([7], cf. [2]) showed that for the Schatten ideal $\mathscr{J} = \mathscr{C}_p$ when $p > 1$, we have $\operatorname{Com}\mathscr{C}_p = \mathscr{C}_p$. They then raised the question whether

$$\operatorname{Com}\mathscr{C}_1 = \{T \in \mathscr{C}_1 : \operatorname{tr} T = 0\}.$$

This question was resolved negatively by Weiss [8], [9]. However, Anderson [1] showed that in the case $p < 1$ we have $\operatorname{Com}\mathscr{C}_p = \{T \in \mathscr{C}_p : \operatorname{tr} T = 0\}$.

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In [6] a complete characterization of $\mathrm{Com}\,\mathscr{C}_1$ was obtained. It was shown that, in the case when $\mathscr{J} = \mathscr{C}_1$ is the trace-class, then $T \in \mathrm{Com}\,\mathscr{C}_1$ if and only if $T \in \mathscr{C}_1$ and its eigenvalues $(\lambda_n(T))_{n=1}^{\infty}$, counted according to algebraic multiplicity and arranged in some order satisfying that $(|\lambda_n(T)|)_{n=1}^{\infty}$ is decreasing, satisfies the inequality

$$\sum_{n=1}^{\infty} \frac{|\lambda_1 + \cdots + \lambda_n|}{n} < \infty . \tag{1.1}$$

If the eigenvalue set of T is finite one may extend the sequence $\lambda_n(T)$ by including infinitely many zeroes. This extended earlier partial results in [8] and [9].

Since, for any ideal $\mathscr{J}$, $\mathrm{Com}\,\mathscr{J}$ is a self-adjoint subspace it is clear that if $T = H + iK$ is split into hermitian and skew-hermitian parts then $T \in \mathrm{Com}\,\mathscr{J}$ if and only if $H \in \mathrm{Com}\,\mathscr{J}$ and $K \in \mathrm{Com}\,\mathscr{J}$. Thus to characterize $\mathrm{Com}\,\mathscr{J}$ it is necessary only to characterize the hermitian operators in $\mathrm{Com}\,\mathscr{J}$. In particular, the result above shows that if $T \in \mathscr{C}_1$ the condition (1.1) is equivalent to the pair of conditions that H and K each satisfy (1.1).

Recently in [3] a very general approach was developed which is applicable to any ideal. It was shown that for any ideal $\mathscr{J}$ a hermitian operator $H \in \mathrm{Com}\,\mathscr{J}$ if and only if $H \in \mathscr{J}$ and the diagonal operator $\mathrm{diag}\left\{\frac{1}{n}(\lambda_1 + \cdots + \lambda_n)\right\}$ belongs to $\mathscr{J}$ where again $\lambda_n = \lambda_n(H)$ is the eigenvalue sequence as above. Although this yields an explicit test for membership in $\mathrm{Com}\,\mathscr{J}$ by the process of splitting into hermitian and skew-hermitian parts, it leaves open the question whether same characterization in terms of eigenvalues extends to all operators in $\mathrm{Com}\,\mathscr{J}$, as in the case of the trace-class.

The aim of this paper is to show that for a fairly broad class of "nice" ideals the answer to this question is positive. The condition we impose on an ideal $\mathscr{J}$ is that it is *geometrically stable*. This means that if a diagonal operator $\mathrm{diag}\{s_1, s_2, \ldots\} \in \mathscr{J}$ where $s_1 \geqq s_2 \geqq \cdots \geqq 0$ then we have $\mathrm{diag}\{u_1, u_2, \ldots\} \in \mathscr{J}$ where $u_n = (s_1 \ldots s_n)^{1/n}$. For any Banach or quasi-Banach ideal (i.e. an ideal equipped with an appropriate ideal quasi-norm) this condition is automatic. Under the hypothesis that $\mathscr{J}$ is geometrically stable we show that $T \in \mathrm{Com}\,\mathscr{J}$ if and only if $\mathrm{diag}\left\{\frac{1}{n}(\lambda_1 + \cdots + \lambda_n)\right\} \in \mathscr{J}$ where $\lambda_n = \lambda_n(T)$.

In a separate note, in collaboration with Ken Dykema [4], we show that then for arbitrary ideals $\mathscr{J}$ this result is false and indeed there is no spectral characterization of the subspace $\mathrm{Com}\,\mathscr{J}$.

Let us note that our results do depend in an essential way on the results of [3], in that we use their result to reduce the problem to discussion of an operator of the form $T = H + iK$ where H, K are hermitian.

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2. The key results

Let T be a bounded operator on a separable Hilbert space $\mathscr{H}$. We denote by $s_n = s_n(T)$, for $n \geqq 1$ the singular values of T. It will be convenient to define s_n for n not an integer by $s_n = s_{[n]+1}$. With this notation we have the inequalities $s_n(S+T) \leqq s_{n/2}(S) + s_{n/2}(T)$ and $s_n(ST) \leqq s_{n/2}(S) s_{n/2}(T)$.

If T is compact we denote by $\lambda_n = \lambda_n(T)$ the eigenvalues of T repeated according to algebraic multiplicity and arranged in decreasing order of absolute value (this arrangement is not unique, so we require some selection to be made). Note that $s_n(T) = \lambda_n(|T|)$.

Let us suppose that $f: \mathbb{C} \to \mathbb{C}$ is any function which vanishes on a neighborhood of the origin. Then we can define a functional $\hat{f}: \mathscr{K}(\mathscr{H}) \to \mathbb{C}$ by the formula:

$$\hat{f}(T) = \sum_{n=1}^{\infty} f(\lambda_n).$$

Lemma 2.1. (1) *If f is continuous then $\hat{f}$ is continuous.*

(2) *If f is a Borel function then $\hat{f}$ is a Borel function.*

(3) *If f is continuous, real-valued and subharmonic then $\hat{f}$ is plurisubharmonic on $\mathscr{K}(\mathscr{H})$, i.e. if $S, T \in \mathscr{K}(\mathscr{H})$ then*

$$\hat{f}(S) \leqq \int_0^{2\pi} \hat{f}(S + e^{i\theta} T) \frac{d\theta}{2\pi}.$$

Proof. (1) Suppose $f(z)$ vanishes for $|z| \leqq \delta$ where $\delta > 0$. If $T \in \mathscr{K}(H)$ then suppose m is the least integer such that $|\lambda_m(T)| < \delta$. Pick $\eta < \delta$ such that $|\lambda_m(T)| < \eta$ and if $m \geqq 2$ then $\eta < |\lambda_{m-1}(T)|$. Now if T_n is a sequence of compact operators with $\lim_{n\to\infty} \|T_n - T\| = 0$ then by results in [5] (see p. 14, 18) we can find n_0 so that an ordering $(\lambda_k'(T_n))_{k=1}^{\infty}$ of the eigenvalues of T_n such that $|\lambda_k'(T_n)| < \eta$ for $n \geqq n_0$ and $k \geqq m$ and $\lim_{n\to\infty} \lambda_k'(T_n) = \lambda_k(T)$ if $k < m$. If follows easily that $\hat{f}(T_n) \to \hat{f}(T)$.

(2) Observe that the set of f such that $\hat{f}$ is Borel is closed under pointwise convergence of sequences on $\mathbb{C}$. (2) follows then from (1).

(3) By (1) $\hat{f}$ is continuous and it therefore suffices to show that

$$\hat{f}(S) \leqq \int_0^{2\pi} \hat{f}(S + e^{i\theta} T) \frac{d\theta}{2\pi},$$

for two finite rank operators S, T. Hence we can suppose S, T are actually $n \times n$ matrices for some n. Then the conclusion is immediate from Proposition 5.2 of [6]. □

We now introduce certain functionals of the above type. We define

$$v(T)=\sum_{|\lambda_n|\geqq 1} 1,\quad \mu(T)=\sum_{n=1}^{\infty}\log_+|\lambda_n|,\quad \text{and}\quad \chi(T)=\sum_{|\lambda_n|\geqq 1}\lambda_n .$$

By the above lemma v and χ are Borel functions while μ is continuous. If $|T|=(T^*T)^{1/2}$ the eigenvalues of $|T|$ correspond to the singular values of T. Notice that if T is normal then $v(T)=v(|T|)$.

Notice also that each of the functionals μ, v and χ are "disjointly additive" in the sense that $\mu(S\oplus T)=\mu(S)+\mu(T)$ etc. Here $S\oplus T$ represents the operator defined on $\mathscr{H}\oplus\mathscr{H}$ by $(S\oplus T)(x,y)=(Sx,Ty)$.

Lemma 2.2. *For any $T\in\mathscr{K}(\mathscr{H})$ we have $0\leqq\mu(T)\leqq\mu(|T|)$.*

Proof. Suppose $v(T)=n$. Then $\mu(T)=\log|\lambda_1\dots\lambda_n|\leqq\log|s_1\dots s_n|\leqq\mu(|T|)$ (see Gohberg-Krein p. 37). □

Lemma 2.3. *If $S,T\in\mathscr{K}(H)$ then $v(|S+T|)\leqq v(2|S|)+v(2|T|)$. In particular if $T=H+iK$ with H, K hermitian then $v(H)\leqq 2v(|T|)$.*

Proof. These follow easily from the Weyl inequalities, that

$$s_{m+n-1}(S+T)\leqq s_m(S)+s_n(T). \quad \square$$

Lemma 2.4. (1) *If T is a compact normal operator with $T=H+iK$ for H, K hermitian then $|\chi(H)-\Re\chi(T)|\leqq v(T)$.*

(2) *If T is any compact operator and $|\alpha|\leqq 1$ then $|\alpha\chi(T)-\chi(\alpha T)|\leqq v(T)$.*

(3) *If $(T_j)_{j=1}^n$ are compact normal operators with $T_1+\cdots+T_n=0$ then*

$$|\chi(T_1)+\cdots+\chi(T_n)|\leqq(n-1)\big(v(T_1)+\cdots+v(T_n)\big).$$

Proof. (1) We have

$$\chi(H)-\Re\chi(T)=\sum_{\Re\lambda_n<1\leqq|\lambda_n|}\Re\lambda_n .$$

The result follows immediately.

(2) For $|\alpha|=1$ this is trivial. If $|\alpha|<1$, we notice that

$$\alpha\chi(T)-\chi(\alpha T)=\alpha\sum_{1\leqq|\lambda_n|<\alpha^{-1}}\lambda_n$$

whence the result follows.

(3) (Compare [3], Lemma 2.2.) Since each T_j is normal there exist self-adjoint projections P_j of rank $\nu(T_j)$ so that $\chi(T_j) = \operatorname{tr}(P_j T P_j)$ and $\| P_j^\perp T_j P_j^\perp \| < 1$. Let Q be a self-adjoint projection of rank $d \leqq \sum_{j=1}^n \nu(T_j)$ whose range includes the range of each P_j. Then

$$\operatorname{tr}(Q T_j Q) = \operatorname{tr}(P_j T_j P_j) + \operatorname{tr}\big((Q - P_j) T_j (Q - P_j)\big)$$

and so

$$\begin{aligned} \left| \sum_{j=1}^n \chi(T_j) \right| &= \left| \sum_{j=1}^n \operatorname{tr}(Q - P_j) T_j (Q - P_j) \right| \\ &\leqq \sum_{j=1}^n \big(d - \nu(T_j)\big) \\ &\leqq (n-1) \sum_{j=1}^n \nu(T_j). \quad \square \end{aligned}$$

Since χ is not a continuous function on $\mathscr{K}(H)$ we will now correct it to make a continuous function. To this end we fix a nondecreasing C^∞-function $\varphi : \mathbb{R} \to \mathbb{R}$ such that $\varphi(x) = 0$ if $x \leqq 0$, and $\varphi(x) = 1$ if $x \geqq 1$. We define

$$\chi_\varphi(T) = \sum_{n=1}^\infty \lambda_n \varphi(|\log \lambda_n|)$$

where $\varphi(-\infty) = 0$. By Lemma 2.1, χ_ϕ is continuous.

Lemma 2.5. *For any compact operator T, we have $|\chi(T) - \chi_\varphi(T)| \leqq e\nu(T)$.*

Proof. $\chi(T) - \chi_\varphi(T) = \sum_{|\lambda_n(T)| \geqq 1} \big(1 - \varphi(\log|\lambda_n|)\big)\lambda_n$. Then the result follows immediately. $\square$

Now we define a second C^2-function $\psi : \mathbb{R} \to \mathbb{R}$ with the properties that $\psi(x) = 0$ if $x \leqq 0$ and $\psi''(x) = e^x\big(|\varphi''(x)| + 2|\varphi'(x)|\big)$ for $x \geqq 0$. ψ is an increasing convex function, which is linear for $x \geqq 1$. Thus there is a constant $C_1 > 0$ such that $\psi(x) \leqq C_1 \max(x, 0)$ for all x.

We now prove a crucial lemma.

Lemma 2.6. *Let $h : \mathbb{C} \to \mathbb{R}$ be defined by $h(0) = 0$ and $h(z) = \psi(\log|z|) - x\varphi(\log|z|)$ for $z \neq 0$. Then h is subharmonic.*

Proof. Note that h vanishes on a neighborhood of the origin. In fact h is C^2 so we check $\nabla^2 h$. It is easy to check that (for $z \neq 0$),

$$\nabla^2\big(\psi(\log|z|)\big) = |z|^{-2}\psi''(\log|z|) = |z|^{-1}\big(|\varphi''(\log|z|)| + 2|\varphi'(\log|z|)|\big).$$

We also have

$$\nabla^2\big(x\varphi(\log|z|)\big) = 2\frac{x}{|z|^2}\varphi'(\log|z|) + \frac{x}{|z|^2}\varphi''(\log|z|).$$

Hence $\nabla^2 h \geqq 0$. □

Theorem 2.7. *Suppose* $T \in \mathscr{K}(\mathscr{H})$. *Suppose* $T = H + iK$ *where* $H = \frac{1}{2}(T + T^*)$ *and* $K = \frac{1}{2i}(T - T^*)$. *Then there is a constant* C_2 *such that*

$$|\chi(H) - \Re\chi(T)| \leqq C_2\,\mu(2|T|)$$

and

$$|\chi(K) - \Im\chi(T)| \leqq C_2\,\mu(2|T|)\,.$$

Proof. For convenience we will define the function $F(z) = \frac{1}{2}(T + zT^*)$. For $0 \leqq \theta \leqq 2\pi$, we have:

$$F(e^{i\theta}) - \frac{1+e^{i\theta}}{2}H - i\,\frac{1-e^{i\theta}}{2}K = 0\,.$$

Note that each operator is normal and we also have from Lemma 2.3 that

$$v\big(|F(e^{i\theta})|\big) \leqq 2v(|T|) \quad \text{and} \quad v(H), v(K) \leqq 2v(|T|)\,.$$

Appealing to Lemma 2.4 (3),

$$\left|\chi\big(F(e^{i\theta})\big) - \chi\left(\frac{1+e^{i\theta}}{2}H\right) - \chi\left(i\frac{1-e^{i\theta}}{2}K\right)\right| \leqq 12v(|T|)\,.$$

On the other hand Lemma 2.4 (2) gives that

$$\left|\chi\left(\frac{1+e^{i\theta}}{2}H\right) - \frac{1+e^{i\theta}}{2}\chi(H)\right| \leqq v(H) \leqq 2v(|T|)$$

and

$$\left|\chi\left(i\frac{1-e^{i\theta}}{2}K\right) - i\,\frac{1-e^{i\theta}}{2}\chi(K)\right| \leqq 2v(|T|)\,.$$

Hence

$$\left|\chi\big(F(e^{i\theta})\big) - \frac{1+e^{i\theta}}{2}\chi(H) - i\,\frac{1-e^{i\theta}}{2}\chi(K)\right| \leqq 16v(|T|)\,.$$

Integrating over θ then gives

$$\left|\int_0^{2\pi} \chi\big(F(e^{i\theta})\big)\frac{d\theta}{2\pi} - \frac{1}{2}\big(\chi(H) + i\chi(K)\big)\right| \leqq 16v(|T|)\,.$$

Taking real parts we have, in particular,

$$\chi(H) \leqq 2\Re \int_0^{2\pi} \chi(F(e^{i\theta}))\frac{d\theta}{2\pi} + 32\nu(|T|).$$

We now replace χ by the smoother function χ_φ, and using Lemma 2.5 (since $e < 3$):

$$\chi(H) \leqq 2\int_0^{2\pi} \Re\chi_\varphi(F(e^{i\theta}))\frac{d\theta}{2\pi} + 44\nu(|T|).$$

Let $g(z) = \psi(\log|z|)$ for $z \neq 0$ and $g(0) = 0$. For any operator S we can write

$$\Re\chi_\varphi(S) = \hat{g}(S) - \hat{h}(S).$$

Note that $0 \leqq \hat{g}(S) \leqq C_1\mu(S)$. Thus

$$\chi(H) \leqq 2C_1\int_0^{2\pi} \mu(F(e^{i\theta}))\frac{d\theta}{2\pi} - 2\int_0^{2\pi} \hat{h}(F(e^{i\theta}))\frac{d\theta}{2\pi} + 44\nu(|T|).$$

Now since h is subharmonic, the functional $\hat{h}$ is plurisubharmonic by Lemma 2.1. Note that $\hat{h}(F(0)) = \hat{h}(T/2) = \hat{g}(T/2) - \Re\chi_\varphi(T/2)$. Hence, by 2.4 (2) and 2.5,

$$\begin{aligned} 2\int_0^{2\pi} \hat{h}(F(e^{i\theta}))\frac{d\theta}{2\pi} &\geqq 2\hat{g}(T/2) - 2\Re\chi_\varphi(T/2) \\ &\geqq -2\Re\chi(T/2) - 6\nu(T) \\ &\geqq -\Re\chi(T) - 8\nu(T). \end{aligned}$$

Hence

$$\chi(H) \leqq 2C_1\int_0^{2\pi} \mu(F(e^{i\theta}))\frac{d\theta}{2\pi} + \Re\chi(T) + 44\nu(|T|) + 8\nu(T).$$

Note that for every n we have $s_n(F(e^{i\theta})) \leqq s_{n/2}(T)$ so that $\mu(F(e^{i\theta})) \leqq 2\mu(|T|)$. We thus can simplify, using Lemma 2.2, to

$$\chi(H) \leqq \Re\chi(T) + 4C_1\mu(|T|) + 44\nu(|T|) + 8\nu(T).$$

Now observe that $\nu(T) \leqq (\log 2)^{-1}\mu(2T)$ so that for a suitable constant C_2 we have

$$\chi(H) \leqq \Re\chi(T) + C_2\mu(2|T|).$$

We now consider $-T$, iT and $-iT$ in place of T and the theorem follows. □

3. The main results

Now suppose that $\mathcal{J}$ is a two-sided ideal contained in $\mathcal{K}(H)$. We denote by Com $\mathcal{J}$ the linear subspace of $\mathcal{J}$ generated by all operators of the form $[S,T]=ST-TS$ for $T\in\mathcal{J}$ and $T\in\mathcal{B}(\mathcal{H})$. It is shown in [3] that if $\mathcal{J}_1$ and $\mathcal{J}_2$ are ideals such that $\mathcal{J}_1\mathcal{J}_2=\mathcal{J}$ then Com $\mathcal{J}$ coincides with the linear span of all $[S,T]$ where $S\in\mathcal{J}_1$ and $T\in\mathcal{J}_2$. It is clear that if $T=H+iK$ with H, K hermitian then $T\in\operatorname{Com}\mathcal{J}$ if and only if $H,K\in\operatorname{Com}\mathcal{J}$. One of the main results of [3] characterizes the hermitian operators in Com $\mathcal{J}$. We now state this result together with a useful rewording.

Theorem 3.1. *Suppose $\mathcal{J}$ is an ideal of compact operators on $\mathcal{H}$. Let N be a normal operator in $\mathcal{J}$, and let $\lambda_n=\lambda_n(N)$. Then the following conditions on N are equivalent*:

(1) $N\in\operatorname{Com}\mathcal{J}$.

(2) $\operatorname{diag}\left\{\frac{1}{n}(\lambda_1+\cdots+\lambda_n)\right\}\in\mathcal{J}$.

(3) *There exists $T\in\mathcal{J}$ so that $\frac{1}{n}|\lambda_1+\cdots+\lambda_n|\leqq s_n(T)$ for each $n\in\mathbb{N}$.*

(4) *There exists $T\in\mathcal{J}$ such that for all $\alpha>0$ we have $|\chi(\alpha N)|\leqq \nu(\alpha|T|)$.*

Proof. The equivalence of (1) and (2) for hermitian operators is proved in [3]. We will first establish the equivalence of (2), (3) and (4).

(3) clearly implies (2). We now check that (2) implies (3). For $m\geqq n$ we have, by an elementary barycentric calculation,

$$\frac{1}{m}|\lambda_1+\cdots+\lambda_m|\leqq\max\left(\frac{1}{n}|\lambda_1+\cdots+\lambda_n|,s_n(N)\right).$$

Let $T=\operatorname{diag}\{u_n\}$ where $u_n=\max\limits_{m\geqq n}\frac{1}{m}|\lambda_1+\cdots+\lambda_m|$. Then the above shows that $T\in\mathcal{J}$ and of course $\frac{1}{n}|\lambda_1+\cdots+\lambda_n|\leqq s_n(T)$.

Next we show (3) implies (4). We can assume that T in (3) satisfies $s_n(T)\geqq s_n(N)$ for every n. Then if $\alpha^{-1}>s_1(T)$ we have $\chi(\alpha N)=0$ and $\nu(\alpha|T|)=0$. Otherwise let n be the largest integer such that $s_n(T)\geqq\alpha^{-1}$. Then $\nu(\alpha|T|)=n$. Now suppose m is the largest integer so that $s_m(N)\geqq\alpha^{-1}$. Then $1\leqq m\leqq n$ and $\chi(\alpha N)=\alpha(\lambda_1+\cdots+\lambda_m)$.

Thus we have

$$\begin{aligned}|\chi(\alpha N)|&\leqq\alpha|\lambda_1+\cdots+\lambda_{n+1}|+n+1\\&\leqq(n+1)\alpha s_{n+1}(T)+n+1\\&\leqq 2(n+1)\leqq 4\nu(\alpha|T|).\end{aligned}$$

This yields (4) with T replaced by $T \oplus T \oplus T \oplus T$.

Now assume we have (4); we may assume T is positive. We again assume $s_n = s_n(T) \geqq s_n(N)$ for all n. Now for any $n \in \mathbb{N}$ we have:

$$\frac{1}{n}|\lambda_1 + \cdots + \lambda_n| \leqq s_n + \frac{1}{n}\left| \sum_{|\lambda_k| > s_n} \lambda_k \right|.$$

Suppose $\sigma > s_n$ is smaller than any $|\lambda_k| > s_n$. Then

$$\frac{1}{n}|\lambda_1 + \cdots + \lambda_n| \leqq s_n + \frac{\sigma}{n}|\chi(\sigma^{-1}N)|.$$

However $|\chi(\sigma^{-1}N)| \leqq v(\sigma^{-1}T) < n$, so that

$$\frac{1}{n}|\lambda_1 + \cdots + \lambda_n| \leqq s_n + \sigma.$$

Letting σ tend to s_n yields

$$\frac{1}{n}|\lambda_1 + \cdots + \lambda_n| \leqq 2s_n$$

so that (3) holds if T is replaced by $2T$.

Finally if $N = H + iK$ where H, K are hermitian then we have by Lemma 2.4 (1) that $|\chi(\alpha H) - \Re\chi(\alpha N)|, |\chi(\alpha K) - \Im\chi(\alpha N)| \leqq v(\alpha N)$. Hence N satisfies (4) if and only if both H and K satisfy (4). As remarked above, the results of [3] imply that for hermitian operators (1) and (2) and hence also (1) and (4) are equivalent. Thus (1) and (4) are also equivalent for normal operators. □

Now let us introduce a stability condition on the ideal $\mathcal{J}$. We will say that $\mathcal{J}$ is *geometrically stable* if whenever $\operatorname{diag}(s_1, s_2, \ldots) \in \mathcal{J}$ with $s_1 \geqq s_2 \geqq \cdots$ then $\operatorname{diag}(t_1, t_2, \ldots) \in \mathcal{J}$ where $t_n = (s_1 \ldots s_n)^{1/n}$.

We say that $\mathcal{J}$ of compact operators is a quasi-Banach ideal (or Schatten ideal) if it can be equipped with a complete quasi-norm $T \to \|T\|_{\mathcal{J}}$ so that we have the ideal property $\|ATB\|_{\mathcal{J}} \leqq \|A\|_\infty \|T\|_{\mathcal{J}} \|B\|_\infty$ whenever $A, B \in \mathcal{B}(\mathcal{H})$. Here we denote the operator norm of A by $\|A\|_\infty$.

Proposition 3.2. *If $\mathcal{J}$ is a quasi-Banach ideal then $\mathcal{J}$ is geometrically stable.*

Proof. We can assume for some $0 < r \leqq 1$ that $\|\cdot\|_{\mathcal{J}}$ is an r-norm i.e.

$$\|S + T\|_{\mathcal{J}}^r \leqq \|S\|_{\mathcal{J}}^r + \|T\|_{\mathcal{J}}^r.$$

Suppose $D = \operatorname{diag}(s_n) \in \mathscr{J}$, where $s_1 \geqq s_2 \geqq \cdots$. We recall our convention that $s_r = s_{[r]+1}$ where $r > 0$ is not an integer. Then for each $k \in \mathbb{N}$ we have $\|\operatorname{diag}(s_{n/2^k})\|_{\mathscr{J}} \leqq 2^{k/r}\|D\|_{\mathscr{J}}$. Pick $\theta > 1/r$. Then by completeness the series $\sum_{k=0}^{\infty} 2^{-\theta k} \operatorname{diag}(s_{n/2^k})$ converges in $\mathscr{J}$. Thus $\operatorname{diag}(u_n) \in \mathscr{J}$ where $u_n = \sum_{k=0}^{\infty} 2^{-\theta k} s_{n/2^k}$. In fact $\|\operatorname{diag}(u_n)\|_{\mathscr{J}} \leqq \left(\sum_{k=0}^{\infty} 2^{k(1-r\theta)}\right)^{1/r} \|D\|_{\mathscr{J}}$. Now suppose $1 \leqq j \leqq n$. Pick $k \in \mathbb{N}$ so that $2^{-k}n \leqq j \leqq 2.2^{-k}n$. Then

$$s_j \leqq s_{n/2^k} \leqq 2^{k\theta} u_n \leqq \left(\frac{2n}{j}\right)^{\theta} u_n .$$

Hence

$$t_n \leqq 2^{\theta} n^{\theta} (n!)^{-\theta/n} u_n \leqq C u_n$$

for some constant C. This implies $\operatorname{diag}(t_n) \in \mathscr{J}$ and further that $\|\operatorname{diag}(t_n)\|_{\mathscr{J}} \leqq C\|D\|_{\mathscr{J}}$ for some constant C depending only on r. □

We now prove the main result of this note, which, for the special case of geometrically stable ideals, answers positively a question posed in [3]. It should be noted that in [4] it is shown that for singly generated ideals geometric stability is a necessary and sufficient condition for the equivalence of (1) and (2) in Theorem 3.3. Thus in general (1) and (2) are not equivalent.

Theorem 3.3. *Suppose $\mathscr{J}$ is a geometrically stable ideal of compact operators on $\mathscr{H}$ (in particular this hold if $\mathscr{J}$ is a quasi-Banach ideal). Let $S \in \mathscr{J}$ and let $\lambda_n = \lambda_n(S)$. Then the following conditions on S are equivalent*:

(1) $S \in \operatorname{Com} \mathscr{J}$.

(2) $\operatorname{diag}\left\{\frac{1}{n}(\lambda_1 + \cdots + \lambda_n)\right\} \in \mathscr{J}$.

(3) *There exists $T \in \mathscr{J}$ so that $\frac{1}{n}|\lambda_1 + \cdots + \lambda_n| \leqq s_n(T)$ for each $n \in \mathbb{N}$.*

(4) *There exists $T \in \mathscr{J}$ such that for all $\alpha > 0$ we have $|\chi(\alpha S)| \leqq \nu(\alpha |T|)$.*

(5) *There exists $T \in \mathscr{J}$ such that for all $\alpha > 0$ we have $|\chi(\alpha S)| \leqq \mu(\alpha |T|)$.*

Proof. We first show that $N = \operatorname{diag}(\lambda_n) \in \mathscr{J}$. Indeed we have $|\lambda_1 \ldots \lambda_n| \leqq s_1 \ldots s_n$ where $s_n = s_n(S)$. Thus $|\lambda_n| \leqq t_n = (s_1 \ldots s_n)^{1/n}$ so that $N \in \mathscr{J}$ by geometric stability. Hence (2), (3) and (4) are equivalent by Theorem 3.1.

It is clear that (4) implies (5) (replace T by eT). Let us prove that (5) implies (3). We can suppose that $s_n = s_n(T) \geqq s_n(S)$ for all n, and let $t_n = (s_1 \ldots s_n)^{1/n}$. Then

$$\begin{aligned} |\lambda_1 + \cdots + \lambda_n| &\leqq \Big|\sum_{|\lambda_k| \geqq s_n} \lambda_k\Big| + n s_n \\ &\leqq s_n |\chi(s_n^{-1} S)| + n s_n \end{aligned}$$

$$\leqq s_n \mu(s_n^{-1}|T|) + ns_n$$

$$\leqq s_n \sum_{k=1}^{n} \log(s_k/s_n) + ns_n$$

$$\leqq ns_n \log(t_n/s_n) + ns_n .$$

Hence since $\log x \leqq x$ for all $x \geqq 1$,

$$\frac{|\lambda_1 + \cdots + \lambda_n|}{n} \leqq t_n + s_n$$

and by the geometric stability of the ideal we have (3).

To conclude the proof we establish equivalence of (1) with (5). To this end note that if $S = H + iK$ with H, K hermitian then by Theorem 2.7, there is a constant C_2 so that, for $\alpha > 0$,

$$|\Re\chi(\alpha S) - \chi(\alpha H)|, |\Im\chi(\alpha S) - \chi(\alpha K)| \leqq C_2 \mu(2\alpha|S|) .$$

Now suppose first that H, K both satisfy (5) so that there are operators $T_1, T_2 \in \mathscr{J}$ with $|\chi(\alpha H)| \leqq \mu(\alpha|T_1|)$ and $|\chi(\alpha K)| \leqq \mu(\alpha|T_2|)$ for $\alpha > 0$. Pick an integer $n > 2C_2$ and consider the operator $W = T_1 \oplus T_2 \oplus V$ where V is the direct sum of n copies of $2|S|$. Then $|\chi(\alpha S)| \leqq \mu(\alpha|W|)$ for all $\alpha > 0$. Conversely if S satisfies (5) for an appropriate operator T then H and K satisfy (5) for T replaced by $T \oplus V$ it follows S satisfies (5) if and only if both H and K satisfy (5). Now if $S \in \mathrm{Com}\,\mathscr{J}$ then H, $K \in \mathrm{Com}\,\mathscr{J}$ so that by Theorem 3.1 H, K satisfy (2)–(4) and hence also (5). Therefore (1) implies (5).

Conversely if (5) holds for S, then both H, K satisfy (5) and hence also (2)–(4); so by Theorem 3.1, $H, K \in \mathrm{Com}\,\mathscr{J}$ and hence $S \in \mathrm{Com}\,\mathscr{J}$ i.e. (5) implies (1). □

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