Spectral characterization of sums of commutators I

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Abstract. Suppose $\mathcal{J}$ is a two-sided quasi-Banach ideal of compact operators an a separable infinite-dimensional Hilbert space $\mathcal{H}$. We show that an operator $T \in \mathcal{J}$ can be expressed as finite linear combination of commutators $[A, B]$ where $A \in \mathcal{J}$ and $B \in \mathcal{B}(\mathcal{H})$ if and only if its eigenvalues $(\lambda_n)$ (arranged in decreasing order of absolute value, repeated according to algebraic multiplicity and augmented by zeros if necessary) satisfy the condition that the diagonal operator $\text{diag} \left\{ \frac{1}{n} (\lambda_1 + \cdots + \lambda_n) \right\}$ is a member of $\mathcal{J}$. This answers (for quasi-Banach ideals) a question raised by Dykema, Figiel, Weiss and Wodzicki.

1. Introduction

Let $\mathcal{H}$ be a separable infinite-dimensional Hilbert space, and let $\mathcal{J}$ be a (two-sided) ideal contained in the ideal of compact operators $\mathcal{K}(\mathcal{H})$ on $\mathcal{H}$. We define the commutator subspace $\text{Com} \mathcal{J}$ to be the closed linear span of commutators $[A, B] = AB - BA$ where $A \in \mathcal{J}$ and $B \in \mathcal{B}(\mathcal{H})$. It has been shown by Dykema, Figiel, Weiss and Wodzicki in [3] that if $\mathcal{J}_1$ and $\mathcal{J}_2$ are any two ideals then the linear span of commutators of the form $[A_1, A_2]$ where $A_j \in \mathcal{J}_j$ for $j = 1, 2$ coincides with the commutator subspace $\text{Com} \mathcal{J}$ where $\mathcal{J} = \mathcal{J}_1 \mathcal{J}_2$.

Peacry and Topping ([7], cf. [2]) showed that for the Schatten ideal $\mathcal{J} = \mathcal{C}_p$ when $p > 1$, we have $\text{Com} \mathcal{C}_p = \mathcal{C}_p$. They then raised the question whether

$$\text{Com} \mathcal{C}_1 = \{ T \in \mathcal{C}_1 : \text{tr} T = 0 \}.$$

This question was resolved negatively by Weiss [8], [9]. However, Anderson [1] showed that in the case $p < 1$ we have $\text{Com} \mathcal{C}_p = \{ T \in \mathcal{C}_p : \text{tr} T = 0 \}$.

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In [6] a complete characterization of Com $\mathcal{C}_1$ was obtained. It was shown that, in the case when $\mathcal{J} = \mathcal{C}_1$ is the trace-class, then $T \in \text{Com } \mathcal{C}_1$ if and only if $T \in \mathcal{C}_1$ and its eigenvalues $(\lambda_n(T))_{n=1}^\infty$, counted according to algebraic multiplicity and arranged in some order satisfying that $(|\lambda_n(T)|)_{n=1}^\infty$ is decreasing, satisfies the inequality

\[
\sum_{n=1}^\infty \frac{|\lambda_1 + \cdots + \lambda_n|}{n} < \infty.
\]

If the eigenvalue set of $T$ is finite one may extend the sequence $\lambda_n(T)$ by including infinitely many zeroes. This extended earlier partial results in [8] and [9].

Since, for any ideal $\mathcal{J}$, Com $\mathcal{J}$ is a self-adjoint subspace it is clear that if $T = H + iK$ is split into hermitian and skew-hermitian parts then $T \in \text{Com } \mathcal{J}$ if and only if $H \in \text{Com } \mathcal{J}$ and $K \in \text{Com } \mathcal{J}$. Thus to characterize Com $\mathcal{J}$ it is necessary only to characterize the hermitian operators in Com $\mathcal{J}$. In particular, the result above shows that if $T \in \mathcal{C}_1$ the condition (1.1) is equivalent to the pair of conditions that $H$ and $K$ each satisfy (1.1).

Recently in [3] a very general approach was developed which is applicable to any ideal. It was shown that for any ideal $\mathcal{J}$ a hermitian operator $H \in \text{Com } \mathcal{J}$ if and only if $H \in \mathcal{J}$ and the diagonal operator $\text{diag} \left\{ \frac{1}{n} (\lambda_1 + \cdots + \lambda_n) \right\}$ belongs to $\mathcal{J}$ where again $\lambda_n = \lambda_n(H)$ is the eigenvalue sequence as above. Although this yields an explicit test for membership in Com $\mathcal{J}$ by the process of splitting into hermitian and skew-hermitian parts, it leaves open the question whether same characterization in terms of eigenvalues extends to all operators in Com $\mathcal{J}$, as in the case of the trace-class.

The aim of this paper is to show that for a fairly broad class of “nice” ideals the answer to this question is positive. The condition we impose on an ideal $\mathcal{J}$ is that it is geometrically stable. This means that if a diagonal operator $\text{diag} \{ s_1, s_2, \ldots \} \in \mathcal{J}$ where $s_1 \geq s_2 \geq \cdots \geq 0$ then we have $\text{diag} \{ u_1, u_2, \ldots \} \in \mathcal{J}$ where $u_n = (s_1 \cdots s_n)^{1/n}$. For any Banach or quasi-Banach ideal (i.e. an ideal equipped with an appropriate ideal quasi-norm) this condition is automatic. Under the hypothesis that $\mathcal{J}$ is geometrically stable we show that $T \in \text{Com } \mathcal{J}$ if and only if $\text{diag} \left\{ \frac{1}{n} (\lambda_1 + \cdots + \lambda_n) \right\} \in \mathcal{J}$ where $\lambda_n = \lambda_n(T)$.

In a separate note, in collaboration with Ken Dykema [4], we show that then for arbitrary ideals $\mathcal{J}$ this result is false and indeed there is no spectral characterization of the subspace Com $\mathcal{J}$.

Let us note that our results do depend in an essential way on the results of [3], in that we use their result to reduce the problem to discussion of an operator of the form $T = H + iK$ where $H$, $K$ are hermitian.

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2. The key results

Let $T$ be a bounded operator on a separable Hilbert space $\mathcal{H}$. We denote by $s_n = s_n(T)$, for $n \geq 1$ the singular values of $T$. It will be convenient to define $s_n$ for $n$ not an integer by $s_n = s_{[n]+1}$. With this notation we have the inequalities $s_n(S + T) \leq s_{n/2}(S) + s_{n/2}(T)$ and $s_n(ST) \leq s_{n/2}(S)s_{n/2}(T)$.

If $T$ is compact we denote by $\lambda_n = \lambda_n(T)$ the eigenvalues of $T$ repeated according to algebraic multiplicity and arranged in decreasing order of absolute value (this arrangement is not unique, so we require some selection to be made). Note that $s_n(T) = \lambda_n(|T|)$.

Let us suppose that $f: \mathbb{C} \to \mathbb{C}$ is any function which vanishes on a neighborhood of the origin. Then we can define a functional $\hat{f}: \mathcal{H}(\mathcal{H}) \to \mathbb{C}$ by the formula:

$$\hat{f}(T) = \sum_{n=1}^{\infty} f(\lambda_n).$$

Lemma 2.1. (1) If $f$ is continuous then $\hat{f}$ is continuous.

(2) If $f$ is a Borel function then $\hat{f}$ is a Borel function.

(3) If $f$ is continuous, real-valued and subharmonic then $\hat{f}$ is plurisubharmonic on $\mathcal{H}(\mathcal{H})$, i.e. if $S, T \in \mathcal{H}(\mathcal{H})$ then

$$\hat{f}(S) \leq \int_0^{2\pi} \hat{f}(S + e^{i\theta}T) \frac{d\theta}{2\pi}.$$

Proof. (1) Suppose $f(z)$ vanishes for $|z| \leq \delta$ where $\delta > 0$. If $T \in \mathcal{H}(H)$ then suppose $m$ is the least integer such that $|\lambda_m(T)| < \delta$. Pick $\eta < \delta$ such that $|\lambda_m(T)| < \eta$ and if $m \geq 2$ then $\eta < |\lambda_{m-1}(T)|$. Now if $T_n$ is a sequence of compact operators with $\lim_{n \to \infty} \|T_n - T\| = 0$ then by results in [5] (see p.14, 18) we can find $n_0$ so that an ordering $(\lambda_k(T_n))_{k=1}^\infty$ of the eigenvalues of $T_n$ such that $|\lambda_k(T_n)| < \eta$ for $n \geq n_0$ and $k \geq m$ and $\lim_{n \to \infty} \lambda_k(T_n) = \lambda_k(T)$ if $k < m$. If follows easily that $\hat{f}(T_n) \to \hat{f}(T)$.

(2) Observe that the set of $f$ such that $\hat{f}$ is Borel is closed under pointwise convergence of sequences on $\mathbb{C}$. (2) follows then from (1).

(3) By (1) $\hat{f}$ is continuous and it therefore suffices to show that

$$\hat{f}(S) \leq \int_0^{2\pi} \hat{f}(S + e^{i\theta}T) \frac{d\theta}{2\pi},$$

for two finite rank operators $S, T$. Hence we can suppose $S, T$ are actually $n \times n$ matrices for some $n$. Then the conclusion is immediate from Proposition 5.2 of [6]. □
We now introduce certain functionals of the above type. We define

\[ v(T) = \sum_{|\lambda_n| \leq 1} 1, \quad \mu(T) = \sum_{n=1}^{\infty} \log_+ |\lambda_n|, \quad \text{and} \quad \chi(T) = \sum_{|\lambda_n| \leq 1} \lambda_n. \]

By the above lemma \( v \) and \( \chi \) are Borel functions while \( \mu \) is continuous. If \( |T| = (T^*T)^{1/2} \) the eigenvalues of \( |T| \) correspond to the singular values of \( T \). Notice that if \( T \) is normal then \( v(T) = v(|T|) \).

Notice also that each of the functionals \( \mu, v \) and \( \chi \) are “disjointly additive” in the sense that \( \mu(S \oplus T) = \mu(S) + \mu(T) \) etc. Here \( S \oplus T \) represents the operator defined on \( \mathcal{H} \oplus \mathcal{H} \) by \( (S \oplus T)(x, y) = (Sx, Ty) \).

**Lemma 2.2.** For any \( T \in \mathcal{B}(\mathcal{H}) \) we have \( 0 \leq \mu(T) \leq \mu(|T|) \).

**Proof.** Suppose \( v(T) = n \). Then \( \mu(T) = \log |\lambda_1 \cdots \lambda_n| \leq \log |s_1 \cdots s_n| \leq \mu(|T|) \) (see Gohberg-Krein p. 37). \( \square \)

**Lemma 2.3.** If \( S, T \in \mathcal{B}(H) \) then \( v(|S + T|) \leq v(2|S|) + v(2|T|) \). In particular if \( T = H + iK \) with \( H, K \) hermitian then \( v(H) \leq 2v(|T|) \).

**Proof.** These follow easily from the Weyl inequalities, that

\[ s_m + s_{m-1}(S + T) \leq s_m(S) + s_n(T). \] \( \square \)

**Lemma 2.4.**

1. If \( T \) is a compact normal operator with \( T = H + iK \) for \( H, K \) hermitian then \( |\chi(H) - \Re \chi(T)| \leq v(T) \).

2. If \( T \) is any compact operator and \( |z| \leq 1 \) then \( |z\chi(T) - \chi(zT)| \leq v(T) \).

3. If \( (T_1)_{n=1}^n \) are compact normal operators with \( T_1 + \cdots + T_n = 0 \) then

\[ |\chi(T_1) + \cdots + \chi(T_n)| \leq (n-1)(v(T_1) + \cdots + v(T_n)). \]

**Proof.**

1. We have

\[ \chi(H) - \Re \chi(T) = \sum_{\Re \lambda_n \leq 1 \leq |\lambda_n|} \Re \lambda_n. \]

The result follows immediately.

2. For \( |z| = 1 \) this is trivial. If \( |z| < 1 \), we notice that

\[ z\chi(T) - \chi(zT) = z \sum_{1 \leq |\lambda_n| < z^{-1}} \lambda_n \]

whence the result follows.
(3) (Compare [3], Lemma 2.2.) Since each $T_j$ is normal there exist self-adjoint projections $P_j$ of rank $v(T_j)$ so that $\chi(T_j) = \text{tr}(P_jT_jP_j)$ and $\|P_j^\perp T_j P_j^\perp\| < 1$. Let $Q$ be a self-adjoint projection of rank $d \leq \sum_{j=1}^n v(T_j)$ whose range includes the range of each $P_j$. Then

$$\text{tr}(QT_jQ) = \text{tr}(P_j T_j P_j) + \text{tr}((Q - P_j) T_j (Q - P_j))$$

and so

$$\left| \sum_{j=1}^n \chi(T_j) \right| = \left| \sum_{j=1}^n \text{tr}(Q - P_j) T_j (Q - P_j) \right|$$

$$\leq \sum_{j=1}^n (d - v(T_j))$$

$$\leq (n - 1) \sum_{j=1}^n v(T_j). \quad \square$$

Since $\chi$ is not a continuous function on $\mathcal{N}(H)$ we will now correct it to make a continuous function. To this end we fix a nondecreasing $C^\infty$-function $\varphi : \mathbb{R} \to \mathbb{R}$ such that $\varphi(x) = 0$ if $x \leq 0$, and $\varphi(x) = 1$ if $x \geq 1$. We define

$$\chi_\varphi(T) = \sum_{n=1}^\infty \lambda_n \varphi(|\log \lambda_n|)$$

where $\varphi(-\infty) = 0$. By Lemma 2.1, $\chi_\varphi$ is continuous.

**Lemma 2.5.** For any compact operator $T$, we have $|\chi(T) - \chi_\varphi(T)| \leq e \varphi(T)$.

**Proof.** $\chi(T) - \chi_\varphi(T) = \sum_{|\lambda_n(T)| \geq 1} (1 - \varphi(\log|\lambda_n|)) \lambda_n$. Then the result follows immediately. $\square$

Now we define a second $C^2$-function $\psi : \mathbb{R} \to \mathbb{R}$ with the properties that $\psi(x) = 0$ if $x \leq 0$ and $\psi''(x) = e^x(2|\varphi''(x)| + 2|\varphi'(x)|)$ for $x \geq 0$. $\psi$ is an increasing convex function, which is linear for $x \geq 1$. Thus there is a constant $C_1 > 0$ such that $\psi(x) \leq C_1 \max(x, 0)$ for all $x$.

We now prove a crucial lemma.

**Lemma 2.6.** Let $h : \mathbb{C} \to \mathbb{R}$ be defined by $h(0) = 0$ and $h(z) = \psi(|\log|z||) - x \varphi(\log|z|)$ for $z \neq 0$. Then $h$ is subharmonic.

**Proof.** Note that $h$ vanishes on a neighborhood of the origin. In fact $h$ is $C^2$ so we check $\nabla^2 h$. It is easy to check that (for $z \neq 0$),

$$\nabla^2(\psi(|\log|z||)) = |z|^{-2} \psi''(\log|z|) = |z|^{-1} (|\varphi''(\log|z||) + 2|\varphi'(\log|z||))$$

We also have

$$\nabla^2(x \varphi(\log|z||)) = 2 \frac{x}{|z|^2} \varphi'(\log|z|) + \frac{x}{|z|^2} \varphi''(\log|z|).$$
Hence $\nabla^2 h \geq 0$. \hfill \square

**Theorem 2.7.** Suppose $T \in \mathcal{H}$ $(\mathcal{H})$. Suppose $T = H + iK$ where $H = \frac{1}{2}(T + T^*)$ and $K = \frac{1}{2i}(T - T^*)$. Then there is a constant $C_2$ such that

$$|\chi(H) - \Re \chi(T)| \leq C_2 \mu(2|T|)$$

and

$$|\chi(K) - \Im \chi(T)| \leq C_2 \mu(2|T|).$$

**Proof.** For convenience we will define the function $F(z) = \frac{1}{2}(T + zT^*)$. For $0 \leq \theta \leq 2\pi$, we have:

$$F(e^{i\theta}) - \frac{1 + e^{i\theta}}{2} H - i \frac{1 - e^{i\theta}}{2} K = 0.$$

Note that each operator is normal and we also have from Lemma 2.3 that

$$v(|F(e^{i\theta})|) \leq 2v(|T|) \quad \text{and} \quad v(H), v(K) \leq 2v(|T|).$$

Appealing to Lemma 2.4 (3),

$$\left| \chi(F(e^{i\theta})) - \chi \left( \frac{1 + e^{i\theta}}{2} H \right) - \chi \left( i \frac{1 - e^{i\theta}}{2} K \right) \right| \leq 12v(|T|).$$

On the other hand Lemma 2.4 (2) gives that

$$\left| \chi \left( \frac{1 + e^{i\theta}}{2} H \right) - \frac{1 + e^{i\theta}}{2} \chi(H) \right| \leq v(H) \leq 2v(|T|)$$

and

$$\left| \chi \left( i \frac{1 - e^{i\theta}}{2} K \right) - i \frac{1 - e^{i\theta}}{2} \chi(K) \right| \leq 2v(|T|).$$

Hence

$$\left| \chi(F(e^{i\theta})) - \frac{1 + e^{i\theta}}{2} \chi(H) - i \frac{1 - e^{i\theta}}{2} \chi(K) \right| \leq 16v(|T|).$$

Integrating over $\theta$ then gives

$$\left| \int_0^{2\pi} \chi(F(e^{i\theta})) d\theta - \frac{1}{2} \left( \chi(H) + i\chi(K) \right) \right| \leq 16v(|T|).$$
Taking real parts we have, in particular,
\[
\chi(H) \leq 2 \mathfrak{R} \int_0^{2\pi} \mu(F(e^{i\theta})) \frac{d\theta}{2\pi} + 32v(|T|).
\]

We now replace \( \chi \) by the smoother function \( \chi^\phi \), and using Lemma 2.5 (since \( e < 3 \)),
\[
\chi(H) \leq 2 \int_0^{2\pi} \mathfrak{R}_{\chi^\phi}(F(e^{i\theta})) \frac{d\theta}{2\pi} + 44v(|T|).
\]

Let \( g(z) = \psi(\log |z|) \) for \( z \neq 0 \) and \( g(0) = 0 \). For any operator \( S \) we can write
\[
\mathfrak{R}_{\chi^\phi}(S) = \hat{g}(S) - \hat{h}(S).
\]

Note that \( 0 \leq \hat{g}(S) \leq C_1 \mu(S) \). Thus
\[
\chi(H) \leq 2C_1 \int_0^{2\pi} \mu(F(e^{i\theta})) \frac{d\theta}{2\pi} - 2 \int_0^{2\pi} \hat{h}(F(e^{i\theta})) \frac{d\theta}{2\pi} + 44v(|T|).
\]

Now since \( h \) is subharmonic, the functional \( \hat{h} \) is plurisubharmonic by Lemma 2.1. Note that \( \hat{h}(F(0)) = \hat{h}(T/2) = \hat{g}(T/2) - \mathfrak{R}_{\chi^\phi}(T/2) \). Hence, by 2.4 (2) and 2.5,
\[
2 \int_0^{2\pi} \hat{h}(F(e^{i\theta})) \frac{d\theta}{2\pi} \geq 2 \hat{g}(T/2) - 2\mathfrak{R}_{\chi^\phi}(T/2)
\]
\[
\geq -2\mathfrak{R}_{\chi}(T/2) - 6v(T)
\]
\[
\geq -\mathfrak{R}_{\chi}(T) - 8v(T).
\]
Hence
\[
\chi(H) \leq 2C_1 \int_0^{2\pi} \mu(F(e^{i\theta})) \frac{d\theta}{2\pi} + \mathfrak{R}_{\chi}(T) + 44v(|T|) + 8v(T).
\]

Note that for every \( n \) we have \( s_n(F(e^{i\theta})) \leq s_{n/2}(T) \) so that \( \mu(F(e^{i\theta})) \leq 2\mu(|T|) \). We thus can simplify, using Lemma 2.2, to
\[
\chi(H) \leq \mathfrak{R}_{\chi}(T) + 4C_1 \mu(|T|) + 44v(|T|) + 8v(T).
\]

Now observe that \( v(T) \leq (\log 2)^{-1} \mu(2T) \) so that for a suitable constant \( C_2 \) we have
\[
\chi(H) \leq \mathfrak{R}_{\chi}(T) + C_2 \mu(2T).
\]

We now consider \(-T, iT\) and \(-iT\) in place of \( T \) and the theorem follows. □
3. The main results

Now suppose that \( \mathcal{J} \) is a two-sided ideal contained in \( \mathcal{B}(H) \). We denote by \( \text{Com} \mathcal{J} \) the linear subspace of \( \mathcal{J} \) generated by all operators of the form \( [S, T] = ST - TS \) for \( T \in \mathcal{J} \) and \( T \in \mathcal{B}(\mathcal{H}) \). It is shown in [3] that if \( \mathcal{J}_1 \) and \( \mathcal{J}_2 \) are ideals such that \( \mathcal{J}_1 \mathcal{J}_2 = \mathcal{J} \) then \( \text{Com} \mathcal{J} \) coincides with the linear span of all \( [S, T] \) where \( S \in \mathcal{J}_1 \) and \( T \in \mathcal{J}_2 \). It is clear that if \( T = H + iK \) with \( H, K \) hermitian then \( T \in \text{Com} \mathcal{J} \) if and only if \( H, K \in \text{Com} \mathcal{J} \). One of the main results of [3] characterizes the hermitian operators in \( \text{Com} \mathcal{J} \). We now state this result together with a useful rewording.

**Theorem 3.1.** Suppose \( \mathcal{J} \) is an ideal of compact operators on \( \mathcal{H} \). Let \( N \) be a normal operator in \( \mathcal{J} \), and let \( \lambda_n = \lambda_n(N) \). Then the following conditions on \( N \) are equivalent:

1. \( N \in \text{Com} \mathcal{J} \).
2. \( \text{diag} \left\{ \frac{1}{n} (\lambda_1 + \cdots + \lambda_n) \right\} \in \mathcal{J} \).
3. There exists \( T \in \mathcal{J} \) so that \( \frac{1}{n} |\lambda_1 + \cdots + \lambda_n| \leq s_n(T) \) for each \( n \in \mathbb{N} \).
4. There exists \( T \in \mathcal{J} \) such that for all \( \alpha > 0 \) we have \( |\chi(\alpha N)| \leq \nu(\alpha |T|) \).

**Proof.** The equivalence of (1) and (2) for hermitian operators is proved in [3]. We will first establish the equivalence of (2), (3) and (4).

(3) clearly implies (2). We now check that (2) implies (3). For \( m \geq n \) we have, by an elementary barycentric calculation,

\[
\frac{1}{m} |\lambda_1 + \cdots + \lambda_m| \leq \max \left( \frac{1}{n} |\lambda_1 + \cdots + \lambda_n|, s_n(N) \right).
\]

Let \( T = \text{diag} \{ u_n \} \) where \( u_n = \max_{m \geq n} \frac{1}{m} |\lambda_1 + \cdots + \lambda_m| \). Then the above shows that \( T \in \mathcal{J} \) and of course \( \frac{1}{n} |\lambda_1 + \cdots + \lambda_n| \leq s_n(T) \).

Next we show (3) implies (4). We can assume that \( T \) in (3) satisfies \( s_n(T) \gtrsim s_n(N) \) for every \( n \). Then if \( \alpha^{-1} > s_1(T) \) we have \( \chi(\alpha N) = 0 \) and \( \nu(\alpha |T|) = 0 \). Otherwise let \( n \) be the largest integer such that \( s_n(T) \gtrsim \alpha^{-1} \). Then \( \nu(\alpha |T|) = n \). Now suppose \( m \) is the largest integer so that \( s_m(N) \gtrsim \alpha^{-1} \). Then \( 1 \leq m \leq n \) and \( \chi(\alpha N) = \alpha (\lambda_1 + \cdots + \lambda_m) \).

Thus we have

\[
|\chi(\alpha N)| \leq \alpha |\lambda_1 + \cdots + \lambda_{n+1}| + n + 1 \\
\leq (n + 1) \alpha s_{n+1}(T) + n + 1 \\
\leq 2(n + 1) \nu(\alpha |T|) .
\]
This yields (4) with $T$ replaced by $T \oplus T \oplus T \oplus T$.

Now assume we have (4); we may assume $T$ is positive. We again assume $s_n = s_n(T) \geq s_n(N)$ for all $n$. Now for any $n \in \mathbb{N}$ we have:

$$\frac{1}{n} |\lambda_1 + \cdots + \lambda_n| \leq s_n + \frac{1}{n} \left| \sum_{|\lambda_k| > s_n} \lambda_k \right| .$$

Suppose $\sigma > s_n$ is smaller than any $|\lambda_k| > s_n$. Then

$$\frac{1}{n} |\lambda_1 + \cdots + \lambda_n| \leq s_n + \frac{\sigma}{n} \chi(\sigma^{-1} N) .$$

However $|\chi(\sigma^{-1} N)| \leq \nu(\sigma^{-1} T) < n$, so that

$$\frac{1}{n} |\lambda_1 + \cdots + \lambda_n| \leq s_n + \sigma .$$

Letting $\sigma$ tend to $s_n$ yields

$$\frac{1}{n} |\lambda_1 + \cdots + \lambda_n| \leq 2s_n$$

so that (3) holds if $T$ is replaced by $2T$.

Finally if $N = H + iK$ where $H, K$ are hermitian then we have by Lemma 2.4 (1) that $|\chi(zH) - \Re\chi(zN)|, |\chi(zK) - \Im\chi(zN)| \leq \nu(zN)$. Hence $N$ satisfies (4) if and only if both $H$ and $K$ satisfy (4). As remarked above, the results of [3] imply that for hermitian operators (1) and (2) and hence also (1) and (4) are equivalent. Thus (1) and (4) are also equivalent for normal operators. □

Now let us introduce a stability condition on the ideal $\mathcal{J}$. We will say that $\mathcal{J}$ is geometrically stable if whenever $\text{diag}(s_1, s_2, \ldots) \in \mathcal{J}$ with $s_1 \geq s_2 \geq \cdots$ then $\text{diag}(t_1, t_2, \ldots) \in \mathcal{J}$ where $t_n = (s_1 \ldots s_n)^{1/n}$.

We say that $\mathcal{J}$ of compact operators is a quasi-Banach ideal (or Schatten ideal) if it can be equipped with a complete quasi-norm $T \rightarrow || T ||_\mathcal{J}$ so that we have the ideal property $|| ATB ||_\mathcal{J} \leq || A ||_\infty || T ||_\mathcal{J} || B ||_\infty$ whenever $A, B \in \mathcal{B}(\mathcal{H})$. Here we denote the operator norm of $A$ by $|| A ||_\infty$.

**Proposition 3.2.** If $\mathcal{J}$ is a quasi-Banach ideal then $\mathcal{J}$ is geometrically stable.

**Proof.** We can assume for some $0 < r \leq 1$ that $|| \cdot ||_\mathcal{J}$ is an $r$-norm i.e.

$$|| S + T ||_\mathcal{J} \leq || S ||_\mathcal{J}^r + || T ||_\mathcal{J}^r .$$
Suppose $D = \text{diag}(s_n) \in \mathcal{F}$, where $s_1 \geq s_2 \geq \cdots$. We recall our convention that $s_n = s_{n+1}$ where $r > 0$ is not an integer. Then for each $k \in \mathbb{N}$ we have $\|\text{diag}(s_{n/2k})\|_{\mathcal{F}} \leq \frac{2^{k/r}}{\|D\|_{\mathcal{F}}}$. Pick $\theta > 1/r$. Then by completeness the series $\sum_{k=0}^{\infty} 2^{-\theta k} \text{diag}(s_{n/2k})$ converges in $\mathcal{F}$. Thus $\text{diag}(u_n) \in \mathcal{F}$ where $u_n = \sum_{k=0}^{\infty} 2^{-\theta k} s_{n/2k}$. In fact $\|\text{diag}(u_n)\|_{\mathcal{F}} \leq \left( \sum_{k=0}^{\infty} 2^{k(1-r\theta)} \right)^{1/r} \|D\|_{\mathcal{F}}$. Now suppose $1 \leq j \leq n$. Pick $k \in \mathbb{N}$ so that $2^{-k} n \leq j \leq 2.2^{-k} n$. Then

$$s_j \leq s_{n/2k} \leq 2^{k} u_n \leq \left( \frac{2n}{j} \right)^{\theta} u_n.$$

Hence

$$t_n \leq 2^{\theta} n^{\theta} (n!)^{-\theta/n} u_n \leq C u_n$$

for some constant $C$. This implies $\text{diag}(t_n) \in \mathcal{F}$ and further that $\|\text{diag}(t_n)\|_{\mathcal{F}} \leq C \|D\|_{\mathcal{F}}$ for some constant $C$ depending only on $r$. \qed

We now prove the main result of this note, which, for the special case of geometrically stable ideals, answers positively a question posed in [3]. It should be noted that in [4] it is shown that for singly generated ideals geometric stability is a necessary and sufficient condition for the equivalence of (1) and (2) in Theorem 3.3. Thus in general (1) and (2) are not equivalent.

**Theorem 3.3.** Suppose $\mathcal{F}$ is a geometrically stable ideal of compact operators on $\mathcal{H}$ (in particular this hold if $\mathcal{F}$ is a quasi-Banach ideal). Let $S \in \mathcal{F}$ and let $\lambda_n = \lambda_n(S)$. Then the following conditions on $S$ are equivalent:

1. $S \in \text{Com} \mathcal{F}$.
2. $\text{diag} \left\{ \frac{1}{n} (\lambda_1 + \cdots + \lambda_n) \right\} \in \mathcal{F}$.
3. There exists $T \in \mathcal{F}$ so that $\frac{1}{n} |\lambda_1 + \cdots + \lambda_n| \leq s_n(T)$ for each $n \in \mathbb{N}$.
4. There exists $T \in \mathcal{F}$ such that for all $\varepsilon > 0$ we have $|\chi_{\varepsilon} S| \leq \nu(\varepsilon |T|)$.
5. There exists $T \in \mathcal{F}$ such that for all $\varepsilon > 0$ we have $|\chi_{\varepsilon} S| \leq \mu(\varepsilon |T|)$.

**Proof.** We first show that $N = \text{diag}(\lambda_n) \in \mathcal{F}$. Indeed we have $|\lambda_1 + \cdots + \lambda_n| \leq s_1 \cdots s_n$ where $s_n = s_n(S)$. Thus $|\lambda_n| \leq t_n = (s_1 \cdots s_n)^{1/n}$ so that $N \in \mathcal{F}$ by geometric stability. Hence (2), (3) and (4) are equivalent by Theorem 3.1.

It is clear that (4) implies (5) (replace $T$ by $eT$). Let us prove that (5) implies (3). We can suppose that $s_n = s_n(T) \geq s_n(S)$ for all $n$, and let $t_n = (s_1 \cdots s_n)^{1/n}$. Then

$$|\lambda_1 + \cdots + \lambda_n| \leq \sum_{|\lambda_k| \leq s_n} |\lambda_k| + ns_n \leq s_n |\chi(s_n^{-1} S)| + ns_n.$$
\[ s_n \mu(s_n^{-1}|T|) + ns_n \leq s_n \sum_{k=1}^{n} \log (s_k/s_n) + ns_n \leq ns_n \log (t_n/s_n) + ns_n. \]

Hence since \( \log x \leq x \) for all \( x \geq 1 \),
\[
\frac{|\hat{\lambda}_1 + \cdots + \hat{\lambda}_n|}{n} \leq t_n + s_n
\]
and by the geometric stability of the ideal we have (3).

To conclude the proof we establish equivalence of (1) with (5). To this end note that if \( S = H + iK \) with \( H, K \) hermitian then by Theorem 2.7, there is a constant \( C_2 \) so that, for \( x > 0 \),
\[
|\Re \chi(xS) - \chi(xH)|, |\Im \chi(xS) - \chi(xK)| \leq C_2 \mu(2x|S|).
\]

Now suppose first that \( H, K \) both satisfy (5) so that there are operators \( T_1, T_2 \in \mathcal{J} \) with \( |\chi(xH)| \leq \mu(x|T_1|) \) and \( |\chi(xK)| \leq \mu(x|T_2|) \) for \( x > 0 \). Pick an integer \( n > 2C_2 \) and consider the operator \( W = T_1 \oplus T_2 \oplus V \) where \( V \) is the direct sum of \( n \) copies of \( 2|S| \). Then \( |\chi(xS)| \leq \mu(x|W|) \) for all \( x > 0 \). Conversely if \( S \) satisfies (5) for an appropriate operator \( T \) then \( H \) and \( K \) satisfy (5) for \( T \) replaced by \( T \oplus V \) it follows \( S \) satisfies (5) if and only if both \( H \) and \( K \) satisfy (5). Now if \( S \in \text{Com} \mathcal{J} \) then \( H, K \in \text{Com} \mathcal{J} \) so that by Theorem 3.1 \( H, K \) satisfy (2)–(4) and hence also (5). Therefore (1) implies (5).

Conversely if (5) holds for \( S \), then both \( H, K \) satisfy (5) and hence also (2)–(4); so by Theorem 3.1, \( H, K \in \text{Com} \mathcal{J} \) and hence \( S \in \text{Com} \mathcal{J} \) i.e. (5) implies (1). □

References


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