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On subspaces of L^1 which embed into ℓ_1

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I. Introduction

The subspaces of $L^p (1 < p < \infty, p \neq 2)$ which are nearly isometric to subspaces of ℓ_p have recently been characterized in [17]. If X is a closed subspace of

$$L^p([0,1]) \quad (1 < p < \infty, p \neq 2),$$

then the unit ball B_X of X is compact in the metric induced by the L^1-norm $\|\,.\,\|_1$ if and only if for any $\varepsilon > 0$, there exists a subspace X_ε of $\ell_p(\mathbb{N}) = \ell_p$ such that $d(X, X_\varepsilon) < 1 + \varepsilon$ ([17], Th. 5.4).

In the present work, we address the corresponding question when $p = 1$. Specific difficulties occur in that setting, namely the conditionality of the Haar basis in L_1, the non-reflexivity of the spaces involved, and the failure of the local convexity in the spaces $L^q (0 \leqq q < 1)$. These obstructions lead us to assume the approximation property, to consider w^*-closed subspaces of ℓ_1, and to assume local convexity of the topology τ_m of convergence in measure on $\|\,.\,\|_1$-bounded sets (note that by Hölder's inequality, τ_m coincide with the topologies induced by L^q, $q < 1$, on $\|\,.\,\|_1$-bounded sets). Examples show that we cannot dispense with the local convexity assumption.

Our main result (Theorem 3.3) asserts that the unit ball B_X of a subspace X of L^1 with the approximation property is compact and locally convex for the topology of convergence in measure if and only if for any $\varepsilon > 0$, there exists a w^*-closed subspace X_ε of $\ell_1(\mathbb{N})$ such that $d(X, X_\varepsilon) < 1 + \varepsilon$. These two conditions are also shown to be equivalent to the fulfillment of the unconditional metric approximation property and the 1-strong Schur property.

As consequence of Theorem 3.3 we obtain a satisfactory description of the subspaces of L^1 whose unit ball is compact locally convex in measure (see Corollary 3.5). However there exist subspaces of L^1 whose unit ball is compact but not locally convex in measure (Theorem 4.1). Our construction follows the lines of an example from [2]. An alternative approach, which consists into replacing the p-stable random variables used in [2] by simpler random variables with the same "tail", was provided by M. Talagrand [28] who kindly allowed us to reproduce it here. The basic idea for constructing such "twisted" spaces is the concept of "needle points" introduced in [23]. We refer to [18] for recent progress related to this technique. How pathological the subspaces of L^1 whose unit ball is τ_m-compact can be is not clear (see Question 2).

We now turn to a detailed description of our results. Section 2 is devoted to investigating, for nicely placed subspaces of L^1 (i.e. for subspaces whose unit ball is τ_m-closed in L^1), the structure of the space $X^\#$ of linear forms on X whose restriction to B_X is τ_m-continuous. This space is an M-ideal in its bidual, and is nearly a subspace of $c_0(\mathbb{N})$ when B_X is τ_m-compact (Proposition 2.1). It is a predual of X when X has the unconditional metric approximation property (Proposition 2.4), and also when B_X is τ_m-compact locally convex (Lemma 2.7). In fact, any separable space with the unconditional metric approximation property and not containing $c_0(\mathbb{N})$ is the dual of a u-ideal (see Proposition 2.8).

Section 3 starts with a representation lemma for the duals of subspaces of $c_0(\mathbb{N})$ with the metric approximation property (Lemma 3.1). A consequence of this lemma is that a subspace of $c_0(\mathbb{N})$ with the metric approximation property is isomorphic to a quotient of $c_0(\mathbb{N})$ if and only if its dual embeds into L^1 (Proposition 3.2). We do not know whether this partial converse to the Alspach-James theorem ([1], [14]) holds without assuming (MAP) (Question 1). Theorem 3.3 provides an analogue for $p = 1$ of [17], Theorem 5.4. Corollary 3.5 somehow means that the subspaces of L^1 whose unit ball is τ_m-compact locally convex are close to the "trivial ones", that is, to w^*-closed subspaces of copies of ℓ_1 generated in L^1 by a sequence of disjoint indicator functions.

Section 4 provides examples of subspaces of L^1 whose unit ball is τ_m-compact but not locally convex (Theorem 4.1). The constructions rely heavily on the weak and strong laws of large numbers, and their proofs. The properties of the spaces we construct are gathered in Theorem 4.2. They are in particular subspaces of L^1 with the 1-strong Schur property, which embed into ℓ_1 isomorphically but not nearly isometrically, and whose unconditional constant is exactly 3. We conclude our work with two open problems.

Notation. We denote by $L^1 = L^1(\Omega, \Sigma, \mathbb{P})$ a separable L^1-space of some probability $\mathbb{P}$. The topology of convergence in measure is denoted τ_m. It is defined by the distance

$$d_m(f,g) = \int_\Omega |f-g|/(1+|f-g|)\,d\mathbb{P}\,.$$

A subspace X of L^1 is nicely placed (see [8], [13]) if its unit ball is τ_m-closed in L^1. The bidual L^{1**} of L^1 can be written

$$L^{1**} = L^1 \oplus_1 L^1_s$$

where $\oplus_1$ means that

$$\|u+s\| = \|u\| + \|s\|$$

for all $u \in L^1$ and $s \in L^1_s$. We denote by $\pi = L^{1**} \to L^1$ the projection with kernel L^1_s (sometimes called the Hewitt-Yoshida projection) obtained by applying the Radon-Nikodym theorem on the spectrum of L^∞. We recall that X is nicely placed if and only if $\pi(X^{\perp\perp}) = X$ ([4]; see [13]). If X is nicely placed, we denote

$$X^{\#} = \{x^* \in X^*;\ x^*_{\upharpoonright B_X} \text{ is } \tau_m\text{-continuous}\}.$$

The space $X^{\#}$ is a closed subspace of X^*. We denote by B_Y the closed unit ball of a Banach space Y, and by w^* the weak-star topology on Y^*. The w^* topology on $\ell_1(\mathbb{N})$ is always the weak-star topology associated to its natural predual $c_0(\mathbb{N})$. The subspaces we consider are always norm-closed.

An approximating sequence $\{R_n\}$ on a Banach space X is a sequence of finite rank operators such that $\lim \|x - R_n x\| = 0$ for all $x \in X$. A separable space X has the unconditional metric approximation property (UMAP) if there exists an approximating sequence on X such that $\lim \|I - 2R_n\| = 1$ (see [5]). The 1-strong Schur property is defined before Theorem 3.3. We refer to [13] for M-ideal theory and the part of isometric theory of Banach spaces which is relevant to our study. Our reference for the classical notions of Banach space theory is [20].

Part of the results of the present paper have been announced in [31].

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II. Preliminary results

Let X be a nicely placed subspace of L^1, i.e. a subspace whose unit ball is τ_m-closed in L^1. We denote

$$X^{\#} = \{x^* \in X^*;\ x^*_{\upharpoonright B_X} \text{ is } \tau_m\text{-continuous}\}.$$

With this notation, we have

Proposition 2.1. *The space $X^{\#}$ is an M-ideal in its bidual. If moreover B_X is τ_m-compact, then for any $\varepsilon > 0$, there exists a subspace Y_ε of $c_0(\mathbb{N})$ such that* $\operatorname{dist}(X^{\#}, Y_\varepsilon) < 1 + \varepsilon$.

Proof. We identify X^{**} with $X^{\perp\perp} \subseteq L^{1**}$. Since X is nicely placed, we have (see [8], [13])

$$X^{**} = X \oplus_1 X_s$$

with $X_s = X^{\perp\perp} \cap L^1_s$, and by [9], Lemma 1.2

$$X^{\#} = (X_s)_\perp .$$

We denote

$$Z = \overline{X_s}^{*} = (X^{\#})^\perp$$

and

$$Y = Z \cap X = (X^{\#})_\perp .$$

The space Y is nicely placed since

$$B_Y = B_X \cap [\bigcap \{x^{*-1}(0); x^* \in X^\#\}].$$

Thus we have

$$Y^{\perp\perp} = Y \oplus_1 (Y^{\perp\perp} \cap L_s^1). \tag{1}$$

We claim that

$$Z = Y \oplus_1 X_s. \tag{2}$$

Indeed, Z contains Y and X_s. On the other hand, any $x^{**} \in Z$ can be written $x^{**} = x + s^{**}$, with $x \in X$ and $s^{**} \in X_s$, and then $x = x^{**} - s^{**} \in Z \cap X = Y$.

By (1), we have

$$(X/Y)^{**} = (X/Y) \oplus_1 (X_s/Y_s)$$

with $Y_s = (Y^{\perp\perp} \cap L_s^1)$. We claim that (X_s/Y_s) is w^*-closed. Let $Q = X \to X/Y$ be the quotient map, and let (t_α) be an ultrafilter in (X_s/Y_s) with $\|t_\alpha\| < 1$ and $t = w^* - \lim(t_\alpha)$. Pick (x_α^{**}) in B_{X_s} such that $Q^{**}(x_\alpha^{**}) = t_\alpha$, and let $x^{**} = w^* - \lim(x_\alpha^{**})$.

Clearly, $x^{**} \in Z$. Hence by (2), $x^{**} = y + s^{**}$ with $y \in Y$ and $s^{**} \in X_s$. We have now

$$\begin{aligned} t = Q^{**}(x^{**}) &= Q(y) + Q^{**}(s^{**}) \\ &= Q^{**}(s^{**}) \end{aligned}$$

hence $t \in Q^{**}(X_s) = (X_s/Y_s)$, and the Banach-Dieudonné theorem concludes the proof of the claim.

We have now that (X/Y) is the dual of an M-ideal, namely the space

$$(X_s/Y_s)_\perp \subseteqq (X/Y)^*.$$

If we identify $(X/Y)^*$ with $Y^\perp \subseteqq X^*$, we can write

$$(X_s/Y_s)_\perp = Y^\perp \cap (X_s)_\perp = (\overline{(X^\#)}^* \cap X^\#) = X^\#$$

and thus $X^\#$ is an M-ideal in its bidual $Y^\perp$.

Let us assume now that B_X is τ_m-compact. By the above, we can identify

$$(X^\#)^* = X/Y.$$

Let (v_n) be a sequence in (X/Y) such that $\lim(\|v_n\|)$ exists and $w^* - \lim(v_n) = 0$. Let (x_n) be a bounded sequence in X such that $Q(x_n) = v_n$. We still denote (x_n) a τ_m-convergent subsequence, and $x = \tau_m - \lim(x_n)$.

For any $x^* \in X^\#$, we have

$$x^*(x) = \lim x^*(x_n) = \lim x^*(v_n) = 0$$

hence $x \in Y$. Since $\tau_m - \lim(x_n - x) = 0$, we have for any $u \in X$ and any $y \in Y$

$$\begin{aligned}\overline{\lim}\,\|u + y + (x_n - x)\| &= \|u + y\| + \overline{\lim}\,\|x_n - x\| \\ &\geqq \|Q(u)\| + \overline{\lim}\,\|v_n\|,\end{aligned}$$

therefore

$$\overline{\lim}\,\|Q(u) + v_n\| \geqq \|Q(u)\| + \overline{\lim}\,\|v_n\|,$$

that is, for any $v \in (X^\#)^* = X/Y$,

$$\overline{\lim}\,\|v + v_n\| \geqq \|v\| + \overline{\lim}\,\|v_n\|. \tag{3}$$

Condition (3) means that $X^\#$ satisfies the property (m_1^*) defined in [17]. It follows that $X^\#$ has the property (m_∞) (see [19], proof of Prop. 2.3). Indeed let $\{x_n\}$ be a sequence in $X^\#$ with $w - \lim(x_n) = 0$. We pick $x \in X^\#$ and $x_n^* \in (X^\#)^*$ with $\|x_n^*\| = 1$ such that

$$\|x_n + x\| = x_n^*(x_n + x).$$

We may and do assume that $w^* - \lim(x_n^*) = x^*$. By (m_1^*), we have

$$1 \geqq \overline{\lim}\,\|x_n^*\| = \|x^*\| + \overline{\lim}\,\|x_n^* - x^*\|$$

and since

$$\begin{aligned}\overline{\lim}\,\|x_n + x\| &= \overline{\lim}\,[x^* + (x_n^* - x^*)](x_n + x) \\ &= \overline{\lim}\,[(x_n^* - x^*)(x_n) + x^*(x)],\end{aligned}$$

it follows that

$$\overline{\lim}\,\|x_n + x\| \leqq \max(\overline{\lim}\,\|x_n\|, \|x\|).$$

On the other hand, we clearly have

$$\|x\| \leqq \underline{\lim}\,\|x + x_n\| \leqq \overline{\lim}\,\|x + x_n\|.$$

Pick $y_n^* \in (X^\#)^*$ with $\|y_n^*\| = 1$ such that

$$y_n^*(x_n) = \|x_n\|.$$

We may and do assume that $w^* - \lim(y_n^*) = y^*$. By (m_1^*), we have

$$\begin{aligned}\overline{\lim}\,\|y_n^* - y^*\| &\leqq \|y^*\| + \overline{\lim}\,\|y_n^* - y^*\| \\ &= \overline{\lim}\,\|y_n^*\| \leqq 1\end{aligned}$$

and thus

$$\begin{aligned}\overline{\lim}\|x_n\| &= \overline{\lim}(y_n^* - y^*)(x_n)\\ &= \overline{\lim}(y_n^* - y^*)(x_n + x)\\ &\leqq \overline{\lim}\|x_n + x\|.\end{aligned}$$

We proved that for all $x \in X^\#$ and any sequence (x_n) in $X^\#$ with $w - \lim(x_n) = 0$, we have

$$\overline{\lim}\|x_n + x\| = \max(\|x\|, \overline{\lim}\|x_n\|).$$

This means that the space $X^\#$ satisfies the property (m_∞) defined in [17], and then an application of [17], Th. 4.5 concludes the proof. □

Throughout this section, we use the notation $X_s = X^{\perp\perp} \cap L_s^1$ with X a nicely placed subspace of L^1.

Lemma 2.2. *For any $x_0 \in X \cap \overline{(B_{X_s})}^*$, there exists a net (x_β) in B_X such that:*

(i) $\tau_m - \lim(x_\beta) = 0$,

(ii) $w - \lim(x_\beta) = x_0$.

Proof. It follows from [4] (see [13]) that for any $x^{**} \in B_{X_s}$, any w^*-neighbourhood W of x^{**} in $B_{X^{**}}$, and any τ_m-neighbourhood V of 0 in B_X, one has $V \cap W \neq \emptyset$.

Since x_0 belongs to the w^*-closure of B_{X_s}, it follows that $V \cap U \neq \emptyset$ for any w-neighbourhood U of x_0 in B_X and any τ_m-neighbourhood V of 0, and this is a restatement of Lemma 2.2. □

Lemma 2.3. *Let X be a nicely placed subspace of L^1 with the bounded approximation property. Let (R_k) be an approximating sequence of finite rank operators, and let $\alpha = \underline{\lim}\|I - 2R_k\|$. Then*

$$X \cap \overline{(B_{X_s})}^* \subseteqq (\alpha - 1)/2 . B_X .$$

Proof. Pick $x_0 \in X \cap \overline{(B_{X_s})}^*$, and let (x_β) be a net in B_X satisfying the conditions of Lemma 2.2. We put

$$z_\beta = x_0 - x_\beta .$$

Since $\lim(z_\beta) = 0$, we have for any $k \geqq 1$

$$\lim_\beta \|R_k z_\beta\| = 0 . \tag{4}$$

Pick $\varepsilon > 0$, and fix $k \geqq 1$ such that $\|x_0 - R_k x_0\| < \varepsilon$ and $\|I - 2R_k\| < \alpha + \varepsilon$. By (4), we have

$$\lim_\beta \|(I - 2R_k)(x_0 - z_\beta) - (x_0 - 2R_k x_0 - z_\beta)\| = 0$$

and thus

$$\overline{\lim}_\beta \|(I - 2R_k)(x_0 - z_\beta) - (-x_0 - z_\beta)\| \leqq 2\varepsilon ,$$

hence

$$\overline{\lim}_\beta \, | \, \|(I - 2R_k)(x_0 - z_\beta)\| - \|x_0 + z_\beta\| \, | \leqq 2\varepsilon \, ,$$

therefore

$$\overline{\lim} \, \|x_0 + z_\beta\| \leqq (\alpha + \varepsilon) \, \overline{\lim} \, \|x_0 - z_\beta\| + 2\varepsilon \, ,$$

and since $\varepsilon > 0$ is arbitrary

$$\overline{\lim} \, \|x_0 + z_\beta\| \leqq \alpha \, \overline{\lim} \, \|x_0 - z_\beta\| \tag{5}$$

Observe now that $x_0 - z_\beta = x_\beta$, and $x_0 + z_\beta = 2x_0 - x_\beta$. Since $\tau_m - \lim(x_\beta) = 0$, we have

$$\overline{\lim} \, \|2x_0 - x_\beta\| = \|2x_0\| + \overline{\lim} \, \|x_\beta\| \, .$$

Therefore (5) implies that

$$\|2x_0\| + \overline{\lim} \, \|x_\beta\| \leqq \alpha \, \overline{\lim} \, \|x_\beta\| \, ,$$

that is,

$$\|x_0\| \leqq (\alpha - 1)/2 . \overline{\lim} \, \|x_\beta\| \leqq (\alpha - 1)/2 \, . \quad \square$$

The following application of Lemma 2.3 should be compared with Prop. 2.8, to be shown below.

Proposition 2.4. *Let X be a subspace of L^1 with the unconditional metric approximation property. Then we have $X^{\perp\perp} = X \oplus_1 X_s$, with $X_s = (X^{\perp\perp} \cap L_s^1)$ a w^*-closed subspace of $X^{\perp\perp}$. Hence $X = (X^*)^*$ is the dual of an M-ideal.*

Proof. By [10], Cor. 6.13 and Prop. 8.2, any subspace X of L^1 with (UMAP) is nicely placed. Let us outline a direct proof. The following claim is implicitely in [5]:

Claim 2.5. *Let Z be a separable Banach space with* (UMAP) *which does not contain $c_0(\mathbb{N})$. Then there exists a projection $P = Z^{**} \to Z$ such that $\|I - 2P\| = 1$.*

Proof of Claim 2.5. By [5], Proof of Th. 3.8, if X has (UMAP) there is an approximating sequence (T_n) such that $T_k T_n = T_n$ if $k > n \geqq 1$ with

$$\prod_{n=1}^{+\infty} (\|I - 2T_n\|) = \gamma < \infty$$

which implies that if we let $A_n = T_n - T_{n-1}$ $(n \geqq 1)$ with $T_0 = 0$, then

$$\sup \left\{ \left\| \sum_{i=1}^{N} \varepsilon_i A_i \right\| ; \, N \geqq 1, \varepsilon_i = \pm 1 \right\} \leqq \gamma \, . \tag{6}$$

Since $Z \not\supset c_0(\mathbb{N})$, (6) shows that

$$Px^{**} = \|.\| - \lim_{N \to +\infty} T_N^{**} x^{**}$$

exists for all $x^{**} \in Z^{**}$. Since (T_n) is an approximating sequence, we have $P^2 = P$, and $\lim \|I - 2T_n\| = 1$ implies that $\|I - 2P\| = 1$. □

We now return to the proof of Proposition 2.4. Let $P = X^{\perp\perp} \to X$ be the projection provided by Claim 2.5. We have to show $\operatorname{Ker}(P) \subseteqq L^1_s$. Pick $s = a + \sigma$ in $\operatorname{Ker}(P)$ with $\|s\| = 1$, $a \in L^1$ and $\sigma \in L^1_s$. It suffices to show $\|\sigma\| = 1$.

Pick $\delta > 0$ arbitrarily. Since $s \in \operatorname{Ker}(P)$, we have $\|x + s\| = \|x - s\|$ for any $x \in X$. The local reflexivity principle now provides a net (x_α) in B_X such that

$$w^* - \lim(x_\alpha) = s\,, \tag{7}$$

$$\|x_\alpha - a\| \leqq (1 + \delta)\|\sigma\| \tag{8}$$

for all α, and for all $x \in X$

$$\lim_\alpha (\|x + x_\alpha\| - \|x - x_\alpha\|) = 0\,. \tag{9}$$

Pick $f^* \in X^*$ with $\|f^*\| = 1$ and $f^*(s) > 1 - \delta$. By (7) we may and do assume that $f^*(x_\alpha) > 1 - \delta$ for all α. For a given α_0, there exists by (9) α_1 such that

$$\|x_{\alpha_0} + x_{\alpha_1}\| - \delta \leqq \|x_{\alpha_0} - x_{\alpha_1}\|\,.$$

Hence

$$\begin{aligned}\|x_{\alpha_0} - x_{\alpha_1}\| &\geqq f^*(x_{\alpha_0} + x_{\alpha_1}) - \delta \\ &\geqq 2 - 3\delta\,.\end{aligned}$$

But by (8), we have

$$\begin{aligned}2(1 + \delta)\|\sigma\| &\geqq \|x_{\alpha_0} - a\| + \|x_{\alpha_1} - a\| \\ &\geqq \|x_{\alpha_0} - x_{\alpha_1}\| \\ &\geqq 2 - 3\delta\end{aligned}$$

and this shows $\|\sigma\| = 1$ since $\delta > 0$ is arbitrary. We have shown that X is nicely placed, that is, $X^{\perp\perp} = X \oplus_1 X_s$.

To conclude the proof of Proposition 2.4, we observe that we may apply Lemma 2.3 with $\alpha = 1$ since X has (UMAP), hence

$$\overline{B_{X_s}}^* \cap X = \{0\}\,. \tag{10}$$

By (10), B_{X_s} is w^*-closed. Indeed, let $v = x + s$ be in $\overline{B_{X_s}}^*$, with $x \in X$ and $s \in X_s$. Since $\|v\| \geqq \|s\|$, we have $(-s) \in B_{X_s}$ and thus $x/2 = (f - s)/2$ belongs to $(\overline{B_{X_s}}^* \cap X)$, hence $x = 0$ and $v \in B_{X_s}$. The Banach-Dieudonné theorem then shows that X_s is w^*-closed. Finally, $X^* = (X_s)_\perp$ (see [9], lemma 1.2) hence $X_s = (X^*)^\perp$ and $X = (X^*)^*$, with X^* an M-ideal in its bidual. □

Lemma 2.6. *For any nicely placed subspace X of L^1, the following sets are identical*:

$$X \cap \overline{B_{X_s}}^{*} = \bigcap \{\overline{V}^{w};\ V\ \tau_m\text{-open in } B_X, 0 \in V\}$$
$$= \bigcap \{\overline{\operatorname{conv}}^{|\cdot|}(V);\ V\ \tau_m\text{-open in } B_X, 0 \in V\}\,.$$

Proof. In the above notation, the inclusion

$$X \cap \overline{B_{X_s}}^{*} \subseteqq \bigcap \{\overline{V}^{w}\}$$

is immediate from Lemma 2.2 and the inclusion

$$\bigcap \{\overline{V}^{w}\} \subseteqq \bigcap \overline{\operatorname{conv}}^{|\cdot|}(V)$$

follows from Mazur's theorem. To prove the last inclusion, pick $x_0 \in \bigcap \overline{\operatorname{conv}}^{|\cdot|}(V)$. If $x_0 \notin X \cap (\overline{B_{X_s}})^{*}$, there is $f^* \in X^*$ such that

$$f^*(x_0) > \gamma = \sup\{f^*(t);\ t \in B_{X_s}\}\,.$$

Since $x_0 \in \bigcap \overline{\operatorname{conv}}^{|\cdot|}(V)$, we can find a sequence (x_n) in B_X such that

$$\underline{\lim}\, f^*(x_n) \geqq f^*(x_0) \tag{11}$$

and

$$d_m(x_n, 0) \leqq 2^{-n}\,. \tag{12}$$

By (12) and [4] (see [13]), every w^*-cluster point t of (x_n) in X^{**} belongs to X_s. But by (11), $f^*(t) \geqq f^*(x_0) > \gamma$, a contradiction. □

Lemma 2.6 is used below for comparing τ_m with locally convex topologies in B_X.

Lemma 2.7. *Let X be a nicely placed subspace of L^1. The following assertions are equivalent*:

(i) *There exists a locally convex Hausdorff topology on X which is coarser than τ_m on B_X.*

(ii) *B_X is $\sigma(X, X^{\#})$-compact.*

(iii) *$\{0\}$ is the intersection of the convex τ_m-neighbourhoods of* 0 *in B_X.*

If moreover B_X is τ_m-compact, then the above conditions are also equivalent to:

(iv) *The weak topology of X is finer than τ_m on B_X.*

(v) *There exists a locally convex Hausdorff topology which coincides with τ_m on B_X.*

(vi) 0 *has a basis of τ_m-neighbourhoods in B_X consisting of convex sets.*

When the conditions (iv), (v), (vi) are satisfied, we say that B_X is τ_m-compact locally convex.

Proof. (i) $\Rightarrow$ (iii) is clear and (ii) $\Rightarrow$ (i) follows from the definition of $X^{\#}$.

(iii) $\Rightarrow$ (ii): We pick $x \in B_X \setminus \{0\}$. By (iii), there exists a convex τ_m-neighbourhood V of 0 in B_X such that $(x/2) \notin V$. We claim that $x \notin \bar{V}^{\|\cdot\|}$. Indeed if not denote (x_n) a sequence in V such that $\lim \|x - x_n\| = 0$. Since $(x - x_n) \in V$ for n large enough, and

$$x/2 = (x_n + (x - x_n))/2$$

we have $(x/2) \in V$ by convexity of V, a contradiction. We have therefore shown that

$$\{0\} = \bigcap \{\bar{V}^{\|\cdot\|};\ V \text{ convex } \tau_m\text{-neigh. of 0 in } B_X\} \tag{13}$$

and thus

$$\bigcap \{\overline{\mathrm{conv}}^{\|\cdot\|}(V);\ V\ \tau_m\text{-open in } B_X, 0 \in V\} = \{0\}\,.$$

Hence by Lemma 2.6 we have

$$X \cap \overline{B_{X_s}}^{*} = \{0\} \tag{10}$$

and (10) implies like in the proof of Proposition 2.4 that $X = (X^{\#})^{*}$, which is (ii).

(iv) $\Rightarrow$ (ii): By Lemma 2.2, (iv) implies (10), and (ii) follows as above.

The implications (v) $\Rightarrow$ (vi) and (vi) $\Rightarrow$ (iii) are obvious.

We now assume that B_X is τ_m-compact.

(ii) $\Rightarrow$ (v): By definition of $X^{\#}$, the topology τ_m is finer than the topology $\sigma(X, X^{\#})$ on B_X, and thus these two topologies agree by compactness.

(v) $\Rightarrow$ (iv): By compactness, τ_m agrees on B_X with the topology of pointwise convergence on the space of τ_m-continuous linear forms, and that topology is coarser than the weak topology. □

Let us mention an amusing consequence of Lemma 2.7. Let us call "a weak topology" on X a topology of pointwise convergence on a separating subspace of X^*. Then, if X is a subspace of L^1 such that B_X is τ_m-compact, the topology τ_m coincides on B_X with a weak topology, if and only if it is coarser on B_X than some weak topology, if and only if it is finer on B_X than some weak topology.

We conclude this section with a general result on spaces with (UMAP) which do not contain $c_0(\mathbb{N})$.

Proposition 2.8. *Let X be a separable Banach space not containing $c_0(\mathbb{N})$ with the unconditional metric approximation property. Then $X^{**} = X \oplus_u X_s$, where X_s is a w^*-closed subspace of X^{**} such that $\|x + s\| = \|x - s\|$ for all $x \in X$ and all $s \in X_s$.*

Proof. We let $X_s = \mathrm{Ker}(P)$, where P is the projection provided by Claim 2.5. It remains to show that X_s is w^*-closed. This amounts to show that $X \cap \overline{B_{X_s}}^* = \{0\}$. Pick $x_0 \in X \cap \overline{B_{X_s}}^*$. By local reflexivity, there exists a filter (x_α) in B_X such that

$$x_0 = w - \lim(x_\alpha)$$

and for all $x \in X$

$$\lim(\|x + x_\alpha\| - \|x - x_\alpha\|) = 0\,. \tag{14}$$

Since X has (UMAP) and $w - \lim(x_0 - x_\alpha) = 0$, the proof of (5) (see the proof of Lemma 2.3) shows that for all $y \in X$,

$$\lim(\|y + (x_0 - x_\alpha)\| - \|y - (x_0 - x_\alpha)\|) = 0\,. \tag{15}$$

Pick now $n \geqq 1$. By (14) and (15), we have

$$\begin{aligned}\overline{\lim}\|2nx_0 - x_\alpha\| &= \overline{\lim}\|(2n-1)x_0 + (x_0 - x_\alpha)\|\\ &= \overline{\lim}\|(2n-1)x_0 - (x_0 - x_\alpha)\|\\ &= \overline{\lim}\|2(n-1)x_0 + x_\alpha\|\\ &= \overline{\lim}\|2(n-1)x_0 - x_\alpha\|\,,\end{aligned}$$

hence $\overline{\lim}\|2nx_0 - x_\alpha\| = \overline{\lim}\|x_\alpha\|$ for any $n \geqq 1$ and it follows that $x_0 = 0$. □

Remark 2.9. By [10], Cor. 9.3, Proposition 2.8 implies that a separable space not containing $c_0(\mathbb{N})$ with (UMAP) actually has the commuting (UMAP). Therefore, Proposition 2.8 answers positively [10], Question 8. In fact, it has recently been shown ([11]) that any separable Banach space with (UMAP) has the commuting (UMAP). More precisely, if a separable Banach space X has (UMAP), then there exists an approximating sequence (R_k) such that $\lim\|I - 2R_k\| = 1$ and $R_k R_n = R_n R_k = R_n$ if $k > n$.

III. Main results

We start with a representation lemma for the duals of certain subspaces of $c_0(\mathbb{N})$.

Lemma 3.1. *Let V be a subspace of $c_0(\mathbb{N})$ with the metric approximation property. Then for any $\eta > 0$, there exists a sequence $\{E_n\}$ of finite-dimensional subspaces of V^* and a $(w^* - w^*)$-continuous linear map*

$$T = V^* \to (\textstyle\sum \oplus E_n)_1$$

such that for all $x^* \in V^*$

$$(1-\eta)\|x^*\| \leqq \|Tx^*\| \leqq (1+\eta)\|x^*\|\,.$$

Proof. Pick $\varepsilon > 0$, to be chosen later. Since $V \subseteq c_0(\mathbb{N})$ has (MAP), there exists ([6]) an approximating sequence $\{S_k; k \geqq 0\}$ with $S_0 = 0$ such that if $R_k = S_k - S_{k-1}$,

$$\sup\left\{\left\|\sum_{j=1}^{N}\varepsilon_j R_j\right\|;\ N\geqq 1,\ |\varepsilon_j|=1\right\}\leqq 1+\varepsilon/2\,. \tag{1}$$

Since $V^*\not\supset c_0(\mathbb{N})$, (1) and Bessaga-Pelczynski's theorem imply that

$$I_{V^*}=\sum_{j=1}^{\infty}R_j^*$$

in the strong operator topology, and this implies that for all $x^*\in V^*$,

$$\lim_{k\to+\infty}\left\{\sup\left\{\left\|\sum_{j=k}^{n}R_j^*(x^*)\right\|;\ n\geqq k\right\}\right\}=0\,. \tag{2}$$

We now use a "skipped blocking" argument (see [3]).

Claim 3.2. *For any $\alpha_1>0$ and any $k_1\geqq 1$, there exists $k_2>k_1$ such that for all $k\geqq k_2$ and all $u,v\in B_V$*

$$\left\|\sum_{j=1}^{k_1}R_j(u)+\sum_{j=k_2}^{k}R_j(v)\right\|<1+\varepsilon/2+\alpha_1\,.$$

Indeed since the R_j's are finite rank operators, there is $k_1'\in\mathbb{N}$ such that for all $i\geqq k_1'$

$$\left\|\sum_{j=1}^{k_1'}R_j^*(e_i^*)\right\|<\alpha_1 \tag{3}$$

where (e_i^*) denotes the coordinate functionals. By (2), there is $k_2>k_1$ such that for all $k\geqq k_2$ and all $i\leqq k_1'$

$$\left\|\sum_{j=k_2}^{k}R_j^*(e_i^*)\right\|<\alpha_1 \tag{4}$$

and then Claim 3.2 follows from (1), (3) and (4).

We proceed along the lines of the proof of Claim 3.2 to find $k_3>k_2$ such that for all $k\geqq k_3$ and all $u,v\in B_V$

$$\left\|\sum_{j=1}^{k_1}R_j(u)+\sum_{j=k_3}^{k}R_j(v)\right\|<1+\varepsilon/2+\alpha_1\,, \tag{5}$$

$$\left\|\sum_{j=1}^{k_2}R_j(u)+\sum_{j=k_3}^{k}R_j(v)\right\|<1+\varepsilon/2+\alpha_1\,. \tag{6}$$

Picking $\alpha_2>0$, we then find $k_4>k_3$ such that for all $k\geqq k_4$ and all $u,v,w\in B_V$, (5) and (6) are still true with k_4 substituted to k_3, and moreover

$$\left\|\sum_{j=1}^{k_1}R_j(u)+\sum_{j=k_2}^{k_3}R_j(v)+\sum_{j=k_4}^{k}R_j(w)\right\|<1+\varepsilon/2+\alpha_1+\alpha_2\,.$$

Continuing along these lines, with $\alpha_n > 0$ such that $\sum \alpha_n < \varepsilon/2$, we show the existence of a sequence $k_1 < k_2 < k_3 \ldots$ such that if we let

$$T_n = \sum_{j=1}^{k_n} R_j$$

then for all $1 \leqq a_1 < b_1 < a_2 < b_2 < \cdots < a_\ell < b_\ell$ and all $u_1, u_2, \ldots, u_\ell$ in B_V,

$$\left\| \sum_{i=1}^{\ell} (T_{b_i} - T_{a_i})(u_i) \right\| < 1 + \varepsilon ,$$

it follows that for any $x^* \in V^*$

$$(7) \qquad (1+\varepsilon) \left\| \sum_{i=1}^{\ell} (T_{b_i}^* - T_{a_i}^*)(x^*) \right\| \geqq \sum_{i=1}^{\ell} \|(T_{b_i}^* - T_{a_i}^*)(x^*)\| .$$

We first prove Lemma 3.1 when V^* has a finite cotype q by the technique of [17], Proof of Th. 5.2. Let $B_k = T_{k+1} - T_k$ and pick $t \in \mathbb{N}$, $t > 1$ to be chosen later. For $0 \leqq s \leqq t-1$, let

$$A_s = \sum_{j \equiv s (\mathrm{mod}.t)} B_j ,$$

$$W_{k,s} = \sum_{(k-1)t+s < j < kt+s} B_j .$$

Pick $x^* \in V^*$ with $\|x^*\| = 1$. By (1), for all $|\lambda_s| = 1$,

$$\left\| \sum_{s=0}^{t-1} \lambda_s A_s^*(x^*) \right\| \leqq (1+\varepsilon) ,$$

thus

$$\left(\sum_{s=0}^{t-1} \|A_s^*(x^*)\|^q \right)^{1/q} \leqq C_q(1+\varepsilon) = C ,$$

therefore

$$\sum_{s=0}^{t-1} \|A_s^*(x^*)\| \leqq C.t^{1-1/q} .$$

By (7) and the triangular inequality, we have for all $s, 0 \leqq s \leqq t-1$,

$$1 - \|A_s^*(x^*)\| \leqq \sum_{k=0}^{+\infty} \|W_{k,s}^*(x^*)\| \leqq (1+\varepsilon)\big(1 + \|A_s^*(x^*)\|\big) .$$

Summing up on s, we get

$$t(1 - Ct^{-1/q}) \leqq \sum_{k=0}^{+\infty} \sum_{s=0}^{t-1} \|W_{k,s}^*(x^*)\| \leqq (1+\varepsilon)\, t(1 + Ct^{-1/q}) ,$$

that is,

$$(8) \qquad 1 - Ct^{-1/q} \leqq \frac{1}{t} \sum_{k=0}^{+\infty} \sum_{s=0}^{t-1} \|W_{k,s}^*(x^*)\| \leqq (1+\varepsilon)(1 + Ct^{-1/q}) .$$

We set now $E_{k,s} = W^*_{k,s}(V^*)$, and we define

$$T = V^* \to (\sum_{k,s} \oplus E_{k,s})_1$$

by

$$T(x^*) = \left(\frac{1}{t} W^*_{k,s}(x^*)\right)_{k,s}.$$

For $\varepsilon > 0$ and $t > 1$ suitably chosen, (8) implies that T satisfies the conclusion of Lemma 3.1.

If V^* has no finite cotype, then (see [29]) it contains arbitrarily close copies of any finite-dimensional space. Since V is arbitrarily close to quotients of C_∞ ([17], Th. 4.2.; see [15]), there is a $(w^* - w^*)$-continuous map $Q^*: V^* \to C^*_\infty$ with

$$(1 - \eta/2)\|x^*\| \leqq \|Q^*(x^*)\| \leqq (1 + \eta/2)\|x^*\|$$

for all $x^* \in V^*$. Lemma 3.1 in the case when V^* has no finite cotype clearly follows from these two facts. □

Note that Lemma 3.1 means that any subspace of $c_0(\mathbb{N})$ with the metric approximation property is arbitrarily close to a quotient of a c_0-sum of its finite-dimensional quotients.

By [1], [14], any quotient of $c_0(\mathbb{N})$ is arbitrarily close to subspaces of $c_0(\mathbb{N})$. The following application of Lemma 3.1 addresses the question of the converse. It is an open problem to know whether Proposition 3.2 holds true without assuming that X has the approximation property.

Proposition 3.2. *Let X be a subspace of $c_0(\mathbb{N})$ with the metric approximation property. Then*:

(i) *X^* is isometric to a subspace of L^1 if and only if for any $\varepsilon > 0$, there exists a quotient X_ε of $c_0(\mathbb{N})$ such that $d(X, X_\varepsilon) < 1 + \varepsilon$.*

(ii) *Pick $K > 1$. Then X is λ-isomorphic to a quotient of $c_0(\mathbb{N})$ for some $\lambda < K$ if and only if X^* is μ-isomorphic to a subspace of L^1 for some $\mu < K$.*

Proof. If X^* is isometric to a subspace of L^1 then for any finite dimensional subspace E of X^* and any $\delta > 0$, there exists $k \geqq 1$ and a subspace V of the k-dimensional ℓ_1^k space such that $d(E, V) < 1 + \delta$. Therefore by Lemma 3.1 there exists for any $\varepsilon > 0$ a $(w^* - w^*)$-continuous map

$$T_\varepsilon: X^* \to \ell_1(\mathbb{N})$$

with $(1 - \varepsilon)\|x^*\| \leqq \|T_\varepsilon x^*\| \leqq (1 + \varepsilon)\|x^*\|$ for all $x^* \in X^*$. The result follows by taking X_ε to be the predual of $T_\varepsilon(X^*)$ equipped with the norm induced by $\ell_1(\mathbb{N})$.

For the converse implication, we observe that $\operatorname{dist}(X^*, X^*_\varepsilon) < 1 + \varepsilon$ and X^*_ε is isometric to a subspace of $\ell_1(\mathbb{N})$. A standard ultraproduct argument concludes the proof of (i). The proof of (ii) is similar and will be ommitted. □

The following theorem provides in particular the analogue for $p = 1$ of the theorem of Kalton and Werner ([17], Th. 5.4). Let us recall that a Banach space X is said to have the 1-strong Schur property if for any $\delta \in (0,2]$ and any $\varepsilon > 0$, all normalized δ-separated sequences in X contain a subsequence which is $(2/\delta + \varepsilon)$-equivalent to the unit vector basis of ℓ_1. With this notation, one has

Theorem 3.3. *Let X be a closed subspace of L^1 with the approximation property. The following assertions are equivalent*:

(i) *The unit ball B_X of X is τ_m-compact locally convex.*

(ii) *The space X has the unconditional metric approximation property and the 1-strong Schur property.*

(iii) *For any $\varepsilon > 0$, there is a quotient Y_ε of $c_0(\mathbb{N})$ such that $d(X, Y_\varepsilon^*) < 1 + \varepsilon$.*

Proof. (i) $\Rightarrow$ (iii): If (i) is satisfied, then by Lemma 2.7 the space X is isometric to the dual of $X^\#$. By Proposition 2.1 there exists for any $\varepsilon > 0$ a subspace Z_ε of $c_0(\mathbb{N})$ such that $d(X^\#, Z_\varepsilon) < (1+\varepsilon)^{1/2}$ and thus $d(X, Z_\varepsilon^*) < (1+\varepsilon)^{1/2}$. Since X has (AP), Z_ε^* is a separable dual with (AP) hence it has (MAP) (see [20], Th. 1.e.15) and thus Z_ε has (MAP). Now we may apply Proposition 3.2 to Z_ε to obtain a quotient Y_ε of $c_0(\mathbb{N})$ with

$$d(Z_\varepsilon, Y_\varepsilon) < (1+\varepsilon)^{1/2}$$

and then

$$d(X, Y_\varepsilon^*) \leqq d(X, Z_\varepsilon^*) . d(Z_\varepsilon^*, Y_\varepsilon^*) < (1+\varepsilon) .$$

(iii) $\Rightarrow$ (ii): By [1], there exists for any $\alpha > 0$ a subspace $Z_{\varepsilon,\alpha}$ of $c_0(\mathbb{N})$ such that

$$d(Y_\varepsilon, Z_{\varepsilon,\alpha}) < 1 + \alpha .$$

Since X has (AP), Y_ε^* and therefore Y_ε have (MAP) as above. By [5], Y_ε has the commuting (MAP) hence if $\alpha < 2$ the space $Z_{\varepsilon,\alpha}$ satisfies the assumptions of [12], Cor. 4.5 and thus it has (MAP). It follows by [10], Th. 9.2 that $Z_{\varepsilon,\alpha}$ has (UMAP) if $\alpha \in (0,2)$ and this easily implies that Y_ε has (UMAP). Another application of [10], Th. 9.2 shows that Y_ε^* has (UMAP), and it follows that X has (UMAP).

The 1-strong Schur property for X easily follows from (iii) and the fact that subspaces of $\ell^1(\mathbb{N})$ satisfy this property.

(ii) $\Rightarrow$ (i): Since X has (UMAP), it is nicely placed by Proposition 2.4. Thus B_X is τ_m-closed. The τ_m-compactness of B_X follows from

Lemma 3.4 ([24]). *Let Z be a subspace of L^1. The following assertions are equivalent*:

(i) *The unit ball B_Z of Z is τ_m-relatively compact in L^1.*

(ii) *Z has the 1-strong Schur property.*

We give for completeness the proof of the implication (ii) $\Rightarrow$ (i) which we use.

Proof of Lemma 3.4. (ii) $\Rightarrow$ (i): By the subsequence splitting lemma any bounded sequence $\{f_n\}$ in L^1 contains a subsequence $\{f_n'\}$ such that $f_n' = u_n + d_n$, with $\{u_n\}$ equi-integrable and $\{d_n\}$ a disjoint sequence. We keep this notation throughout the proof.

We claim that any sequence $\{f_n\}$ in B_Z has a subsequence $\{f_n' = u_n + d_n\}$ such that $\{u_n\}$ is norm-convergent. If not, there is a sequence $\{f_n' = u_n + d_n\}$ with $\|f_n'\| = 1$ for all n, $\{u_n\}$ α-separated with $\alpha > 0$, and $\eta = \lim \|d_n\|$ exists. Since Z has the Schur property, necessarily $\eta > 0$.

There is $N \geqq 1$ such that $\{f_n'; n \geqq N\}$ is $(\alpha/2 + 2\eta)$ separated. Indeed, pick $\delta > 0$ such that for all $n \geqq 1$ and $E \in \Sigma$,

$$\mathbb{P}(E) < \delta \Rightarrow \int_E |u_n| \, d\mathbb{P} < \alpha/12 .$$

Let $\{E_j; j \geqq 1\}$ be disjoint sets supporting $\{d_j\}$, and pick $N \geqq 1$ such that

$$\mathbb{P}\Big(\bigcup_{j \geqq N} E_j\Big) < \delta$$

and

$$\|d_j\| \geqq \eta - \alpha/12 \quad \text{for all } j \geqq N .$$

Let $E = \bigcup_{j \geqq N} E_j$. Then if $j, k \geqq N$, $j \neq k$,

$$\begin{aligned} \|f_j' - f_k'\| &= \int_E |u_j - u_k + d_j - d_k| \, d\mathbb{P} + \int_{E^c} |u_j - u_k| \, d\mathbb{P} \\ &\geqq (\|d_j\| + \|d_k\| - \alpha/6) + \alpha - \alpha/6 \\ &\geqq (2\eta - 2\alpha/6) + \alpha - \alpha/6 = 2\eta + \alpha/2 . \end{aligned}$$

Since Z has the 1-strong Schur property, there exists a subsequence $\{f_n'' = u_n' + d_n'\}$ of $\{f_n'\}$ which is $\beta = 2/(2\eta + \alpha/4)$-equivalent to ℓ_1. Obviously, $\eta < 1/\beta$.

Let $\varepsilon = (\beta^{-1} - \eta)/3$, and $K \geqq 1$ be such that $\|d_n'\| \leqq \eta + \varepsilon$ for all $n \geqq K$. Since $\{u_n'\}$ is equi-integrable, there exist scalars $\lambda_1, \lambda_2, \ldots, \lambda_L$ such that

$$\sum_{j=1}^{L} |\lambda_j| = 1$$

and

$$\Big\| \sum_{j=1}^{L} \lambda_j u_{K+j}' \Big\| < \varepsilon .$$

It follows that

$$\beta^{-1} \leqq \Big\| \sum_{j=1}^{L} \lambda_j f_{K+j}'' \Big\|$$

$$\leqq \left\| \sum_{j=1}^{L} \lambda_j u'_{K+j} \right\| + \left\| \sum_{j=1}^{L} \lambda_j d'_{K+j} \right\|$$

$$\leqq \varepsilon + (\eta + \varepsilon) = \eta + 2\varepsilon$$

and this contradicts our choice of ε and shows (ii) $\Rightarrow$ (i) in Lemma 3.4. □

Coming back to Theorem 3.3 we have shown that if (ii) is satisfied then B_X is τ_m-compact. By Proposition 2.4 we also have that B_X is $\sigma(X, X^\#)$-compact. By Lemma 2.7 it follows that (i) holds true. This concludes the proof of Theorem 3.3. □

The spaces which satisfy the conditions of Theorem 3.3 can be characterized with approximation conditions.

Corollary 3.5. *Let X be a subspace of L^1 with the approximation property. The following assertions are equivalent*:

(i) *B_X is τ_m-compact locally convex.*

(ii) *There is a sequence $\{R_n\}$ of finite rank operators such that* $\lim \|f - R_n f\| = 0$ *for all $f \in X$,* $\lim \|I - 2R_n\| = 1$, *every R_n is $(\tau_m - \|.\|)$-continuous on B_X, and for any $\varepsilon > 0$, there exists $n \geqq 1$ such that*

$$d_m(f, R_n f) < \varepsilon$$

for all $f \in B_X$.

(iii) *X is nicely placed, and there is a sequence $\{R_n\}$ of finite rank operators such that* $\lim \|R_n\| = 1$ *and for any $\varepsilon > 0$, there exists $n \geqq 1$ such that*

$$d_m(f, R_n f) < \varepsilon$$

for all $f \in B_X$.

(iv) *X is nicely placed, and for any $\varepsilon > 0$, there is a sub σ-field Σ_ε of Σ generated by disjoint sets such that*

$$d_m(f, \mathbb{E}_{\Sigma_\varepsilon} f) < \varepsilon$$

for all $f \in B_X$.

Proof. (i) $\Rightarrow$ (ii): By Theorem 3.3, X has (UMAP) and by Lemma 2.7 we have $X = (X^\#)^*$, with $X^\#$ M-ideal in X^*. It follows by [10], Th. 9.2 that X has (UMAP) with conjugate finite rank operators. That is, there is a sequence T_n of finite rank operators on $X^\#$ such that $\lim \|x - T_n x\| = 0$ for all $x \in X^\#$, $\lim \|I - 2T_n\| = 1$ and $\lim \|f - T_n^* f\| = 0$ for all $f \in X$. Pick $\{x_k; k \geqq 1\}$ a dense sequence in $X^\#$. Since τ_m coincide with $\sigma(X, X^\#)$ on B_X, there exist for all $\varepsilon > 0$, $\delta = \delta(\varepsilon) > 0$ and $N = N(\varepsilon) \geqq 1$ such that if $f, g \in B_X$ and

$$\langle x_k, f - g \rangle \leqq \delta \quad \text{for all } k \geqq N,$$

then

$$d_m(f, g) < \varepsilon .$$

Now (ii) follows with $R_n = T_n^*$.

(ii) ⇒ (iii) since by Proposition 2.4 any subspace of L^1 with (UMAP) is nicely placed.

(iii) ⇒ (iv): It is classical and easily checked that for any finite-dimensional subspace F of L^1 and any $\delta > 0$, there exists a sub σ-field Σ_δ of Σ generated by finitely many disjoint sets such that

$$\|g - \mathbb{E}_{\Sigma_\delta} g\| < \delta \|g\|$$

for all $g \in F$. Now (iv) follows from (iii) by applying this observation to $F = R_n(X)$.

(iv) ⇒ (i): If we let $Y_\varepsilon = \mathbb{E}_{\Sigma_\varepsilon}(L^1)$, then the unit ball of Y_ε is τ_m-compact. Thus it follows from (iv) that B_X is τ_m-compact.

To show that B_X is τ_m-locally convex, it suffices by Lemma 2.7 to show that the weak topology is finer than τ_m on B_X. To prove this, observe that we may assume without loss of generality that Σ_ε is generated by finitely many sets. Pick now (x_α) in B_X with

$$w - \lim (x_\alpha) = x_0 .$$

We have

$$\begin{aligned} d_m(x_\alpha, x_0) &\leqq d_m(x_\alpha, \mathbb{E}_{\Sigma_\varepsilon} x_\alpha) + d_m(\mathbb{E}_{\Sigma_\varepsilon} x_\alpha, \mathbb{E}_{\Sigma_\varepsilon} x_0) \\ &\quad + d_m(\mathbb{E}_{\Sigma_\varepsilon} x_0, x_0) \\ &\leqq 2\varepsilon + d_m(\mathbb{E}_{\Sigma_\varepsilon} x_\alpha, \mathbb{E}_{\Sigma_\varepsilon} x_0) . \end{aligned} \tag{9}$$

But we have

$$w - \lim (\mathbb{E}_{\Sigma_\varepsilon} x_\alpha) = \mathbb{E}_{\Sigma_\varepsilon} x_0$$

and the space $\mathbb{E}_{\Sigma_\varepsilon}(L^1)$ is finite dimensional. It follows that

$$\lim_\alpha \|\mathbb{E}_{\Sigma_\varepsilon} x_0 - \mathbb{E}_{\Sigma_\varepsilon} x_0\| = 0 . \tag{10}$$

Since $\varepsilon > 0$ is arbitrary, (9) and (10) imply that

$$\tau_m - \lim (x_\alpha) = x_0 .$$

This shows (i) and concludes the proof. □

Our next statement is essentially Theorem 3.3 in the (much simpler) atomic case. We provide a simple proof.

Corollary 3.6. *Let X be a subspace of $\ell_1(\mathbb{N})$ with the approximation property. Then X has the unconditional metric approximation property if and only if X is w^*-closed.*

Proof. If X has (UMAP), then by Proposition 2.5 (or by [10], Th. 6.9) we have

$$X^{\perp\perp} = X \oplus_1 (X^{\perp\perp} \cap c_0^\perp) \tag{11}$$

and (11) means that X is w^*-closed. Conversely if X is w^*-closed, we have $X = X_0^*$ with X_0 a quotient of $c_0(\mathbb{N})$. By Theorem 3.3 (iii) ⇒ (ii), the space X has (UMAP). □

It follows from [27] that $\ell_1(\mathbb{N})$ contains w^*-closed subspaces failing (AP), Corollary 3.7 below should be compared with the main result of [22] (itself extending a result from [16]).

Corollary 3.7. *Let X be a subspace of L^1. If X has the 1-strong Schur property, and if for any $\varepsilon > 0$, X is $(1+\varepsilon)$-isomorphic to a $(1+\varepsilon)$-complemented subspace of a space with a $(1+\varepsilon)$-unconditional* (FDD), *then for any $\delta > 0$, X is $(1+\delta)$-isomorphic to a subspace of $\ell_1(\mathbb{N})$.*

Proof. It suffices to observe that the assumptions clearly imply that X has (UMAP), and to apply Theorem 3.3. □

We conclude this section with two simple facts.

Proposition 3.8. *Let X be a nicely placed subspace of L_1. We denote $\pi = X^{**} \to X$ the projection with kernel $(X^{\perp\perp} \cap L_s^1)$. Then*

(i) *π is $(w^* - \tau_m)$ sequentially continuous on $B_{X^{**}}$ if and only if X has the Schur property,*

(ii) *π is $(w^* - \tau_m)$ continuous on $B_{X^{**}}$ if and only if B_X is τ_m-compact locally convex.*

Proof. (i) Since X^{**} is a subspace of $L^{\infty *}$ and L^∞ has the Grothendieck property, any w^*-convergent sequence (x_n^{**}) in X^{**} is weakly convergent, and thus $\pi(x_n^{**})$ is weakly convergent. If X has the Schur property, $\pi(x_n^{**})$ is then norm-convergent and thus τ_m-convergent, hence π is $(w^* - \tau_m)$ sequentially continuous. Conversely if π is $(w^* - \tau_m)$ sequentially continuous on $B_{X^{**}}$, any weakly convergent sequence in B_X is τ_m-convergent, and therefore norm-convergent since it is equi-integrable.

(ii) If π is $(w^* - \tau_m)$ continuous on $B_{X^{**}}$ then clearly B_X is τ_m-compact. Since π is the identity on its range B_X, the weak topology is finer than τ_m on B_X and by Lemma 2.7, B_X is τ_m-locally convex. Conversely if B_X is τ_m-compact locally convex, then π coincide with the canonical projection from $(X^*)^{***}$ onto $(X^*)^* = X$, and the $(w^* - \tau_m)$ continuity of π on $B_{X^{**}}$ follows since the topologies τ_m and $\sigma(X, X^*)$ coincide on B_X. □

Proposition 3.9. *Let X be a subspace of L^1 with the approximation property and whose unit ball is τ_m-compact. Then the following statements are equivalent*:

(i) *B_X is τ_m-compact locally convex.*

(ii) *For any $\varepsilon > 0$, there exists a subspace Y_ε of $\ell_1(\mathbb{N})$ such that $d(X, Y_\varepsilon) < 1 + \varepsilon$.*

Proof. (i) ⇒ (ii) follows immediately from Theorem 3.3.

(ii) ⇒ (i): Denote by $J_\varepsilon = X \to Y_\varepsilon$ an isomorphism such that $\|J_\varepsilon\| . \|J_\varepsilon^{-1}\| < 1 + \varepsilon$, and by $\pi : X^{**} \to X$ the L-projection. Let $P_\varepsilon = J_\varepsilon \pi J_\varepsilon^{-1} : Y_\varepsilon^{**} \to Y_\varepsilon$. It follows from [13], Lemma IV.1.4, and the w^*-closedness of the L-complement of ℓ_1 in ℓ_1^{**} that

$$(12) \qquad \lim_{\varepsilon \to 0} (\sup \{\|y\|; y \in Y_\varepsilon \cap \overline{B_{\operatorname{Ker}(P_\varepsilon)}}^{*}\}) = 0$$

and (12) and the Banach-Dieudonné theorem easily imply that $\mathrm{Ker}(\pi)$ is w^*-closed. Since $\mathrm{Ker}(\pi)_\perp = X^\#$, it follows as above that $X = (X^\#)^*$ and (i) holds. □

IV. Examples

We first exhibit examples showing that we cannot dispense in Theorem 3.3 with the assumption that τ_m is locally convex on B_X.

Theorem 4.1. *There exist subspaces X of L^1 such that B_X is τ_m-compact, but not locally convex. That is, $X^\#$ fails to separate X and B_X does not satisfy the equivalent conditions of Lemma* 2.7.

Proof. We will actually provide two examples of this phenomenon. Our first construction relies heavily on the proof of [2], Th. 1.2. Upon reading this construction, M. Talagrand ([28]) produced the second (and simpler) approach, and he kindly allowed us to reproduce it here.

Example 4.1.1. We work for convenience with $\Omega = [0,1]^{\mathbb{N}}$ equipped with the natural product probability $\mathbb{P}$.

For each $p \in (1,2]$, we let $Y_p \in L^1([0,1])$ be a symmetric p-stable r.v. so that $\|Y_p\|_1 = 1$ and whose characteristic function $\hat{Y}_p$ satisfies

$$\hat{Y}_p(t) = \exp(-c_p|t|^p)$$

for some $c_p > 0$. Then ([2], Lemma 1.8) we have

$$(1) \qquad \lim_{p \to 1} c_p = 0\,,$$

$$(2) \qquad \tau_m - \lim_{p \to 1} (Y_p) = 0\,.$$

For each $j \in \mathbb{N}$ we let $Z_j = Y_{p_j} \circ \pi_j$, where $\pi_j = [0,1]^{\mathbb{N}} \to [0,1]$ is the jth coordinate map, and $p_j > 1$ will be chosen later. The Z_j's are symmetric p_j-stable independent elements of $L^1(\Omega)$ with $\|Z_j\|_1 = 1$.

Set now $X_j = |Z_j|$ and $U_j = X_j - \mathbb{1}$.

The space X is

$$X = \overline{\mathrm{span}}^{\|\cdot\|}\{\mathbb{1}, \{U_j\}_{j \geq 1}\}$$

with a proper choice of the p_j's.

If $Z_1, Z_2, \ldots, Z_k$ are p-stable, then $\{Z_i, 1 \leqq i \leqq k\}$ is isometrically equivalent to the canonical basis of ℓ_p^k (see [2], p. 61). By [2], Lemma 1.5 there is a constant K which is independent of the choice of the p_j's, and such that

$$K^{-1}\left\| \sum_{j=1}^{n} \alpha_j U_j \right\|_1 \leqq \left\| \sum_{j=1}^{n} \alpha_j Z_j \right\|_1 \leqq K \left\| \sum_{j=1}^{n} \alpha_j U_j \right\|_1 \tag{3}$$

for all $n \geqq 1$ and scalars (α_j). Following [2], we observe that if

$$V = \sum_{k=1}^{n} a_k U_k \tag{4}$$

with $p_1 = p_2 = \cdots = p_n = p$ and $\|V\|_1 \leqq 1$, we have for all $t \in \mathbb{R}$

$$\begin{aligned} |\hat{V}(t)| &= \prod_{k=1}^{n} |(X_k - \mathbb{1})\hat{\ }(a_k t)| \\ &= \prod_{k=1}^{n} ||Z_k|\hat{\ }(a_k t)| \\ &\geqq \prod_{k=1}^{n} \hat{Z}_k(a_k t) \end{aligned}$$

since Z_j is symmetric. Hence for all $t \in [-1, 1]$ we have by (3)

$$\begin{aligned} |\hat{V}(t)| &\geqq \exp\left(-c_p\left(\sum_{k=1}^{n} |a_k|^p |t|^p\right)\right) \\ &\geqq \exp(-c_p K^p). \end{aligned} \tag{5}$$

By (1) and (5), there exists for any $\delta > 0$ some $\alpha > 0$ such that $p \in (1, 1+\alpha)$ implies

$$|\hat{V}(t)| > 1 - \delta$$

for all $t \in [-1, 1]$. By [2], Lemma 1.6, it follows that for any $\varepsilon > 0$, there is $\alpha > 0$ such that $p \in (1, 1+\alpha)$ implies

$$\tau_m - \operatorname{dist}(V, \operatorname{span}[\mathbb{1}]) < \varepsilon \tag{6}$$

for all V as in (4) with $\|V\|_1 \leqq 1$. Observe that $\alpha = \alpha(\varepsilon)$ does not depend upon n. By the weak law of large numbers, we have

$$\lim_{n \to +\infty} \left\| \frac{1}{n}\left(\sum_{i=1}^{n} X_i\right) - \mathbb{1} \right\|_1 = 0. \tag{7}$$

We choose the p_j's as follows: by (6) and (7), there exists a strictly increasing sequence (b_k) in $\mathbb{N}$ with $b_1 = 1$, and $p_k > 1$ with $\lim(p_k) = 1$, such that if we denote $I_k = [b_k, b_{k+1}) \cap \mathbb{N}$, and if we pick Z_i to be p_k-stable if $i \in I_k$, then

(a) for all $V \in \operatorname{span}\{U_i, i \in I_k\}$ with $\|V\| \leqq 1$,

$$\tau_m - \operatorname{dist}(V, \operatorname{span}(\mathbb{1})) < 2^{-k}, \tag{8}$$

(b)

$$\left\| \frac{1}{|I_k|}\Big(\sum_{i \in I_k} X_i\Big) - \mathbb{1} \right\|_1 < \frac{1}{k}. \tag{9}$$

Note that by (2) we have

$$\tau_m - \lim(X_i) = 0. \tag{10}$$

The space X is now well-defined. Let us check that B_X is τ_m-compact. First, since the U_j's are independent with mean zero, the sequence $\{U_i; i \geqq 1\}$ is 2-unconditional and thus $\mathscr{B} = \{\mathbb{1}\} \cup \{U_i; i \geqq 1\}$ is a C-unconditional basis of X, for some $C \in \mathbb{R}$ (in fact $C = 3$, see Proposition 4.2 below).

Any $V \in X$ has a unique expansion on $\mathscr{B}$, denoted

$$V = \alpha_0(V)\mathbb{1} + \sum_{j=1}^{\infty} \alpha_j(V) U_j.$$

For any sequence $\{V_n\}$ in B_X, there is a subsequence, still denoted $\{V_n\}$, such that

$$\lim_n \alpha_j(V_n) = \alpha_j \tag{11}$$

exists for all $j \geqq 0$. Since $\mathscr{B}$ is boundedly complete, we can define

$$W = \alpha_0 \mathbb{1} + \sum_{j=1}^{\infty} \alpha_j U_j$$

and $W \in X$, with $\|W\| \leqq C$. An application of (8) to $(C^{-1}V)$ shows that for any $\varepsilon > 0$, we can find $k(\varepsilon) \geqq 1$ such that for all $V \in B_X$

$$\tau_m - \operatorname{dist}\left(\sum_{j=b_{k(\varepsilon)}}^{+\infty} \alpha_j(V) U_j, \operatorname{span}(\mathbb{1}) \right) < \frac{\varepsilon}{4}. \tag{12}$$

It follows from (11) and (12) that there exists $N(\varepsilon) \in \mathbb{N}$ such that for all $n \geqq N(\varepsilon)$

$$\tau_m - \operatorname{dist}(V_n - W, \operatorname{span}[\mathbb{1}]) < \varepsilon.$$

We can therefore write

$$V_n = W + \gamma_n \mathbb{1} + H_n \tag{13}$$

with (γ_n) scalars and

$$\tau_m - \lim(H_n) = 0. \tag{14}$$

By (13) and (14), we have $\overline{\lim}|\gamma_n| \leqq 1 + C$. Therefore there exists γ scalar and a subsequence $\{V_n'\}$ of $\{V_n\}$ such that

$$\tau_m - \lim(V_n') = W + \gamma \mathbb{1}.$$

Clearly $(W + \gamma \mathbb{1}) \in X$. Moreover $\| W + \gamma \mathbb{1} \|_1 \leqq 1$, since $\| . \|_1$ is τ_m – l.s.c. This shows the τ_m-compactness of B_X.

Finally we have

$$\mathbb{1} \in \cap \{x^{*-1}(0); x^* \in X^\#\}.$$

Indeed for any $x^* \in X^\#$ we have by (9)

$$x^*(\mathbb{1}) = \lim_k |I_k|^{-1} \Big(\sum_{i \in I_k} x^*(X_i) \Big).$$

But by (10) and $x^* \in X^\#$, we have $\lim_i x^*(X_i) = 0$, hence $x^*(\mathbb{1}) = 0$.

Example 4.1.2 ([28]). This self-contained approach is similar but aims at avoiding the use of the crucial inequality (3). We work as above on $\Omega = [0,1]^{\mathbb{N}}$ and write $t \in \Omega$ as $t = (t_i)$. We let

$$U_i(t) = q t_i^{q-1} - 1 = W_q(t_i)$$

where $q \in (0,1)$ is a parameter that will be chosen small enough later. Observe that W_q has mean zero for all $q > 0$. We claim that if $U_i(t) = W_q(t_i)$ for $1 \leqq i \leqq n$ with $q \in (0, 1/2)$ and

$$\Big\| \sum_{i=1}^n \alpha_i U_i \Big\|_1 \leqq 1$$

then

(15) $$S = \sum |\alpha_i|^{1/1-q} \leqq 1000.$$

In order to see this, we let

(16) $$a_i = \frac{|\alpha_i|^{1/1-q}}{10 S}$$

and define

$$A_i = \{t \in \Omega; t_i \leqq a_i\}$$

and

$$A_i' = \bigcup_{j \neq i} A_j; \quad B_i = A_i \setminus A_i'.$$

We compute

$$\int_{B_i} \operatorname{sign}(\alpha_i) \Big(\sum_{j=1}^n \alpha_j U_j \Big) d\mathbb{P} \geqq |\alpha_i| \int_{B_i} U_i d\mathbb{P} - \sum_{j \neq i} |\alpha_j| \Big| \int_{B_i} U_j d\mathbb{P} \Big|$$

and by independence

$$\begin{aligned}\int_{B_i} U_i \, d\mathbb{P} &= \int \mathbb{1}_{(A_i')^c} \cdot \mathbb{1}_{A_i} U_i \, d\mathbb{P} = \mathbb{P}\big((A_i')^c\big) \int_{A_i} U_i \, d\mathbb{P} \\ &\geqq \frac{9}{10} \int_{A_i} U_i \, d\mathbb{P} \\ &= \frac{9}{10} \int_0^{a_i} (q t^{q-1} - 1) \, dt \\ &= \frac{9}{10} (a_i^q - a_i) \geqq \frac{3}{5} a_i^q\end{aligned}$$

since $\sum a_i \leqq 1/10$ by (16) and $q \in (0, 1/2)$ so $a_i^{q-1} > 3$. Using independence as above, we find

$$\int_{B_i} U_j \, d\mathbb{P} = \Big(\int_{A_j^c} U_j \, d\mathbb{P}\Big) . \mathbb{P}\Big(A_i \cap \Big(\bigcap_{\substack{\ell \neq i \\ \ell \neq j}} A_\ell^c\Big)\Big),$$

hence

$$\begin{aligned}\Big|\int_{B_i} U_j \, d\mathbb{P}\Big| &\leqq \mathbb{P}(A_i) \Big|\int_{A_j^c} U_j \, d\mathbb{P}\Big| \\ &\leqq a_i \Big|\int_{A_j^c} U_j \, d\mathbb{P}\Big| \leqq a_i a_j^q .\end{aligned}$$

So we have

$$\int_{B_i} \operatorname{sign}(\alpha_i) \Big(\sum_{j=1}^n \alpha_j U_j\Big) d\mathbb{P} \geqq \frac{1}{4} |\alpha_i| a_i^q - a_i \Big(\sum_{j \neq i} |\alpha_j| a_j^q\Big).$$

Since

$$1 \geqq \Big\| \sum_{i=1}^n \alpha_i U_i \Big\|_1 \geqq \sum_{i=1}^n \int_{B_i} \operatorname{sign}(\alpha_i) \Big(\sum_{j=1}^n \alpha_j U_j\Big) d\mathbb{P}$$

it follows that

$$\begin{aligned}1 &\geqq \frac{1}{2} \sum_{i=1}^n |\alpha_i| a_i^q - \Big(\sum_{i=1}^n |a_i|\Big)\Big(\sum_{j=1}^n |\alpha_j| a_j^q\Big) \\ &\geqq \frac{1}{10} \Big(\sum_{i=1}^n |\alpha_i| a_i^q\Big).\end{aligned}$$

But we have

$$\sum_{i=1}^n |\alpha_i| a_i^q = \frac{\sum_{i=1}^n |\alpha_i|^{1/1-q}}{(10 S)^q} = \frac{S^{1-q}}{10^q},$$

therefore

$$S \leqq 10^{1/1-q} 10^{q/1-q}$$

which shows (15) since $q < \frac{1}{2}$.

We claim now that if

$$Y = \sum \alpha_i U_i$$

with

$$\sum |\alpha_i|^{1/1-q} \leqq 1000$$

for some $q \in (0, 1/4)$, then

$$Y = Y_1 + Y_2 + Y_3 \tag{17}$$

where

$$\mathbb{P}(Y_1 \neq 0) \leqq Kq\,, \tag{18}$$

$$\|Y_2\|_2^2 \leqq Kq\,, \tag{19}$$

$$Y_3 \in \mathbb{R}\mathbb{1}\,, \tag{20}$$

there and below, K denotes a universal constant (which may differ from line to line).

Indeed let

$$b_i = q|\alpha_i|^{1/1-q}$$

so that

$$\sum b_i \leqq 1000\,q \tag{21}$$

and define

$$Y_1 = \sum b_i U_i \mathbb{1}_{\{t_i \leqq b_i\}}$$

and with

$$Z_i = U_i \mathbb{1}_{\{t_i \geqq b_i\}}$$

let

$$Y_2 = \sum \alpha_i (Z_i - (\mathbb{E} Z_i).\mathbb{1})$$

and

$$Y_3 = (\sum \alpha_i \mathbb{E} Z_i)\mathbb{1}\,.$$

With this notation, (17) and (20) are clear and (18) follows from (21). To show (19), we estimate

$$\mathbb{E}((Z - \mathbb{E} Z)^2) = \mathbb{E}(Z^2) - \mathbb{E}(Z)^2$$

where $Z = W_q \mathbb{1}_{\{t \geqq b\}}$, $b = q|\alpha|^{1/1-q}$ with $|\alpha| \leqq 1000$ and $q \in (0, 1/4)$.

$$\mathbb{E}(Z^2) = \int_b^1 (q^2 t^{2q-2} - 2qt^{q-1} + 1)\,dt$$

$$= \frac{q^2}{2q-1} t^{2q-1} - 2t^q + t]_b^1$$

$$= \frac{q^2}{2q-1} - 1 - \frac{q^2}{2q-1} b^{2q-1} + 2b^q - b$$

and

$$(\mathbb{E}Z)^2 = \left(\int_b^1 (qt^{q-1} - 1)\,dt\right)^2 = (b - b^q)^2$$
$$= b^2 + b^{2q} - 2b^{q+1}.$$

We claim that

$$\mathbb{E}((Z - \mathbb{E}Z)^2) \leqq Kq^2 b^{2q-1}. \tag{22}$$

Indeed we have

$$\mathbb{E}((Z - \mathbb{E}Z)^2) = \frac{q^2}{2q-1} - \frac{q^2}{2q-1} b^{2q-1} - (1 - b^q)^2 - b - b^2 + 2b^{q+1} \tag{23}$$
$$\leqq \frac{q^2}{1-2q} b^{2q-1} + 2b^{q+1}$$

since $(2q-1) < 0$. We observe now that

$$b^{q+1} \leqq Kq^2 b^{2q-1}. \tag{24}$$

Indeed

$$q^2 b^{2q-1} = b^{q+1}(q^2 b^{q-2}) = b^{q+1}(q^q |\alpha|^{(q-2)/(1-q)})$$

and we have

$$q^q \geqq \exp(-1/e)$$

and

$$\beta = \frac{q-2}{1-q} > -3$$

hence

$$|\alpha|^\beta \geqq 10^{-9}.$$

This shows (24). Since $q \in (0, 1/4)$, we have

$$\frac{q^2}{1-2q} b^{2q-1} \leqq 2q^2 b^{2q-1} \tag{25}$$

and (22) follows from (23), (24) and (25). We now come back to the proof of (19). By (22), we have

$$\mathbb{E}(Y_2^2) = \sum \alpha_i^2 \mathbb{E}((Z_i - \mathbb{E}Z_i)^2)$$
$$\leqq Kq^2 \sum \alpha_i^2 b_i^{2q-1}$$
$$= Kq^2 q^{2q-1} \sum \alpha_i^2 |\alpha_i|^{\frac{2q-1}{1-q}}$$

$$= Kq^{1+2q}\sum|\alpha_i|^{1/1-q}$$
$$\leqq Kq$$

and (19) is shown. Note that (19) implies

$$(26) \qquad Kq^{1/2} \geqq \|Y_2\|_2 \geqq \|Y_2\|_1 .$$

We can now conclude by following the lines of the proof of Example 4.1.1. Pick $k \geqq 1$ a positive integer. Using (15), (18), (20) and (26), we find $\varepsilon_k \in (0, 1/4)$ such that if $0 < q < \varepsilon_k$ and $U_i(t) = W_q(t_i)$, we have for any $n \geqq 1$ and any (α_i) with

$$\left\| \sum_{i=1}^{n} \alpha_i U_i \right\|_1 \leqq 1$$

that

$$(27) \qquad \tau_m - \operatorname{dist}\left(\left(\sum_{i=1}^{n} \alpha_i U_i\right), \mathbb{1}\right) < 2^{-k} .$$

Fixing $q = q'_k \in (0, \varepsilon_k)$, we find by the weak law of large numbers $n \geqq 1$ such that if $X_i = U_i + \mathbb{1}$,

$$(28) \qquad \left\| \frac{1}{n}\left(\sum_{i=1}^{n} X_i\right) - \mathbb{1} \right\|_1 < \frac{1}{k} .$$

The inequalities (27) and (28) are identical to (8) and (9) in the construction of example 4.1.1. It is now clear that we can achieve the construction along the previous lines, by letting

$$U_i(t) = W_{q_i}(t_i)$$

with $q_i = q'_k$ for $i \in [c_k, c_{k+1})$ with (c_k) properly chosen. □

A careful examination of Examples 4.1.1 and 4.1.2 provides the following statement.

Theorem 4.2. *There exists a subspace X of L^1 which satisfies the following conditions*:

(1) *The unit ball B_X of X is τ_m-compact. In particular X has the* 1-*strong Schur property.*

(2) $\mathbb{1}_\Omega$ *belongs to the weak closure of any τ_m-neighbourhood of* 0 *in B_X.*

(3) *For any approximating sequence $\{R_n\}$ of finite rank operators, one has*

$$\underline{\lim}\, \|I - 2R_n\| \geqq 3 .$$

(4) *X has a monotone unconditional basis with unconditional constant $C = 3$.*

(5) *X is isomorphic to an ℓ_1-sum of finite dimensional spaces. In particular X is isomorphic to a subspace of ℓ_1.*

(6) *There exists $\alpha > 1$ such that $d(X, Y) > \alpha$ for any subspace Y of ℓ_1.*

Note that (2) implies that B_X is not τ_m-locally convex, that (3) implies that the unconditional constant of any unconditional basis is at least 3, and that (4) implies that X has the metric approximation property.

Proof. The space X constructed in Example 4.1.1 satisfies (1). By (9), (10) and Lemma 2.6 we have (2) and also

$$\mathbb{1}_\Omega \in X \cap \overline{B_{X_s}}^*$$

and then Lemma 2.3 shows (3).

For showing (4), we use the classical

Fact 4.3. If X and Y are independent random variables and $\mathbb{E}(Y) = 0$, then $\|X + Y\|_1 \geqq \|X\|_1$.

Indeed we have

$$\begin{aligned} \|X+Y\|_1 &= \iint |x+y| \, d\mu_X(x) \, d\mu_Y(y) \\ &\geqq \int \left| \int (x+y) \, d\mu_Y(y) \right| d\mu_X(x) \\ &\geqq \int |x| \, d\mu_X(x) = \|X\|_1 . \end{aligned}$$

We observe now that the basis $\mathscr{B} = \{\mathbb{1}\} \cup \{U_i; i \geqq 1\}$ consists of independent r.v. with $\mathbb{E}(U_i) = 0$, hence if $0 \leqq k < n$ we have by Fact 4.3 and with $U_0 = \mathbb{1}$

$$\left\| \sum_{j=0}^{k} \alpha_j U_j \right\| \leqq \left\| \sum_{j=0}^{n} \alpha_j U_j \right\|$$

hence $\mathscr{B}$ is a monotone basis. If now $\varepsilon_j \in \{-1, 1\}$ and if we let, as in the proof of Lemma 2(a) in [25]

$$X = \sum_{\{i; \varepsilon_0 \varepsilon_i = 1\}} \varepsilon_i \alpha_i U_i$$

and

$$Y = \sum_{\{i; \varepsilon_0 \varepsilon_i = -1\}} \varepsilon_i \alpha_i U_i .$$

Then applying again Fact 4.3:

$$\begin{aligned} \left\| \sum_{i=0}^{\infty} \varepsilon_i \alpha_i U_i \right\|_1 &= \|X+Y\|_1 \\ &\leqq \|X\|_1 + \|Y\|_1 \\ &\leqq 2\|X\|_1 + \|X-Y\|_1 \\ &\leqq 3\|X-Y\|_1 \\ &= 3 \left\| \sum_{i=0}^{\infty} \alpha_i U_i \right\|_1 . \end{aligned}$$

Therefore the unconditional constant of $\mathscr{B}$ is at most 3, and thus is exactly 3 by condition (3). This shows (4).

Finally, condition (5) follows from (1), (4) and [16], Appendix, and (6) follows from Proposition 3.9. □

Remark 4.4. We have no example of a subspace of L^1 satisfying the conditions of Theorem 4.2 and which is isomorphic to ℓ_1. Let us outline why the above technique fails to provide isomorphic copies of ℓ_1. Clearly we may replace (9) by

$$\lim_k \left\| \frac{1}{|I_k|} \sum_{i\in I_k} X_i - \mathbb{1} \right\|_1 = 0 . \tag{29}$$

By [2], p. 62, we have for some absolute constant K

$$K \left\| \frac{1}{n} \sum_{i=1}^{n} X_i - \mathbb{1} \right\| \geqq n^{\frac{1}{p}-1}$$

with the X_i's independent and p-stable. To ensure (29), we must therefore pick $n = n(p)$ such that

$$\lim_{p\to 1} n(p).e^{-p/p-1} = +\infty . \tag{30}$$

Thus if the spaces $\ell_p^{n(p)}$, with $n(p)$ satisfying (30), are not uniformly complemented in isomorphs of ℓ_1, the construction provides a space which is not isomorphic to ℓ_1. Applying [26] (see [29], p. 259) to $\ell_q^{n(p)}$, with $p^{-1} + q^{-1} = 1$, shows that if X is isomorphic to ℓ_1 and

$$\pi = X \to \ell_p^{n(p)}$$

is a projection, then

$$\operatorname{dist}(\ell_p^{n(p)}, \ell_1^{n(p)}) \leqq C[1 + \log\|\pi\|]^{7/2}\|\pi\| . \tag{31}$$

But we have (see [29])

$$\operatorname{dist}(\ell_p^k, \ell_1^k) = k^{1-\frac{1}{p}} .$$

Therefore if $n(p)$ satisfies (30) we have

$$\lim_{p\to 1} (\ell_p^{n(p)}, \ell_1^{n(p)}) = +\infty$$

and this contradicts (31). It follows that our Examples 4.1.1 are not isomorphic to ℓ_1.

Note that this conclusion would also follow from (30) and the uniqueness, up to equivalence, of the unconditional basis of ℓ_1 (see [20]).

Our last statement is a simple observation on (UMAP) for natural spaces which could qualify for this property.

Proposition 4.5. (1) *Let X a separable predual of a von Neumann algebra. Then X has* (UMAP) *if and only if X is isometric to $(\Sigma \oplus N(H_n))_1$, where the H_n's are finite dimensional Hilbert spaces.*

(2) *The Hardy space $H^1(D)$ fails* (UMAP).

(3) *There exist infinite subsets Λ of $\mathbb{Z}$ such that $L^1_\Lambda(\mathbb{T})$ has* (UMAP).

Proof. It follows from [7], Theorem VII. 8, that a predual of a von Neumann algebra has the Radon-Nikodym property if and only if it is isometric to an ℓ_1-sum of spaces $N(H_i)$ of nuclear operators on Hilbert spaces H_i. By Proposition 2.8 any separable space not containing $c_0(\mathbb{N})$ with (UMAP) is a separable dual and therefore has the (RNP). Since (UMAP) is clearly stable under contractive projections, (1) will follow if we show that $N(H)$ fails (UMAP) when H is an infinite-dimensional separable Hilbert space. To see this, we denote $\{e_n, n \geqq 1\}$ an orthonormal basis of H, and we let

$$T_n = e_1 \otimes e_n + e_n \otimes e_1 + e_n \otimes e_n .$$

An easy computation leads to

$$\| e_1 \otimes e_1 + T_n \|_{K(H)} = \sqrt{2}$$

and

$$\| e_i \otimes e_1 - T_n \|_{K(H)} = 1 .$$

Clearly

$$w - \lim (T_n) = 0$$

in the space $K(H)$ of compact operators on H although

$$\lim (\| e_1 \otimes e_1 + T_n \| - \| e_1 \otimes e_1 - T_n \|) \neq 0$$

and by the proof of Lemma 2.3 this shows that $K(H)$ fails (UMAP). It now follows from [10], Th. 9.2, that $N(H) = K(H)^*$ fails (UMAP).

To prove (2), we exhibit as above a sequence (f_n) in $H^1(D)$ such that $w - \lim(f_n) = 0$ but $(\|f+f_n\| f-f_n\|)$ fails to converge to 0 for some $f \in H^1(D)$. For doing so we reproduce an argument from [30]: for any $n \geqq 1$, we have

$$\| 1 + 2e^{int} + e^{2int} \|_1 = \| 2(1 + \cos nt) \|_1 = 2$$

while

$$\| 1 - 2e^{int} - e^{2int} \|_1 = \| 2(1 + i \sin(nt)) \|_1$$

and since

$$|1 + i \sin(nt)| \geqq 1 + (\sqrt{2} - 1) \sin^2(nt)$$

we have

$$\liminf \| 1 - 2e^{int} - e^{2int} \| > 2$$

and this concludes the proof of (2).

We denote $e_n(z) = z^n$ for $n \in \mathbb{Z}$ and $z \in \mathbb{T}$. For proving (3), we observe the

Fact 4.6. Let F be a finite subset of $\mathbb{Z}$, and pick $\varepsilon > 0$. Then there exists an odd integer N such that if $P_N = \{(2k+1)N; k \in \mathbb{Z}\}$, then for all $f \in L^1_F(\mathbb{T})$, $g \in L^1_{P_N}(\mathbb{T})$,

$$\|f+g\| \leqq (1+\varepsilon)\|f-g\|. \tag{32}$$

Fact 4.6 is shown by approximating an $\varepsilon/3$-net of the unit ball of $L^1_F(\mathbb{T})$ by step functions (up to $\varepsilon/3$), by checking (32) when f is a step function (using the fact that $\|1+g\|_1 = \|1-g\|_1$ when g is odd) and by picking N large enough.

To construct Λ such that $L^1_\Lambda(\mathbb{T})$ has (UMAP), we proceed by induction. We pick $\varepsilon_n > 0$ with $\lim(\varepsilon_n) = 0$. For any finite subset F of $\mathbb{Z}$, we denote by π_F the projection from $L^1(\mathbb{T})$ to $L^1_F(\mathbb{T})$ whose restriction to trigonometric polynomials satisfies

$$\pi_F\left(\sum{}' a_n e_n\right) = \sum_{n \in F} a_n e_n .$$

Starting with $n_1 = 0$, we find $N_1 \geqq 1$ such that (32) is satisfied with $F_1 = \{n_1\}$ and $\varepsilon = \varepsilon_1$. We let $n_2 = N_1$ and $F_2 = \{n_1, n_2\}$. We find $N_2 \geqq 1$ such that (32) is satisfied with

$$F_2 = \{n_1, n_2\} \quad \text{and} \quad \varepsilon = \varepsilon_2 ,$$

and we let $n_3 = N_1 N_2$. If $F_k = \{n_1, \ldots, n_k\}$ is constructed, we pick N_k such that (32) is satisfied with F_k and $\varepsilon = \varepsilon_k$, and we let $n_{k+1} = N_1 N_2 \cdots N_k$. If we let now

$$\Lambda = \{n_i; i \geqq 1\}$$

we clearly have for all $k \geqq 1$

$$\|I - 2\pi_{F_k}\|_{L^1_\Lambda} \leqq (1+\varepsilon_k)$$

and this shows (UMAP) for L^1_Λ since $\{\pi_{F_k}; k \geqq 1\}$ is an approximating sequence for this space.

Note that it follows from Proposition 4.5.2 and [10], Th. 8.3, that the space

$$K(\mathscr{C}(\mathbb{T})/A_0(D))$$

of compact operators on the natural predual $\mathscr{C}(\mathbb{T})/A_0(D) (= VMO)$ of $H^1(D)$ is not an h-ideal in the space $L(\mathscr{C}(\mathbb{T})/A_0(D))$ of bounded operators.

We do not know how lacunary an infinite subset Λ of $\mathbb{Z}$ such that $L^1_\Lambda(\mathbb{T})$ has (UMAP) must be. Note that it follows from [10], Th. 9.2, and its proof that if $L^1_\Lambda(\mathbb{T})$ has (UMAP) then there exists an approximating sequence R_n of convex combinations of the operators $C_k(f) = f * \sigma_k$ of convolution with Fejer kernels such that $\lim \|I - 2R_n\| = 1$. It follows that $L^1_S(\mathbb{T})$ has (UMAP) for any subset S of Λ.

We conclude our work with two questions.

Question 1. Let X be a subspace of $c_0(\mathbb{N})$ such that X^* is isomorphic to a subspace of L^1. Is the space X isomorphic to a quotient of $c_0(\mathbb{N})$?

Note that Proposition 3.2 provides a positive answer when we assume that X has (MAP). By [21], there exists a subspace X of $\mathscr{C}(\omega^\omega)$ such that X^* embeds into L^1 but not into ℓ_1.

Question 2. Does there exist a subspace X of L^1 whose unit ball B_X is compact for the topology τ_m of convergence in measure but which fails the Radon-Nikodym property?

By [2], this question has a positive answer if we replace "compact" by "relatively compact". Note that according to [2], there exists a subspace of L^1 with a τ_m-relatively compact unit ball and the R.N.P., but which does not embed into ℓ_1.

The most ambitious problem in this direction would be to find X in L^1 with B_X τ_m-compact and $\mathrm{Ext}(B_X) = \emptyset$. Let us mention that such a space would satisfy $X^\# = \{0\}$. Indeed, since $X^{**} = X \oplus_1 X_s$ and $\mathrm{Ext}(B_X) = \emptyset$, we would have $\mathrm{Ext}(B_{X^{**}}) \subseteq X_s$. Then applying Krein-Milman theorem to $B_{X^{**}}$ would show that X_s is w^*-dense in $B_{X^{**}}$, and then $X^\# = (X_s)_\perp$ implies $X^\# = \{0\}$. Let us recall that although there exists Banach spaces Z such that B_Z is compact for a t.v.s. topology but has no extreme point ([23]), it is not known whether there exists a separable Banach space with this property. Actually, Theorem 4.1 provides the first examples of separable Banach spaces whose unit ball is compact but not locally convex for a Hausdorff topological vector space topolgy.

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