J. reine angew. Math. 461 (1995), 137—178

Journal für die reine und angewandte Mathematik

Property (M), M-ideals, and almost isometric structure of Banach spaces

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1. Introduction

In his investigation of M-ideals of compact operators (we will define this notion shortly) the first-named author [30] introduced two properties of Banach spaces, called (M) and (M^*), and pointed out their relevance in Banach space theory. Property (M) is the requirement that

$$\limsup \|u + x_n\| = \limsup \|v + x_n\|$$

whenever $\|u\| = \|v\|$ and $x_n \to 0$ weakly, and (M^*) means that

$$\limsup \|u^* + x_n^*\| = \limsup \|v^* + x_n^*\|$$

whenever $\|u^*\| = \|v^*\|$ and $x_n^* \to 0$ weak*. In this paper we will continue studying (M), (M^*) and some of their variants, especially in the context of L_p-spaces. It turns out that apart from having some impact on M-ideals these properties are significant in the general theory as well and lead to new results concerning the almost isometric structure of Banach spaces.

We now describe the contents of this paper in detail. In section 2 we first deal with the relation between (M) and (M^*). It is relatively easy to check that (M^*) implies (M) (see [30], Prop. 2.3), and it is clear that the converse is false (consider ℓ_1). However in Theorem 2.6 we prove that this is essentially the only counterexample: If X does not contain ℓ_1 and has (M), then X has (M^*). It should be noted that, due to the weak* compactness of the dual unit ball, (M^*) is the technically more convenient property of the two to work with. In the second part of section 2 we investigate the approximation property in Banach spaces with (M^*). We show in Theorem 2.7 that a separable Banach space with the metric compact approximation property and property (M^*) actually admits of a sequence (L_n) of compact operators such that $L_n \to \mathrm{Id}$ strongly (that is, $L_n x \to x$ for all $x \in X$), $L_n^* \to \mathrm{Id}_{X^*}$ strongly and $\|\mathrm{Id} - 2L_n\| \to 1$. Here the norm condition on $\mathrm{Id} - 2L_n$ can be thought of as an unconditionality property since it is fulfilled if X has a 1-unconditional (shrinking) basis and L_n are the canonical finite rank projections associated to that basis.

These theorems enable us to improve the main result of [30] in Theorem 2.13: For a separable Banach space X, the space of compact operators $\mathscr{K}(X)$ is an M-ideal in $\mathscr{L}(X)$, the space of bounded operators, if and only if X does not contain a copy of ℓ_1, X has (M) and X has the metric compact approximation property.

In section 3 we study a particular version of property (M), namely property (m_p) which requires that

$$\limsup \|x + x_n\| = (\|x\|^p + \limsup \|x_n\|^p)^{1/p}$$

whenever $x_n \to 0$ weakly. (The modification for $p = \infty$ is obvious.) It is known from results in [30] and [46] that, for $1 < p \leqq \infty$, $\mathscr{K}(X \oplus_p X)$ is an M-ideal in $\mathscr{L}(X \oplus_p X)$ if and only if $\mathscr{K}(X)$ is an M-ideal in $\mathscr{L}(X)$ and X has (m_p). Such spaces have been investigated in [30], [46] and [47]; and it has been conjectured (cf. [20], p. 336) that a Banach space with the above properties must be almost isometric to a subspace of a quotient of an ℓ_p-sum of finite-dimensional spaces. (If $\mathfrak{Y}$ is a class of Banach spaces, we say that X is almost isometric to a member of $\mathfrak{Y}$ if for every $\varepsilon > 0$ there is a member Y_ε of $\mathfrak{Y}$ with Banach-Mazur distance to X less than $1 + \varepsilon$.) For separable Banach spaces not containing a copy of ℓ_1 a stronger result than this conjecture is proved in Theorems 3.2 and 3.3, namely, X has (m_p) if and only if X is almost isometric to a subspace of some ℓ_p-sum of finite-dimensional spaces if and only if X is almost isometric to a quotient of some ℓ_p-sum of finite-dimensional spaces. As a corollary we obtain an almost isometric version of a result due to Johnson and Zippin [28] that the class of subspaces of ℓ_p-sums coincides with the class of quotients of ℓ_p-sums. Previously the validity of the above conjecture has been proved only for special cases if $p = \infty$; see [57] and [20], p. 337. Theorem 3.5 now presents a necessary and sufficient condition for a Banach space to embed almost isometrically into c_0.

The following two sections deal with property (M) and in particular property (m_p) for subspaces of $L_p[0,1]$ (section 4) resp. for subspaces of the Schatten classes c_p of operators on a separable Hilbert space $\mathscr{H}$ (section 5). It turns out (Theorem 4.4) that a subspace X of L_p, $1 < p < \infty$, $p \neq 2$, has (m_p) if and only if its unit ball is compact for the topology inherited from L_1 if and only if X embeds almost isometrically into ℓ_p. Natural examples of this situation are the Bergman spaces. If $2 < p < \infty$, Theorem 4.4 allows us to deduce a more precise variant of a result of Johnson and Odell [26]: X embeds almost isometrically into ℓ_p if and only if X contains no hilbertian subspace (Corollary 4.6). In connection with the Bergman spaces we also consider the Bloch space β_0 and show that it is almost isometric to a subspace of c_0 and that $\mathscr{K}(\beta_0)$ is an M-ideal in $\mathscr{L}(\beta_0)$, thus answering a question raised in [36].

We derive similar results in the context of the Schatten classes c_p in section 5. Here $\ell_p(c_p^n)$ plays the rôle of ℓ_p, and for $X \subset c_p$ we now obtain that X has (m_p) if and only if X embeds almost isometrically into $\ell_p(c_p^n)$ (Theorem 5.2), and if $p > 2$ this happens if and only if X does not contain a hilbertian subspace (Corollary 5.4). This improves a result of Arazy and Lindenstrauss [5].

Section 6 is devoted to M-ideals of compact operators. Let us now recall the definition of an M-ideal. According to Alfsen and Effros [1], who introduced this notion in 1972, a subspace J of a Banach space E is called an M-ideal if there is an ℓ_1-direct decomposition of the dual space E^* of E into the annihilator $J^\perp$ of J and some subspace $V \subset E^*$:

$$E^* = J^\perp \oplus_1 V.$$

Since that time M-ideal theory has proved useful in Banach space geometry, approximation theory and harmonic analysis; see [20] for a detailed account.

A number of authors have studied the M-ideal structure in $\mathscr{L}(X, Y)$, the space of bounded linear operators between Banach spaces X and Y, with special emphasis on the question whether $\mathscr{K}(X, Y)$, the subspace of compact operators, is an M-ideal; see e.g. [7], [10], [11], [17], [30], [31], [34], [44], [46], [47], [53], [56] or Chapter VI in [20]. Examples of M-ideals of compact operators include $\mathscr{K}(\mathscr{H})$ for a Hilbert space $\mathscr{H}$ (this is due to Dixmier [13]), $\mathscr{K}(\ell_p, \ell_q)$ for $1 < p \leqq q < \infty$ [34], and $\mathscr{K}(X)$ for certain renormings of Orlicz sequence spaces with separable duals [30].

Necessary and/or sufficient conditions for $\mathscr{K}(X)$ to be an M-ideal in $\mathscr{L}(X)$ were studied for instance in [19], [30] and [56]. In section 6 we propose a notion of property (M) for a single operator $T: X \to Y$ and employ it in connection with the problem of giving necessary and sufficient conditions for $\mathscr{K}(X, Y)$ to be an M-ideal in $\mathscr{L}(X, Y)$. Such a condition is proved for $X = \ell_p$ in Corollary 6.4; it permits us to deduce that $\mathscr{K}(\ell_p, L_p[0,1])$ is an M-ideal for $2 < p < \infty$.

A list of open questions concludes the paper.

The results and proofs in this paper are – for the most part – formulated for the case of real scalars. They extend, however, to complex scalars and are in fact tacitly used in this context. Our notation largely follows [39]; in particular, B_X denotes the closed unit ball of a Banach space X. The symbol $d(X, Y)$ stands for the Banach-Mazur distance between two Banach spaces, i.e., $d(X, Y) = \inf \|T\| \, \|T^{-1}\|$ where the infimum extends over all surjective isomorphism $T: X \to Y$.

Acknowledgement. This work was done while the second-named author was visiting the University of Missouri, Columbia. It is his pleasure to express his gratitude to all those who made this stay possible.

2. Implications of properties (M) and (M^*)

The first objective of this section is to show that property (M) implies (M^*) for separable Banach spaces containing no copies of ℓ_1; the definitions of properties (M) and (M^*), introduced in [30], have been recalled in the introduction. Actually, we will also deal with particular versions of (M) and (M^*) that will be studied in more detail in the next sections, but we formulate them already here.

Definition 2.1. Let $1 \leqq p \leqq \infty$.

(a) A separable Banach space is said to have *property* (m_p) if

$$\limsup \|x + x_n\| = \|(\|x\|, \limsup \|x_n\|)\|_p$$

whenever $x_n \to 0$ weakly.

(b) A separable Banach space is said to have *property* (m_p^*) if

$$\limsup \|x^* + x_n^*\| = \|(\|x^*\|, \limsup \|x_n^*\|)\|_p$$

whenever $x_n^* \to 0$ weak*.

The following lemmas will lead to the first main result, Theorem 2.6.

Lemma 2.2. *Suppose X is a separable Banach space and that (x_n^*) is a bounded sequence in X^*. Let F be a finite-dimensional subspace of X^*. Then for any*

$$\alpha > \liminf d(x_n^*, F)$$

there exist a subsequence (u_n^*) of (x_n^*) and $f^* \in F$ so that $\|u_n^* - f^*\| \leqq \alpha$ for all n.

Proof. Obvious. □

Lemma 2.3. *Suppose X is a separable Banach space and that (x_n^*) is a weak*-null sequence in X^*. Then for any finite-dimensional subspace F of X^* we have*

$$\liminf d(x_n^*, F) \geqq \frac{1}{2} \liminf \|x_n^*\|.$$

Proof. Assume α is such that there exist a subsequence (u_n^*) of (x_n^*) and $f^* \in F$ with $\|u_n^* - f^*\| \leqq \alpha$. Since $w^*\text{-}\lim_{n\to\infty} (u_n^* - f^*) = -f^*$ we obtain $\|f^*\| \leqq \alpha$ and hence

$$\|u_n^*\| \leqq 2\alpha$$

for all n. The result follows by Lemma 2.2. □

Lemma 2.4. *Suppose X is a separable Banach space containing no copy of ℓ_1 and that (x_n^*) is a weak*-null sequence. Then there are a subsequence (u_n^*) of (x_n^*) and a weakly null sequence (w_n) in X with $\|w_n\| \leqq 1$ so that* $\lim \langle w_n, u_n^* \rangle \geqq \frac{1}{4} \liminf \|x_n^*\|$.

Proof. Let $\alpha = \liminf \|x_n^*\|$. By Lemma 2.3, we can pass to a subsequence (u_n^*) so that $\liminf d(u_n^*, \operatorname{lin}\{u_1^*, \dots, u_{n-1}^*\}) \geqq \alpha/2$. Hence we can find a sequence (x_n) in B_X so that $\liminf \langle x_n, u_n^* \rangle \geqq \alpha/2$ and $\langle x_n, u_k^* \rangle = 0$ for $k < n$. By passing to a subsequence we can suppose that (x_n) is weakly Cauchy by Rosenthal's ℓ_1-theorem [39], Th. 2.e.5. Let $w_n = \frac{1}{2}(x_n - x_{n+1})$, and the result follows. □

We say that a Banach space has the weak* Kadec-Klee property if the weak* and norm topologies agree on its dual unit sphere. (Occasionally this has been called property (**).)

Lemma 2.5. *Assume X is a separable Banach space containing no subspace isomorphic to ℓ_1 and with property (M). Then X has the weak* Kadec-Klee property, X^* has no proper norming subspace and X^* is separable.*

Proof. Suppose that (x_n^*) converges weak* to x^* and that $\lim\|x_n^*\| = \|x^*\| \neq 0$. We show that $\lim\|x_n^* - x^*\| = 0$. Suppose not. Then, after passing to some subsequence, we would have $\lim\|x_n^* - x^*\| = \alpha > 0$. By Lemma 2.4 there are a weakly null sequence (w_n) in B_X and a subsequence (u_n^*) of (x_n^*) so that $\liminf\langle w_n, x_n^* - x^*\rangle \geqq \alpha/4$. Hence

$$\liminf x_n^*(w_n) \geqq \alpha/4 .$$

Suppose $x \in X$ with $\|x\| = 1$. Then for any $t \geqq 0$ we have

$$\liminf x_n^*(x + tw_n) \geqq x^*(x) + t\alpha/4.$$

By property (M) we can pass to a subsequence (v_n) of (w_n) so that there is an absolute norm N on $\mathbb{R}^2$ with $N(1,0) = 1$ and $\lim\|u + tv_n\| = N(\|u\|, |t|)$ for any $u \in X$ and $t \in \mathbb{R}$. It follows that $N(1,t) - 1 \geqq \alpha|t|/4$. But this implies (cf. [30], Lemma 3.6) that ℓ_1 embeds into X contrary to our assumption. Hence X has the weak* Kadec-Klee property.

We also deduce that X^* has no proper norming subspace and must be separable. Indeed, if $Y \subset X^*$ is a norming (closed) subspace (i.e., $\|x\| = \sup_{y^* \in B_Y} |y^*(x)|$ for all $x \in X$), then B_Y is weak*-dense in B_{X^*}. Hence, for each $x^* \in X^*$, $\|x^*\| = 1$, there is a sequence $(y_n^*) \subset Y$, $\|y_n^*\| = 1$, such that $y_n^* \to x^*$ in the weak* sense. By the weak* Kadec-Klee property, $\|y_n^* - x^*\| \to 0$ and thus $x^* \in Y$, which proves that there are no proper norming subspaces. Now, if $A \subset X$ is a countable dense set, there is a countable set $B \subset X^*$ that norms A. Hence $\overline{\text{lin}}\, B$ is a separable norming subspace, and we infer that X^* is separable. □

Theorem 2.6. *Let X be a separable Banach space with property (M) and containing no copy of ℓ_1. Then X has property (M^*). Furthermore if X has property (m_p) where $1 < p \leqq \infty$, then X has property (m_q^*) where $1/p + 1/q = 1$.*

Proof. By Lemma 2.5, X^* is separable. It therefore suffices, for the first part, to prove that if (x_n^*) is a weak*-null sequence in X^* with the property that $\lim\|x^* + x_n^*\|$ exists for all $x^* \in X^*$, then the function

$$\phi(x^*) = \lim\|x^* + x_n^*\|$$

satisfies $\phi(x^*) = \phi(y^*)$ whenever $\|x^*\| = \|y^*\|$. For the second part we will need additionally to show that $\phi(x^*) = (\|x^*\|^q + \phi(0)^q)^{1/q}$.

We observe first that ϕ is a convex and norm-continuous function: in fact

$$|\phi(x^*) - \phi(y^*)| \leqq \|x^* - y^*\| .$$

We also note that $\phi(x^*) \geqq \|x^*\| - \phi(0)$. For $\tau \geqq 0$ we define $g(\tau) = \inf\{\phi(x^*) : \|x^*\| = \tau\}$. Then g is also continuous: in fact $|g(\tau) - g(\sigma)| \leqq |\tau - \sigma|$ and $g(\tau) \geqq \tau - g(0)$. It follows that g attains its minimum and that there exists τ_0 so that $g(\tau_0) \leqq g(\tau)$ for all τ with strict inequality if $\tau_0 < \tau$.

Now suppose that $\tau \geqq 0$ and u^* is a weak*-strongly exposed point of B_{X^*}; this means there exists $u \in B_X$ with $u^*(u) = 1$ such that $\lim\|u_n^* - u^*\| = 0$ if $\lim\|u_n^*\| = \lim u_n^*(u) = 1$.

The existence of such points follows from the separability of X^*, see e.g. [48], p. 80. Let $Z = \{x^* \in X^* : x^*(u) = 0\}$. We will define

$$\theta = \inf\{\phi(\tau v^*) : v^* \in u^* + Z\} .$$

Let (F_n) be an increasing sequence of finite-dimensional subspaces of Z so that $\bigcup F_n$ is dense in Z. We claim that $\liminf_{n\to\infty} d(\tau u^* + x_n^*, F_k) \geqq \theta$ for each k. Indeed if for some k this fails, a simple compactness argument gives a subsequence (ξ_n^*) of (x_n^*) and $f^* \in F_k$ so that $\lim \|f^* + \tau u^* + \xi_n^*\| < \theta$, i.e., $\phi(\tau u^* + f^*) < \theta$. This would contradict the choice of θ.

Now we may pass to a subsequence (y_n^*) of (x_n^*) so that we obtain

$$\liminf d(\tau u^* + y_n^*, F_n) \geqq \theta .$$

Hence we can find $y_n \in B_X$ so that $f^*(y_n) = 0$ for $f^* \in F_n$ and

$$\liminf(\tau u^*(y_n) + y_n^*(y_n)) \geqq \theta . \tag{1}$$

Clearly any weak* cluster point, say χ, of (y_n) in X^{**} satisfies $\chi|_Z = 0$ and so $\chi \in \operatorname{lin}\{u\}$. In particular, the sequence (y_n) is relatively weakly compact, and passing to a subsequence we can suppose that (y_n) converges weakly to αu for some α. We thus write $y_n = \alpha u + f_n$ where (f_n) is weakly null.

We first use the sequence to estimate ϕ. If $\|x\| \leqq 1$, then by property (M),

$$\limsup \|\alpha x + f_n\| \leqq \limsup \|\alpha u + f_n\| \leqq 1$$

and hence for any $x^* \in X^*$.

$$\phi(x^*) \geqq \alpha x^*(x) + \limsup y_n^*(f_n) .$$

Let $\beta = \limsup y_n^*(f_n)$; then

$$\phi(x^*) \geqq |\alpha| \|x^*\| + \beta \tag{2}$$

for all $x^* \in X^*$. Now if $v^* \in u^* + Z$, then $\|v^*\| \geqq v^*(u) = 1$, and so we obtain

$$\theta \geqq |\alpha|\tau + \beta .$$

On the other hand by (1)

$$\theta \leqq \liminf(\tau u^*(y_n) + y_n^*(y_n)) \leqq \alpha\tau + \beta .$$

We conclude that $\theta = \alpha\tau + \beta$ and $\alpha \geqq 0$ if $\tau > 0$.

At this point we specialize to the assumption that $\tau > \tau_0$. Since

$$\phi(x^*) \geqq \alpha \|x^*\| + \beta \geqq \beta$$

we also have by definition of τ_0 that $\beta \leqq g(\tau_0)$. On the other hand,

$$\theta \geqq \inf\{g(\sigma) : \sigma \geqq \tau\} > g(\tau_0),$$

and thus $\alpha > 0$. Now choose $v_n^* \in u^* + Z$ so that $\phi(\tau v_n^*) < \theta + \frac{1}{n}$. Then

$$\alpha\tau\|v_n^*\| + \beta < \theta + \frac{1}{n}$$

and so

$$\|v_n^*\| - 1 \leqq \frac{1}{\alpha\tau n}.$$

Since $v_n^*(u) = u^*(u) = 1$ we conclude that $\lim\|v_n^*\| = 1$, and by choice of u^*,

$$\lim\|v_n^* - u^*\| = 0.$$

Thus $\phi(\tau u^*) = \theta = \alpha\tau + \beta$. By (2) this implies that $\phi(\tau u^*) \leqq \phi(\tau v^*)$ whenever $\|v^*\| = 1$ and hence $\phi(\tau u^*) = g(\tau)$.

Now from the convexity of ϕ we have $\phi(\tau x^*) \leqq g(\tau)$ whenever x^* is in the convex hull C of the weak*-strongly exposed points of B_{X^*}. However since X^* is separable C is weak*-dense in B_{X^*} [48], p. 80. By the weak* Kadec-Klee property (Lemma 2.5) C is norm-dense in B_{X^*}. Hence $\phi(\tau x^*) \leqq g(\tau)$ whenever $\|x^*\| \leqq 1$. In particular $\phi(\tau x^*) = g(\tau)$ whenever $\|x^*\| = 1$ and $\tau > \tau_0$.

We can now conclude the argument for the first part. If $\tau \leqq \tau_0$ and $\|x^*\| = 1$ we have by the above argument $\phi(\tau x^*) \leqq g(\sigma)$ for any $\sigma > \tau_0$. Hence $\phi(\tau x^*) \leqq g(\tau_0)$ and consequently $\phi(\tau x^*) = g(\tau_0) = g(\tau)$. Thus in either case $\phi(\tau x^*) = g(\tau)$, and the first part is proved.

For the second part we observe that by what we have already shown $\phi(u^*) \leqq \phi(v^*)$ whenever $\|u^*\| \leqq \|v^*\|$. Therefore if u^* is a weak*-strongly exposed point of B_{X^*} and $\tau \geqq 0$, we can deduce that in the construction above we have $\phi(\tau u^*) = \theta = \alpha\tau + \beta$ without restriction on τ. Hence by (1) $\phi(\tau u^*) \leqq \|(\tau, \phi(0))\|_q \liminf\|(\alpha, f_n)\|_p$. Therefore

$$g(\tau) \leqq (\tau^q + g(0)^q)^{1/q}.$$

For the converse direction we perform the above construction for $\tau = 0$ to produce α, (f_n) so that $\|\alpha u + f_n\| \leqq 1$ and $g(0) = \lim y_n^*(f_n)$. Then $\limsup\|f_n\| \leqq 1$. Now for any $\tau > 0$ we have $\limsup\|\gamma u + \delta f_n\| \leqq \|(\gamma, \delta)\|_p$ and $g(\tau) \geqq \gamma\tau + \delta g(0)$ whenever $\|(\gamma, \delta)\|_p \leqq 1$ which yields $g(\tau) \geqq (\tau^q + g(0)^q)^{1/q}$. The proof is then complete. □

In the second part of this section we study the approximation property in Banach spaces with property (M^*). We shall show that in separable Banach spaces with (M^*) and the metric compact approximation property actually an unconditional version of this approximation property is valid. As in [30] we call a sequence (K_n) of compact operators a shrinking compact approximating sequence if $K_n \to \mathrm{Id}_X$ and $K_n^* \to \mathrm{Id}_{X^*}$ strongly.

Theorem 2.7. *Let X be a separable Banach space with property (M^*) and the metric compact approximation property. Then there exists a shrinking compact approximating sequence (L_n) such that*

$$\lim_{n\to\infty} \|L_n\| = \lim_{n\to\infty} \|\mathrm{Id}_X - 2L_n\| = 1 .$$

Proof. By Proposition 2.3 of [30] X is an M-ideal in X^{**}, X^* is separable, and a result of Godefroy and Saphar ([18], p. 678 and p. 681) shows that X has a shrinking compact approximating sequence (K_n) with $\|K_n\| = 1$. It suffices to show that for given $\varepsilon > 0$ we can find a shrinking compact approximating sequence (S_n) satisfying $\limsup \|\mathrm{Id} - 2S_n\| < 1 + \varepsilon$. The operators S_n will be obtained by averaging a certain subsequence of the K_n. This will be achieved in a series of lemmas. For notational convenience we assume $K_0 = 0$.

Let $\mathcal{N}_X$ be the collection of norms on $\mathbb{R}^2$ such that there is a weak*-null sequence $(x_n^*) \subset X^*$ satisfying $\|x_n^*\| = 1$ and $N(\alpha, \beta) = \lim_{n\to\infty} \|\alpha u^* + \beta x_n^*\|$ whenever $\|u^*\| = 1$. It is easy to see that $\mathcal{N}_X$ is a compact subset of the space of all continuous real-valued functions on $\mathbb{R}^2$ with the topology of uniform convergence on compact sets.

Suppose some sequence $(N_j)_{j=1}^{\infty}$ with $N_j \in \mathcal{N}_X$ is given. For each n we define a norm Φ, depending on the sequence (N_j), on $\mathbb{R}^{n+1}$ by the inductive formula

$$\Phi(\xi_0, \ldots, \xi_n) = N_n(\Phi(\xi_0, \ldots, \xi_{n-1}), \xi_n) .$$

We can then define a space $\Lambda(N_j)$ or $\Lambda(\Phi)$ as the completion of the space c_{00} of finitely supported sequences under the norm

$$\Phi(\alpha_0, \ldots, \alpha_n, 0, \ldots) = \Phi(\alpha_0, \alpha_1, \ldots, \alpha_n) .$$

The space $\Lambda(\Phi)$ is then isomorphic to a subspace of X^*. The collection of such norms Φ is denoted $\mathcal{F}_X$.

Lemma 2.8. *There exist constants $C > 0$, $p < \infty$ so that for any $\phi \in \mathcal{F}_X$, $m \in \mathbb{N}$ and any disjoint $v_1, \ldots, v_m \in \Lambda(\Phi)$, we have*

$$\left(\sum_{j=1}^{m} \Phi(v_j)^p\right)^{1/p} \leqq C\Phi\left(\sum_{j=1}^{m} v_j\right).$$

Proof. Let (N_j') be any sequence which is dense in $\mathcal{N}_X$ and form a sequence (N_j) in which each N_j' is repeated infinitely often; then (N_j) induces a norm $\Psi \in \mathcal{F}_X$. Any $\Lambda(\Phi)$ for $\Phi \in \mathcal{F}_X$ is (lattice-)finitely representable in $\Lambda(\Psi)$. Thus it suffices to show that $\Lambda(\Psi)$ has a lower p-estimate for some p. Now $\Lambda(\Psi)$ is a modular sequence space $h_{(F_j)}$ where

$$F_j(t) = N_j(1, t) - 1 ,$$

for $j \geqq 1$ and $F_0(t) = |t|$ (cf. [30]). However, $\Lambda(\Psi)$ is isomorphic to a subspace of X^* and thus contains no copy of ℓ_∞. By a result of Woo [58], $h_{(F_k)}$ coincides with a modular space

$h_{(G_k)}$ where the G_k satisfy a uniform Δ_2-condition, and this in turn implies that $h_{(G_k)}$ has a lower p-estimate (by calculations similar to those in [40], p. 139f.). □

At this point we suppose $0 < \delta < \varepsilon/8$ and that $\eta_j > 0$ are chosen for $j \geqq 1$ so that $\sum_{j \geqq 1} \eta_j < \delta$.

Lemma 2.9. *There exists a subsequence $(T_n)_{n=0}^{\infty} = (K_{r_n})_{n=0}^{\infty}$ of (K_n) so that $r_0 = 0$ and if we put*

$$A_n = \left\{ \sum_{j=1}^{n} \lambda_j (T_j^* x^* - T_{j-1}^* x^*) : |\lambda_j| \leqq 1, \|x^*\| \leqq 1 \right\}$$

and

$$B_n = \{ T_m^* x^* - T_l^* x^* : \|x^*\| \leqq 1, m > l \geqq n \},$$

then:

(a) $\left\| \sum_{k=1}^{n} (-1)^k T_k \right\| < 3$ *for* $n = 1, 2, \ldots$.

(b) *For any $v^* \in B_n$ there exists $N \in \mathcal{N}_X$ so that if $|\alpha| \leqq 1$ and $u^* \in A_{n-1}$, then*

$$N(\|u^*\|, |\alpha| \|v^*\|) - \eta_n \leqq \|u^* + \alpha v^*\| \leqq N(\|u^*\|, |\alpha| \|v^*\|) + \eta_n .$$

Proof. We will construct (r_n) by induction so that:

(c) $\left\| \sum_{k=1}^{n} (-1)^k T_k \right\| < 3$ and $\left\| \sum_{k=1}^{n} (-1)^k T_k + (-1)^{n+1} K_l \right\| < 3$ whenever $l > r_n$.

(d) If $D_n = \{ K_m^* x^* - K_l^* x^* : \|x^*\| \leqq 1, m > l \geqq r_n \}$, then for any $v^* \in D_n$ there exists $N \in \mathcal{N}_X$ so that if $|\alpha| \leqq 1$ and $u^* \in A_{n-1}$, then $|N(\|u^*\|, |\alpha| \|v^*\|) - \|u^* + \alpha v^*\|| \leqq \eta_n$.

To begin the induction simply take $r_0 = 0$. Then (c) and (d) are trivial.

Now suppose $r_0, \ldots, r_{n-1}$ have been chosen. We show that one can choose r_n large enough so that (c) and (d) hold. We consider each case separately. If one cannot satisfy (c) for all large enough choices of r_n then by induction one can find a sequence (x_k^*) in X^* with $\|x_k^*\| = 1$ and increasing sequences of natural numbers (a_k), (b_k) so that

$$r_{n-1} < a_1 < b_1 < a_2 < b_2 < \ldots,$$

for which if $S = \sum_{j=1}^{n-1} (-1)^j T_j$,

$$\|S^* x_k^* + (-1)^n (K_{b_k}^* - K_{a_k}^*) x_k^*\| \geqq 3 .$$

By passing to a subsequence we can assume that (x_k^*) converges weak* to some x^*. Since S is compact, $(S^* x_k^*)$ converges in norm to $S^* x^*$. Let $d_k = a_k$ or b_k. Then $(K_{d_k}^*(x_k^* - x^*))$ converges weak* to 0 and so

$$\begin{aligned}\limsup \|S^*x_k^* \pm 2K_{d_k}^*(x_k^*-x^*)\| &= \limsup \|S^*x^* \pm 2K_{d_k}^*(x_k^*-x^*)\| \\ &= \limsup \|K_{d_k}^*(S^*x^* \pm 2(x_k^*-x^*))\| \\ &\leqq \limsup \|S^*x^* \pm 2(x_k^*-x^*)\| \\ &\leqq \max(\|S\|, 2) \limsup \|x^* \pm (x_k^*-x^*)\| \\ &\leqq \max(\|S\|, 2)\,.\end{aligned}$$

(The last estimate used property (M^*).) By averaging

$$\limsup \|S^*x_k^* + (-1)^n(K_{b_k}^* - K_{a_k}^*)(x_k^*-x^*)\| < 3\,.$$

It remains to observe that $\lim \|(K_{b_k}^* - K_{a_k}^*)x^*\| = 0$; then we arrive at a contradiction and conclude that if r_n is chosen large enough (c) will hold.

We turn now to (d). If this cannot be satisfied by taking r_n large enough, then we can find a sequence (x_k^*) with $\|x_k^*\| = 1$ and two increasing sequences of positive integers (a_k), (b_k) so that $r_{n-1} < a_1 < b_1 < a_2 < \ldots$ and if $v_k^* = (K_{b_k}^* - K_{a_k}^*)x_k^*$ then for every $N \in \mathcal{N}_X$

$$\max_{|\alpha| \leqq 1} \max_{u^* \in A_{n-1}} |N(\|u^*\|, |\alpha|\|v_k^*\|) - \|u^* + \alpha v_k^*\|| > \eta_n\,.$$

Clearly we have that $\liminf \|v_k^*\| > 0$. We may pass to a subsequence so that $(\|v_k^*\|)$ converges to some $\beta > 0$ and $\lim \|x^* + \alpha v_k^*\| = N(\|x^*\|, |\alpha|\beta)$ for every $x^* \in X^*$, where $N \in \mathcal{N}_X$. Now pick $|\alpha_k| \leqq 1$ and $u_k^* \in A_{n-1}$ so that

$$|N(\|u_k^*\|, |\alpha_k|\|v_k^*\|) - \|u_k^* + \alpha_k v_k^*\|| > \eta_n\,.$$

By passing to a further subsequence we can suppose (u_k^*) converges in norm to some u^* (since A_{n-1} is norm compact) and that (α_k) converges to some α. This gives a contradiction and so (d) holds for all large enough choices of r_n. This enables us to complete the inductive construction, and so the lemma is proved. □

Lemma 2.10. *Suppose $(a_k)_{k \geqq 0}$ and $(b_k)_{k \geqq 0}$ are two sequences of nonnegative integers so that*

$$0 \leqq a_0 < b_0 < a_1 < b_1 < \ldots\,.$$

Suppose further that $\|x^*\| \leqq 1$. Then there exists $\Phi \in \mathcal{F}_X$ so that for any finite sequence $(\lambda_k)_{0 \leqq k \leqq n}$, with $|\lambda_k| \leqq 1$, if we let $v_k^* = (T_{b_k}^* - T_{a_k}^*)x^*$, we have

$$\Phi(\lambda_0\|v_0^*\|, \ldots, \lambda_n\|v_n^*\|) - \delta \leqq \left\| \sum_{k=0}^{n} \lambda_k v_k^* \right\| \leqq \Phi(\lambda_0\|v_0^*\|, \ldots, \lambda_n\|v_n^*\|) + \delta\,.$$

In particular

$$\left\| \sum_{k=1}^{n} \lambda_k v_k^* \right\| \leqq \left\| \sum_{k=1}^{n} v_k^* \right\| + 2\delta\,. \tag{3}$$

Proof. It follows from (b) in Lemma 2.9 that we can choose $N_j \in \mathcal{N}_X$ for $j \geqq 1$ so that if $u^* \in A_{b_{j-1}}$ and $|\alpha| \leqq 1$, then

$$|N_j(\|u^*\|, |\alpha| \|v_j^*\|) - \|u^* + \alpha v_j^*\|| < \eta_{a_j}.$$

Choose Φ corresponding to the sequence (N_j). Then we prove by induction that

$$\Phi(\lambda_0\|v_0^*\|, \ldots, \lambda_k\|v_k^*\|) - \sum_{j=1}^{k} \eta_{a_j} \leqq \left\| \sum_{j=0}^{k} \lambda_j v_j^* \right\|$$
$$\leqq \Phi(\lambda_0\|v_0^*\|, \ldots, \lambda_k\|v_k^*\|) + \sum_{j=1}^{k} \eta_{a_j}$$

for $0 \leqq k \leqq n$. This is trivial for $k = 0$. Assume it is true for $k = s - 1$. Then

$$\left| \left\| \sum_{j=0}^{s} \lambda_j v_j^* \right\| - N_s\left(\left\| \sum_{j=0}^{s-1} \lambda_j v_j^* \right\|, \lambda_s \|v_s^*\| \right) \right| \leqq \eta_{a_s}.$$

The inductive hypothesis follows easily. The last statement is trivial from the unconditionality of the norm Φ. □

Now let $V_n = T_{n+1} - T_n$.

Lemma 2.11. *For any finite sequence $(\lambda_k)_{k=0}^n$ we have*

$$\left\| \sum_{k=0}^{n} \lambda_k V_k \right\| \leqq 8 \max |\lambda_k|.$$

Proof. For convenience we suppose n is even, say $n = 2m$; we also suppose $\max|\lambda_k| = 1$. Let $\|x^*\| = 1$. Then by Lemma 2.10,

$$\left\| \sum_{k=0}^{m} \lambda_{2k} V_{2k}^* x^* \right\| \leqq \left\| \sum_{k=0}^{m} V_{2k}^* x^* \right\| + 2\delta.$$

However by (a) in Lemma 2.9 we have $\left\| \sum_{k=0}^{m} V_{2k}^* \right\| < 3$. Hence

$$\left\| \sum_{k=0}^{m} \lambda_{2k} V_{2k}^* x^* \right\| \leqq 4.$$

Similarly

$$\left\| \sum_{k=1}^{m} \lambda_{2k-1} V_{2k-1}^* x^* \right\| \leqq 4$$

and the lemma is proved. □

It follows now that for any $x^* \in X^*$ the series $\sum_{n=0}^{\infty} \lambda_n V_n^* x^*$ converges weak* in X^*. In fact since X^* is separable the series must converge in norm, although we do not need this observation.

Lemma 2.12. *Suppose C and p are given by Lemma 2.8, and that $t \in \mathbb{N}$. Then for $\|x^*\| = 1$ and any sequence (λ_k) with $\max|\lambda_k| \leqq 1$ we have*:

$$\left\| \sum_{k=0}^{\infty} \lambda_k V_k^* x^* \right\| \leqq 1 + 18Ct^{-1/p} + 8\delta + 16t\Delta$$

where $\Delta = \sup_{n \geqq 0} |\lambda_n - \lambda_{n+1}|$.

Proof. We begin by noting that according to Lemma 2.10 there exists $\Phi \in \mathscr{F}_X$ so that if $(\alpha_k)_{k=0}^n$ is a finite sequence satisfying $\max_{0 \leqq k \leqq n} |\alpha_k| \leqq 1$, then

$$\Phi(\alpha_0 \|V_0^* x^*\|, \ldots, \alpha_n \|V_{2n}^* x^*\|) - \delta \leqq \left\| \sum_{j=0}^{n} \alpha_j V_{2j}^* x^* \right\|$$
$$\leqq \Phi(\alpha_0 \|V_0^* x^*\|, \ldots, \alpha_n \|V_{2n}^* x^*\|) + \delta .$$

We can apply Lemma 2.8 to obtain for any $n \in \mathbb{N}$

$$\left(\sum_{s=0}^{t-1} \left(\left\| \sum_{k=0}^{n} V_{2tk+2s}^* x^* \right\| - \delta \right)^p \right)^{1/p} \leqq C \left(\left\| \sum_{k=0}^{nt+t-1} V_{2k}^* x^* \right\| + \delta \right) \leqq 9C .$$

It follows that there exists $0 \leqq s \leqq t-1$ so that

$$\left\| \sum_{k=0}^{n} V_{2tk+2s}^* x^* \right\| \leqq \delta + 9Ct^{-1/p} .$$

Now, by Lemma 2.10,

$$\left\| \sum_{k=0}^{n} \lambda_{2tk+2s} V_{2tk+2s}^* x^* \right\| \leqq \left\| \sum_{k=0}^{n} V_{2tk+2s}^* x^* \right\| + 2\delta \leqq 9Ct^{-1/p} + 3\delta .$$

For $0 \leqq k \leqq n+1$ we set

$$v_k^* = \sum_{2(k-1)t+2s<j<2kt+2s} V_j^* x^*$$

and

$$w_k^* = \sum_{2(k-1)t+2s<j<2kt+2s} \lambda_j V_j^* x^* ,$$

where each range of summation consists only of j with $0 \leqq j \leqq 2nt + 2t - 1$. Then for $0 \leqq k \leqq n$ we have

$$w_k^* - \lambda_{2kt+2s} v_k^* = \sum_{2(k-1)t+2s<j<2kt+2s} (\lambda_j - \lambda_{2kt+2s}) V_j^* x^* .$$

Similarly

$$w_{n+1}^* - \lambda_{2nt+2s} v_{n+1}^* = \sum_{2nt+2s<j<2nt+2t-1} (\lambda_j - \lambda_{2nt+2s}) V_j^* x^* .$$

Combining we have that

$$\sum_{k=0}^{n+1} w_k^* - \sum_{k=0}^{n+1} \alpha_k v_k^* = 2t\Delta \sum_{j=0}^{2nt+2t-1} \beta_j V_j^* x^*$$

where $\max_{0 \leqq k \leqq n+1} |\alpha_k| \leqq 1$ and $\max_{0 \leqq j \leqq 2nt+2t-1} |\beta_j| \leqq 1$. Thus by (3) and Lemma 2.11

$$\left\| \sum_{k=0}^{n+1} w_k^* \right\| \leqq \left\| \sum_{k=0}^{n+1} v_k^* \right\| + 2\delta + 16t\Delta .$$

Now

$$\begin{aligned} \left\| \sum_{j=0}^{2nt+2t-1} \lambda_j V_j^* x^* \right\| &\leqq \left\| \sum_{k=0}^{n+1} w_k^* \right\| + \left\| \sum_{k=0}^{n} \lambda_{2tk+2s} V_{2tk+2s}^* x^* \right\| \\ &\leqq \left\| \sum_{k=0}^{n+1} v_k^* \right\| + 9Ct^{-1/p} + 16t\Delta + 5\delta \\ &\leqq \left\| \sum_{j=0}^{2nt+2t-1} V_j^* x^* \right\| + 18Ct^{-1/p} + 16t\Delta + 6\delta \\ &\leqq 1 + 18Ct^{-1/p} + 16t\Delta + 8\delta . \end{aligned}$$

Finally letting $n \to \infty$ we obtain the result. □

We complete the proof of Theorem 2.7 by showing that if $S_n = \frac{1}{n}(T_1 + \cdots + T_n)$, then $\limsup \|\mathrm{Id} - 2S_n\| < 1 + \varepsilon$. To see this we write

$$\mathrm{Id} - 2S_n = \sum_{k=1}^{n} \frac{2k-n-2}{n} V_k + \sum_{k=n+1}^{\infty} V_k$$

so that by Lemma 2.12 for any $t \in \mathbb{N}$

$$\|\mathrm{Id} - 2S_n\| \leqq 1 + 18Ct^{-1/p} + 32tn^{-1} + 8\delta .$$

Hence $\limsup \|\mathrm{Id} - 2S_n\| \leqq 1 + 18Ct^{-1/p} + 8\delta$. However, as t is arbitrary we obtain $\limsup \|\mathrm{Id} - 2S_n\| < 1 + \varepsilon$. □

Under the assumptions of Theorem 2.7 we have thus established condition (6) of Theorem 2.4 in [30]. Combining Theorems 2.6 and 2.7 we therefore obtain the following improvement of the main result of that paper. (For the necessity of the conditions below see [19] and [30].)

Theorem 2.13. *Let X be a separable Banach space. Then $\mathscr{K}(X)$ is an M-ideal in $\mathscr{L}(X)$ if and only if X does not contain a copy of ℓ_1, X has (M) and X has the metric compact approximation property.*

Since a reflexive space with the compact approximation property automatically has the metric compact approximation property [10] (see also [18]) we can refine Theorem 2.13 in the case of reflexive X.

Corollary 2.14. *Let X be a separable reflexive Banach space. Then $\mathscr{K}(X)$ is an M-ideal in $\mathscr{L}(X)$ if and only if X has (M) and the compact approximation property.*

We remark that an alternative approach to Theorem 2.7 was recently developed by Lima [35].

3. Implications of property (m_p)

In this section we prove that a separable Banach space that does not contain a copy of ℓ_1 and enjoys property (m_p), which was introduced in Definition 2.1, is almost isometric to a subspace of an ℓ_p-sum of finite-dimensional spaces if $1 < p < \infty$ (Theorem 3.3), respectively almost isometric to a subspace of c_0 if $p = \infty$ (Theorem 3.5). At the end of this section we point out the relevance of the (m_p)-condition in the theory of M-ideals.

We first recall that, if $1 \leqq p < \infty$, the space C_p is defined to be $\ell_p(E_n)$ where (E_n) is a sequence of finite-dimensional Banach spaces dense in all finite-dimensional Banach spaces in the sense of Banach-Mazur distance. If $p = \infty$ we define $C_\infty = c_0(E_n)$. Now if (F_n) is any sequence of finite-dimensional Banach spaces and $\varepsilon > 0$, it is easy to see that there is a 1-complemented subspace X_0 of C_p so that $d(\ell_p(F_n), X_0) < 1 + \varepsilon$, or $d(c_0(F_n), X_0) < 1 + \varepsilon$ when $p = \infty$. For short we say that $\ell_p(F_n)$ is almost isometric to a 1-complemented subspace of C_p. It is clear that C_∞ is almost isometric to a subspace of c_0. The spaces C_p have been investigated by Johnson and Zippin ([22], [23], [27], [28]).

Although the definition of C_p depends on the choice of the sequence (E_n), it is known that any two such choices yield essentially the same space C_p. More precisely, if C_p and C_p' are built on two dense sequences (E_n) and (E_n'), then $d(C_p, C_p') = 1$ (see [22]). Since the statements we are going to make are almost isometric in character rather than isometric, this ambiguity does not affect the following theorems. So we fix one model of C_p once and for all.

Some of the ideas in the proof of the following Theorem 3.2 are borrowed from those of Johnson and Zippin [28]. We also need a lemma.

Lemma 3.1. *Let $1 < p \leqq \infty$ and let X be a separable Banach space with (m_p) that does not contain a copy of ℓ_1. Let q be the exponent conjugate to p, i.e., $1/p + 1/q = 1$.*

(a) *If E is a finite-dimensional subspace of X and $\eta > 0$, then there is a finite-codimensional subspace U of X such that*

$$(1-\eta)(\|x\|^p + \|y\|^p)^{1/p} \leqq \|x + y\| \leqq (1+\eta)(\|x\|^p + \|y\|^p)^{1/p}$$

for all $x \in E$, $y \in U$.

(b) *If F is a finite-dimensional subspace of X^* and $\eta > 0$, then there is a finite-codimensional weak*-closed subspace V of X^* such that*

$$(1-\eta)(\|x^*\|^q + \|y^*\|^q)^{1/q} \leqq \|x^* + y^*\| \leqq (1+\eta)(\|x^*\|^q + \|y^*\|^q)^{1/q}$$

for all $x^* \in F$, $y^* \in V$.

Proof. (a) A compactness argument shows that it is enough to give the proof in the case where $\dim E = 1$, say $E = \mathrm{lin}\{x\}$. By Lemma 2.5, X^* is separable; let (u_n^*) be a dense sequence in X^*. Suppose now that the lemma does not hold. Then there exists some $\eta > 0$ so that there is a bounded sequence (y_n), $y_n \in U_n = \{y : u_k^*(y) = 0 \text{ for } k = 1, \ldots, n\}$, satisfying

$$|\|x + y_n\| - (\|x\|^p + \|y_n\|^p)^{1/p}| > \eta .$$

Since $y_n \to 0$ weakly, this is a contradiction to property (m_p).

The proof of (b) is similar; note that X has (m_q^*) by Theorem 2.6. □

Theorem 3.2. *Suppose $1 < p \leqq \infty$, and suppose X is a separable Banach space not containing a copy of ℓ_1. Then X has property (m_p) if and only if given any $\varepsilon > 0$ there is a quotient X_0 of C_p so that $d(X, X_0) < 1 + \varepsilon$.*

Proof. It is clear by Theorem 2.6 that – in the case $p < \infty$ – a space which is almost isometric to a quotient of C_p has (m_p), since it is reflexive and its dual has (m_q). (Here and in what follows we suppose $1/p + 1/q = 1$.) In the case $p = \infty$ we use a result due to Alspach [2] to conclude that a quotient of C_∞ is almost isometric to a subspace of c_0 and thus has (m_∞).

We now prove the converse. Suppose $\delta > 0$ is chosen so that $\delta < \frac{1}{4}\varepsilon$ and

$$\frac{1-\delta}{1+\delta} > 1 - \frac{1}{8}\varepsilon .$$

We will also need a positive integer t selected so that $32 t^{-1/q} < \frac{1}{8}\varepsilon$. Suppose (η_n) is a sequence of real numbers satisfying

$$0 < \eta_n < \frac{1}{2}\delta, \quad \prod_{n \geqq 1} (1 - \eta_n) > 1 - \delta \quad \text{and} \quad \prod_{n \geqq 1} (1 + \eta_n) < 1 + \delta .$$

Let (u_n) be a dense sequence in X. For $F \subset X^*$ we let $F_\perp = \{x \in X : x^*(x) = 0 \; \forall x^* \in F\}$, and we denote the span of $u_1, \ldots, u_n$ by $[u_1, \ldots, u_n]$. We will inductively choose two sequences of subspaces (F_n) and (F_n') of X^* and subspaces $E(m, n)$, $1 \leqq m \leqq n$, in X so that:

(a) $\dim F_n < \infty$, $\dim E(m, n) < \infty$ for all $m \leqq n$.

(b) $F_n' \subset [u_1, \ldots, u_n]^\perp \cap \bigcap_{j \leqq k < n} E(j, k)^\perp$ is weak*-closed and $X^* = F_1 \oplus \cdots \oplus F_n \oplus F_n'$.

(c) $F_n' = F_{n+1} \oplus F_{n+1}'$.

(d) If $x^* \in F_1 + \cdots + F_n$ and $y^* \in F'_{n+1}$, then

$$(1-\eta_n)(\|x^*\|^q + \|y^*\|^q)^{1/q} \leqq \|x^* + y^*\| \leqq (1+\eta_n)(\|x^*\|^q + \|y^*\|^q)^{1/q}.$$

(e) If $x \in (F_1 + \cdots + F_n)_\perp$ and $y \in \sum_{j \leqq k < n} E(j,k)$, then

$$(1-\eta_n)(\|x\|^p + \|y\|^p)^{1/p} \leqq \|x + y\| \leqq (1+\eta_n)(\|x\|^p + \|y\|^p)^{1/p}$$

when $p < \infty$ or

$$(1-\eta_n)\max(\|x\|, \|y\|) \leqq \|x+y\| \leqq (1+\eta_n)\max(\|x\|, \|y\|)$$

when $p = \infty$.

(f) We have $(F_1 + \cdots + F_{m-1} + F'_n)_\perp \subset E(m,n)$ and $E(m,n) \subset (F_1 + \cdots + F_{m-2})_\perp$ if $1 \leqq m \leqq n$.

(g) If $x^* \in F_m + \cdots + F_n$, then there exists $x \in E(m,n)$ so that $\|x\| \leqq 1$ and

$$x^*(x) \geqq (1-\delta)\|x^*\|.$$

(Here we understand $F_1 + \cdots + F_r = \{0\}$ if $r < 1$.)

Let us describe the inductive construction. We pick $F'_1 = [u_1]^\perp$ and F_1 to be any complement of F'_1 to start the procedure.

Now suppose we have defined $(F_j)_{j \leqq n}$, $(F'_j)_{j \leqq n}$ and $E(j,k)$ for $1 \leqq j \leqq k \leqq n-1$ (at the first step no E-subspaces exist).

We first must define the spaces $E(m,n)$ for $1 \leqq m \leqq n$. To do this we notice that if $m \geqq 3$, $x^* \in F_1 + \cdots + F_{m-2}$ and $y^* \in F_m + \cdots + F_n$ we have

$$\|x^* + y^*\| \geqq (1-\eta_{m-2})(\|x^*\|^q + \|y^*\|^q)^{1/q} \geqq \left(1 - \frac{\delta}{2}\right)\|y^*\|.$$

Hence we can find a finite-dimensional subspace $G = G(m,n)$ so that

$$G \subset (F_1 + \cdots + F_{m-2})_\perp$$

and

$$\sup_{g \in G, \|g\| = 1} y^*(g) \geqq (1-\delta)\|y^*\|$$

if $y^* \in F_m + \cdots + F_n$. We then augment G to $E(m,n) = G + (F_1 + \cdots + F_{m-1} + F'_n)_\perp$. For $m = 1$ or $m = 2$ the procedure is similar.

We now turn to the definitions of F_{n+1} and F'_{n+1}. Using Lemma 3.1 we see that there exist finite-dimensional subspaces $H \subset X$ and $K \subset X^*$ so that:

(h) If $x \in \sum_{1 \leqq m \leqq n} E(m,n)$ and $y \in K_\perp$, then

$$(1-\eta_{n+1})(\|x\|^p + \|y\|^p)^{1/p} \leqq \|x+y\| \leqq (1+\eta_{n+1})(\|x\|^p + \|y\|^p)^{1/p}.$$

(i) If $x^* \in F_1 + \cdots + F_n$ and $y^* \in H^\perp$, then

$$(1-\eta_n)(\|x^*\|^q + \|y^*\|^q)^{1/q} \leqq \|x^*+y^*\| \leqq (1+\eta_n)(\|x^*\|^q + \|y^*\|^q)^{1/q}.$$

Let D be any weak*-closed complement of $F_1 + \cdots + F_n + K$ which is contained in F_n' (which is a complement of $F_1 + \cdots + F_n$). Then let F_{n+1}' be defined as the intersection

$$F_{n+1}' = D \cap \left(H + \sum_{1 \leqq m \leqq n} E(m,n) + [u_1, \ldots, u_n]\right)^\perp.$$

Now there is a subspace K' of F_n' so that $F_1 + \cdots + F_n + K = F_1 + \cdots + F_n + K'$. Pick $F_{n+1} \supset K'$ so that $F_n' = F_{n+1} \oplus F_{n+1}'$. This completes the inductive construction.

Now suppose $(a_n)_{n \geqq 0}$ and $(b_n)_{n \geqq 0}$ are two increasing sequences of integers with $a_0 \leqq b_0 < a_1 \leqq b_1 < a_2 \leqq \ldots$. First suppose $b_n + 2 \leqq a_{n+1}$ for all n. We observe that if $x_n^* \in F_{a_n} + \cdots + F_{b_n}$ for all n, then by induction, using (d), we have

$$\prod_{k=1}^{n-1}(1-\eta_{b_k})\left(\sum_{k=1}^{n}\|x_k^*\|^q\right)^{1/q} \leqq \left\|\sum_{k=1}^{n} x_k^*\right\| \leqq \prod_{k=1}^{n-1}(1+\eta_{b_k})\left(\sum_{k=1}^{n}\|x_k^*\|^q\right)^{1/q}$$

and hence

$$(1-\delta)\left(\sum_{k=1}^{n}\|x_k^*\|^q\right)^{1/q} \leqq \left\|\sum_{k=1}^{n} x_k^*\right\| \leqq (1+\delta)\left(\sum_{k=1}^{n}\|x_k^*\|^q\right)^{1/q}.$$

Similarly if $b_n + 3 \leqq a_{n+1}$ and $x_n \in E(a_n, b_n)$ then by (e) and (f) and induction

$$(1-\delta)\left(\sum_{k=1}^{n}\|x_k\|^p\right)^{1/p} \leqq \left\|\sum_{k=1}^{n} x_k\right\| \leqq (1+\delta)\left(\sum_{k=1}^{n}\|x_k\|^p\right)^{1/p}$$

when $p < \infty$ or

$$(1-\delta)\max(\|x_1\|, \ldots, \|x_n\|) \leqq \left\|\sum_{k=1}^{n} x_k\right\| \leqq (1+\delta)\max(\|x_1\|, \ldots, \|x_n\|)$$

when $p = \infty$.

We also notice that for each n there is a weak*-continuous projection, P_n^* say, of X^* onto F_n with kernel $F_1 + \cdots + F_{n-1} + F_n'$. Then $P_n^* P_m^* = 0$ if $m \neq n$, and $\operatorname{ran} P_k \subset E(m,n)$ whenever $m \leqq k \leqq n$. If $x^* \in X$ and $1 \leqq n < \infty$ then we can write $x^* = \sum_{k=1}^{n} P_k^* x^* + w^*$ with $w^* \in F_n'$. Then $x^*(u_n) = \sum_{k=1}^{n} x^*(P_k u_n)$ and $P_k u_n = 0$ for $k > n$.

Now if $0 \leqq s \leqq t-1$ we will define two operators, T_s and R_s. Suppose $p < \infty$. We define

$$T_s : Y_s := \ell_p\big(E(4(n-1)t + 4s + 4, 4nt + 4s + 1)_{n \geqq 0}\big) \to X$$

by $T_s((x_n)) = \sum_{n \geqq 0} x_n$ where the first space with $n = 0$ is to be understood as $E(1, 4s+1)$. We also define

$$R_s : Z_s := \ell_p\big(E(4nt + 4s + 2, 4nt + 4s + 3)_{n \geqq 0}\big) \to X$$

by $R_s((x_n)) = \sum_{n \geqq 0} x_n$. In the case $p = \infty$ we let Y_s and Z_s be c_0-sums.

It is clear that $(1-\delta)\|\xi\| \leqq \|T_s \xi\| \leqq (1+\delta)\|\xi\|$ for $\xi \in Y_s$ and similarly

$$(1-\delta)\|\xi\| \leqq \|R_s \xi\| \leqq (1+\delta)\|\xi\|$$

for $\xi \in Z_s$.

At the same time we introduce two operators

$$T : Y := \ell_p\big((Y_s)_{s=0}^{t-1}\big) \to X \quad \text{and} \quad R : Z := \ell_p\big((Z_s)_{s=0}^{t-1}\big) \to X$$

defined by

$$T(\xi_0, \ldots, \xi_{t-1}) = \frac{1}{t^{1/q}} \sum_{s=0}^{t-1} T_s \xi_s$$

and

$$R(\xi_0, \ldots, \xi_{t-1}) = \sum_{s=0}^{t-1} R_s \xi_s .$$

By construction we have

$$\|T(\xi_0, \ldots, \xi_{t-1})\| \leqq \frac{1}{t^{1/q}} \sum_{s=0}^{t-1} \|T_s\| \, \|\xi_s\| \leqq (1+\delta) \left(\sum_{s=0}^{t-1} \|\xi_s\|^p \right)^{1/p}$$

and thus $\|T\| \leqq 1 + \delta$. Also, we can consider R as an operator on the space

$$\ell_p\big(E(4n+2, 4n+3)_{n \geqq 0}\big)$$

given by $R((x_n)) = \sum_{n \geqq 0} x_n$ and deduce that $\|R\| \leqq 1 + \delta$.

Now suppose $x^* \in X^*$ and $R_s^* x^* = 0$. Then it is clear that $P_k^* x^* = 0$ whenever $k \equiv 4s + 2$ or $k \equiv 4s + 3 \bmod 4t$. Let

$$x_n^* = \sum_{4(n-1)t + 4s + 4 \leqq k \leqq 4nt + 4s + 1} P_k^* x^*$$

for $n \geqq 0$. Then for each $n \geqq 0$ there exists $x_n \in E\big(4(n-1)t + 4s + 4, 4nt + 4s + 1\big)$ by (g) so that $\|x_n\| = 1$ and $x_n^*(x_n) \geqq (1-\delta)\|x_n^*\|$. We also have $x_n^*(x_m) = 0$ if $n \neq m$ by condition (f).

Hence if $(\alpha_n)_{n\geqq 0}$ is any finitely nonzero sequence with $\|(\alpha_n)\|_p = 1$, then

$$\|T_s^* x^*\| \geqq |\langle x^*, \sum \alpha_n x_n\rangle| \geqq (1-\delta)\sum \alpha_n \|x_n^*\|.$$

We draw the conclusion that

$$(\sum \|x_n^*\|^q)^{1/q} \leqq \frac{1}{1-\delta}\|T_s^* x^*\| \leqq \frac{1+\delta}{1-\delta}\|x^*\|.$$

By the remarks above it follows that $\sum_{k=1}^{n} x_k^*$ is bounded in norm by

$$(1+\delta)^2(1-\delta)^{-1}\|x^*\|.$$

Clearly $\sum_n x_n^*(u_k)$ converges for each k and so $\sum x_n^*$ converges weak* (and actually also in norm, but we do not need this). Since $P_i^*(x^* - \sum x_n^*) = 0$ for all i it follows that $x^*(u_k) = \sum x_n^*(u_k)$ for all k. Hence $x^* = \sum x_n^*$ weak*.

Now

$$\begin{aligned}\|x^*\| &\leqq \sup_n \left\|\sum_{k=1}^{n} x_k^*\right\| \\ &\leqq (1+\delta)\left(\sum_{k=1}^{\infty}\|x_k^*\|^q\right)^{1/q} \\ &\leqq \frac{1+\delta}{1-\delta}\|T_s^* x^*\|\end{aligned}$$

provided, of course, that $R_s^* x^* = 0$.

Now let us relax this assumption. In general the norm of x^* restricted to the range of R_s is at most $(1-\delta)^{-1}\|R_s^* x^*\|$. Hence by the Hahn-Banach theorem there exists $y^* \in X^*$ with $\|y^*\| \leqq (1-\delta)^{-1}\|R_s^* x^*\| \leqq 4\|R_s^* x^*\|$ and so that $R_s^* y^* = R_s^* x^*$. But by the above

$$\|y^* - x^*\| \leqq \frac{1+\delta}{1-\delta}\|T_s^*(y^* - x^*)\|.$$

Hence

$$\begin{aligned}\|T_s^* x^*\| &\geqq \|T_s^*(y^* - x^*)\| - \|T_s^* y^*\| \\ &\geqq \frac{1-\delta}{1+\delta}\|y^* - x^*\| - (1+\delta)\|y^*\| \\ &\geqq \frac{1-\delta}{1+\delta}\|x^*\| - 4\|y^*\| \\ &\geqq \frac{1-\delta}{1+\delta}\|x^*\| - 16\|R_s^* x^*\|.\end{aligned}$$

From this we deduce using the triangle inequality in ℓ_q^t that

$$\left(\frac{1}{t}\sum_{s=0}^{t-1}\|T_s^*x^*\|^q\right)^{1/q} \geqq \frac{1-\delta}{1+\delta}\|x^*\| - 16\left(\frac{1}{t}\sum_{s=0}^{t-1}\|R_s^*x^*\|^q\right)^{1/q}.$$

Hence

$$\|T^*x^*\| \geqq \frac{1-\delta}{1+\delta}\|x^*\| - \frac{16}{t^{1/q}}\|R^*x^*\|,$$

and this gives

$$\|T^*x^*\| \geqq \left(1 - \frac{1}{4}\varepsilon\right)\|x^*\|.$$

Since $\|T\| \leqq 1 + \frac{1}{4}\varepsilon$ this implies that $d(X, Y/\ker T) < 1 + \varepsilon$. Since Y is an ℓ_p-sum of finite-dimensional spaces we are done. □

Theorem 3.3. *Suppose $1 < p < \infty$ and X is a separable Banach space not containing a copy of ℓ_1. Then X has property (m_p) if and only if, given any $\varepsilon > 0$, there is a subspace X_0 of C_p so that $d(X, X_0) < 1 + \varepsilon$.*

Proof. In this case X, being a quotient of C_p by the preceding result, is reflexive and so Theorem 3.2 can be applied to X^* to give the result. □

The following corollary is a more precise statement of a classical result of Johnson and Zippin [28] who have proved the corresponding isomorphic results.

Corollary 3.4. *Let $1 < p < \infty$ and $\varepsilon > 0$.*

(a) *If X is a quotient of C_p, then there is a subspace X_0 of C_p with $d(X, X_0) < 1 + \varepsilon$.*

(b) *If X is a subspace of C_p, then there is a quotient X_0 of C_p with $d(X, X_0) < 1 + \varepsilon$.*

Proof. (a) We have that X^* has property (m_q), since it is a subspace of C_q where as usual $1/p + 1/q = 1$, and hence X^* has (m_p^*) by Theorem 2.6. It follows that X has (m_p), and it remains to apply Theorem 3.3.

The proof of (b) is similar. □

Theorem 3.5. *Let X be a separable Banach space not containing a copy of ℓ_1. Then X has property (m_∞) if and only if, for any $\varepsilon > 0$, there is a subspace X_0 of c_0 with $d(X, X_0) < 1 + \varepsilon$.*

Proof. If X has (m_∞), then Theorem 3.2 shows that there is a quotient of C_∞, X_1 say, with $d(X, X_1) < 1 + \varepsilon$. However, this implies that there is a quotient of a subspace of c_0 and hence a subspace of a quotient of c_0, X_2 say, with $d(X, X_2) < 1 + \varepsilon$. The result now follows from Alspach's theorem [2] that quotients of c_0 are almost isometric to subspaces of c_0. □

We finally combine results from [30], [46] and Theorems 2.13 and 3.3 (resp. 3.5) to obtain the following corollary. If X satisfies condition (i) below, it was said to be an (M_p)-space in [46]; see also Chapter VI.5 in [20] for information on (M_p)-spaces.

Corollary 3.6. *Let $1 < p \leqq \infty$ and let X be a separable Banach space. Then the following conditions are equivalent.*

(i) *$\mathscr{K}(X \oplus_p X)$ is an M-ideal in $\mathscr{L}(X \oplus_p X)$.*

(ii) *The space X has the metric compact approximation property, enjoys property (m_p) and fails to contain a copy of ℓ_1.*

(iii) *The space X has the metric compact approximation property and is almost isometric to a subspace of an ℓ_p-sum of finite-dimensional spaces when $p < \infty$, respectively, to a subspace of c_0 when $p = \infty$.*

Remarks. (1) The equivalence (i) $\Leftrightarrow$ (iii) has been conjectured by M-idealists for some time, cf. [20], p. 336. A special case of the equivalence (i) $\Leftrightarrow$ (iii) for $p = \infty$ was proved by W. Werner [57]. Actually, it now turns out that the condition considered by him is equivalent to the (M_∞)-condition.

(2) One can prove the implication (ii) $\Rightarrow$ (iii) employing techniques as in the proof of Theorem 2.7. For simplicity suppose $p < \infty$. In this case one concludes first that there is a sequence (S_n) in $\mathscr{K}(X)$ such that

$$(1-\varepsilon)\|x\| \leqq \left(\sum_{n=0}^{\infty} \|S_n x\|^p\right)^{1/p} \leqq (1+\varepsilon)\|x\|$$

for all $x \in X$. Then one embeds X into a space Y with the approximation property, say via an embedding j. Then jS_n can be approximated up to $\varepsilon/2^n$ by finite-rank operators F_n. Thus X embeds into $\ell_p(\operatorname{ran} F_n)$.

(3) For examples illustrating Corollary 3.6 we refer to Corollaries 4.8 and 4.9 below.

4. Property (M) for subspaces of L_p

The subject-matter of the present section is to refine the results of the preceding one for subspaces of L_p. Here L_p stands for a separable nonatomic $L_p(\mu)$-space. It is known that such a space is isometric to $L_p[0,1]$ [51], p. 321; in fact, the 'same' isomorphism works for all values of p. We are first going to prove a technical result, Theorem 4.2, that will be applied to subspaces of L_p in this section and to subspaces of the Schatten classes in the following one. In order to formulate it we need the notions of a finite-dimensional decomposition (FDD) and of cotype for which we refer to [39], p. 48, and [40], p. 72ff., respectively. We also found it convenient to introduce the following technical definition.

Definition 4.1. Let Y be a separable Banach space and let τ be a topology on Y making it a topological vector space. If $1 < p < \infty$ we will say that Y has *property $(m_p(\tau))$*

if whenever $\|u\| = 1$ and (y_n) is a normalized sequence with $\lim y_n = 0$ for the topology τ, then

$$\lim_{n \to \infty} \|\alpha u + \beta y_n\|^p = |\alpha|^p + |\beta|^p$$

for all scalars α and β.

We remark that Y has $(m_p(\tau))$ if and only if

$$\limsup \|u + y_n\| = \|(\|u\|, \limsup \|y_n\|)\|_p$$

whenever τ-$\lim y_n = 0$. We will take advantage of the fact L_p has $(m_p(\tau))$ for the topology of convergence in measure.

Theorem 4.2. *Suppose Y is a separable Banach space with nontrivial cotype. Suppose Y has an unconditional* (FDD) *and denote the partial sum operators by S_n. Suppose further that τ is a vector topology on Y weaker than the norm topology and such that Y has property $(m_p(\tau))$ for some $p > 1$.*

Now let X be a closed infinite-dimensional subspace of Y so that $S_n x \to x$ for the topology τ uniformly for $x \in B_X$. Then, given $\varepsilon > 0$ there are a sequence of finite-dimensional subspaces (E_n) of Y and a subspace X_0 of $\ell_p(E_n)$ so that $d(X_0, X) \leqq 1 + \varepsilon$.

Proof. The argument is a variant of Theorem 2.7. We suppose Y has cotype q where $q \geqq p$, with constant C_q, and we denote the unconditional constant of the (FDD) by C_u. We select $0 < \delta < \frac{1}{8}\varepsilon$ and choose $\eta_n > 0$ so that $\sum \eta_n < \delta$, and finally we pick an integer t so that $C_u C_q t^{-1/q} \leqq \frac{1}{8}\varepsilon$.

For convenience we index (S_n) from zero with $S_0 = 0$. We will construct a subsequence $T_n = S_{r_n}$ (for $n \geqq 0$) of (S_n) with the property that if

$$A_n = \left\{ \sum_{j=1}^{n} \lambda_j (T_j x - T_{j-1} x) : |\lambda_j| \leqq 1, x \in B_X \right\}$$

and $D_n = \{S_m x - S_l x : x \in B_X, m > l \geqq r_n\}$, then for any $v \in D_n$, $u \in A_{n-1}$ we have:

$$(\|u\|^p + \|v\|^p)^{1/p} - \eta_n \leqq \|u + v\| \leqq (\|u\|^p + \|v\|^p)^{1/p} + \eta_n .$$

To start the induction set $r_0 = 0$. Now suppose $r_0, \dots, r_{n-1}$ have been chosen. If no successful choice of $r_n > r_{n-1}$ can be made, then we can find sequences

$$r_{n-1} < a_1 < b_1 < a_2 < \dots$$

and $x_k \in B_X$ so that if $v_k = S_{b_k} x_k - S_{a_k} x_k$, then there exist $u_k \in A_{n-1}$ with

$$|\|u_k + v_k\| - (\|u_k\|^p + \|v_k\|^p)^{1/p}| > \eta_n .$$

Passing to a subsequence we can suppose that (u_k) converges in norm to some $u \in A_{n-1}$. However (v_k) then converges to 0 for τ by assumption on X, and we obtain from property $(m_p(\tau))$

$$\lim_{k \to \infty} (\|u + v_k\| - (\|u\|^p + \|v_k\|^p)^{1/p}) = 0$$

which leads to a contradiction. Thus the induction can be carried out.

Now it follows by induction that, if $x \in B_X$ and $(a_k)_{k \geqq 0}$, $(b_k)_{k \geqq 0}$ are two sequences satisfying $0 \leqq a_0 < b_0 < a_1 < b_1 < \dots$, then, letting $v_k = T_{b_k}x - T_{a_k}x$, we have

$$\left(\sum_{j=1}^{n} \|v_j\|^p\right)^{1/p} - \sum_{j=1}^{n} \eta_{a_j} \leqq \left\|\sum_{j=1}^{n} v_j\right\| \leqq \left(\sum_{j=1}^{n} \|v_j\|^p\right)^{1/p} + \sum_{j=1}^{n} \eta_{a_j}.$$

Since $\sum v_j$ converges as the (FDD) is unconditional this means that

$$\left(\sum_{j=1}^{\infty} \|v_j\|^p\right)^{1/p} - \delta \leqq \left\|\sum_{j=1}^{\infty} v_j\right\| \leqq \left(\sum_{j=1}^{\infty} \|v_j\|^p\right)^{1/p} + \delta.$$

Now let $V_k = T_{k+1} - T_k$ for $k \geqq 0$, and for $0 \leqq s \leqq t-1$ define $A_s y = \sum_{j \equiv s} V_j y$ (congruence mod t). We also let

$$W_{k,s} = \sum_{(k-1)t + s < j < kt + s} V_j$$

for $k \geqq 0$ and $0 \leqq s \leqq t-1$.

Then for any $|\lambda_s| = 1$ and any $x \in X$ with $\|x\| = 1$ we have

$$\left\|\sum_{s=0}^{t-1} \lambda_s A_s x\right\| \leqq C_u$$

and hence by the cotype q condition

$$\left(\sum_{s=0}^{t-1} \|A_s x\|^q\right)^{1/q} \leqq C_u C_q.$$

From this we deduce since $p \leqq q$,

$$\left(\sum_{s=0}^{t-1} \|A_s x\|^p\right)^{1/p} \leqq C_u C_q t^{1/p - 1/q}.$$

Now

$$\left(\sum_{k=0}^{\infty} \|W_{k,s} x\|^p\right)^{1/p} - \delta \leqq \|x - A_s x\| \leqq \left(\sum_{k=0}^{\infty} \|W_{k,s} x\|^p\right)^{1/p} + \delta.$$

Hence

$$1 - \|A_s x\| - \delta \leqq \left(\sum_{k=0}^{\infty} \|W_{k,s} x\|^p\right)^{1/p} \leqq 1 + \|A_s x\| + \delta.$$

It follows that

$$t^{1/p}(1-\delta-C_uC_qt^{-1/q})\leqq\left(\sum_{k=0}^{\infty}\sum_{s=0}^{t-1}\|W_{k,s}x\|^p\right)^{1/p}\leqq t^{1/p}(1+\delta+C_uC_qt^{-1/q}).$$

By choice of δ and t this implies that

$$1-\frac{1}{4}\varepsilon\leqq\left(\frac{1}{t}\sum_{k=0}^{\infty}\sum_{s=0}^{t-1}\|W_{k,s}x\|^p\right)^{1/p}\leqq 1+\frac{1}{4}\varepsilon.$$

Hence X has Banach-Mazur distance less than $1+\varepsilon$ from a closed subspace of $\ell_p(E_{k,s})$ where $E_{k,s}=\operatorname{ran}W_{k,s}$. □

It remains to give concrete examples.

Lemma 4.3. *Suppose $1<p<\infty$ and $p\neq 2$. Suppose $f\in L_p$ and (g_n) is a bounded sequence in L_p. Then the following statements are equivalent:*

(i) *For any scalars α and β we have*

$$\lim_{n\to\infty}(\|\alpha f+\beta g_n\|^p-|\alpha|^p\|f\|^p-|\beta|^p\|g_n\|^p)=0.$$

(ii) $\lim_{n\to\infty}\int_{\{|f|>0\}}|g_n|\,d\mu=0.$

Proof. (i) ⇒ (ii): It suffices to consider the case $\|f\|_p=\|g_n\|_p=1$. Then by a result of Dor [14] (cf. the discussion in Alspach [3]), there exist for each n disjoint Borel sets A_n, B_n so that $\|f\chi_{A_n}\|_p\to 1$ and $\|g\chi_{B_n}\|_p\to 1$. Hence $\lim\|g\chi_{A_n}\|_p=0$. Now let

$$C_n=\{\omega:|f(\omega)|>0,\omega\notin A_n\}.$$

Then $\lim\mu(C_n)=0$. Hence (g_n) tends to 0 in measure on $\{|f|>0\}$ and is equi-integrable, so that (ii) follows.

(ii) ⇒ (i): This is standard and we omit it. □

Theorem 4.4. *Suppose $1<p<\infty$, $p\neq 2$, and let X be a closed infinite-dimensional subspace of L_p. Then the following conditions on X are equivalent:*

(i) *B_X is compact in L_1.*

(ii) *X has property (m_p).*

(iii) *For any $\varepsilon>0$, there is a subspace X_0 of ℓ_p so that $d(X_0,X)<1+\varepsilon$.*

Proof. (iii) ⇒ (ii): This is immediate.

(ii) ⇒ (i): We start by producing a function f in X with maximal support. Let (f_n) be a dense sequence in the unit ball of X. Then $\sum_{n=1}^{\infty}t_n2^{-n}f_n$ converges in L_p and absolutely a.e. for every $(t_n) \in [0,1]^{\mathbb{N}}$. By a standard Fubini argument there is a choice of $t = (t_n)$ so that if $f = \sum t_n 2^{-n} f_n$ then f is nonzero a.e. on the set $S = \bigcup \{|f| > 0\}$. (In fact, for almost every $\omega \in S$ the set of those t for which $\sum t_n 2^{-n} f_n(\omega) = 0$ has measure 0, and for $A = \{(t,\omega) : \omega \in S, \sum t_n 2^{-n} f_n(\omega) \neq 0\}$, $\int dt \int d\mu \chi_A = \int d\mu \int dt \chi_A = \mu(S)$.) It is then easy to see that any $g \in X$ vanishes a.e. on the set $\{f = 0\}$.

Now suppose (g_n) is a weakly null sequence in X. Then by (m_p) and the preceding lemma, $\lim_{n\to\infty} \int_{\{|f|>0\}} |g_n| d\mu = 0$. Thus $\lim \|g_n\|_1 = 0$ and, as X is reflexive, the inclusion $X \hookrightarrow L_1$ is compact and B_X is therefore compact in L_1.

(i) $\Rightarrow$ (iii): We let τ be the L_1-norm topology on L_p. Then L_p has $(m_p(\tau))$ and nontrivial cotype. Also L_p has an unconditional basis (the Haar basis) [40], Th. 2.c.5. The Haar basis is also a basis for L_1 [40], Prop. 2.c.1. Hence if (S_n) denotes the sequence of partial sum operators, then $S_n f \to f$ L_1-uniformly on B_X. We are thus in a position to apply Theorem 4.2, and since each E_n produced there is a finite-dimensional subspace of L_p we obtain (iii). □

Next we note that in this context property (M) necessarily reduces to property (m_r) for some r.

Corollary 4.5. *Suppose $1 \leqq p < \infty$ and let X be a closed infinite-dimensional subspace of L_p with property (M) and, if $p = 1$, failing to contain a copy of ℓ_1. Then there exists $1 < r < \infty$ so that X has (m_r), and for any $\varepsilon > 0$ there exists a closed subspace X_0 of ℓ_r with $d(X, X_0) < 1 + \varepsilon$. If $p \leqq 2$, then $p \leqq r \leqq 2$; if $p > 2$, then $r = 2$ or $r = p$.*

Proof. Since X is stable [32] it follows that X has property (m_r) for some $1 < r < \infty$. (This is what the proof of [30], Th. 3.10, actually shows.) Theorem 3.3 implies that X contains a copy of ℓ_r. Now L_p contains only those ℓ_r as stated [38], p. 132ff., hence we get the final assertion.

Since there is a weakly null sequence (u_n) in X with $\lim \|x + \alpha u_n\| = (\|x\|^r + |\alpha|^r)^{1/r}$ for any $x \in X$ [30], Prop. 3.9, the space $X \oplus_r \mathbb{R}$ is finitely representable in X and thus embeds isometrically into an ultraproduct of X and hence also into L_p. We wish to conclude that X is isometric to a subspace of L_r.

If $p \leqq 2$, this is a result independently due to Dor [15] and Raynaud [50], Prop. 3. If $p > 2$ and $r = 2$, we observe that Raynaud's argument still works if p is not an even integer; and the assertion is trivial if $r = p$. Suppose now that $p = 2m$ is an even integer and $r = 2$. We claim that a Banach space Y is isometric to a Hilbert space provided $Y \oplus_2 \mathbb{R}$ is isometric to a subspace of L_p. Indeed, if $Y \subset L_p$ and $g \in L_p$ satisfy

$$\|y + tg\|_p = (\|y\|_p^2 + |t|^2)^{1/2}$$

for all $y \in Y$, $t \in \mathbb{R}$, then

$$\|y + tg\|_p^p = \int |y + tg|^{2m} d\mu = (\|y\|_p^2 + |t|^2)^m$$

is a polynomial in t, the coefficient of t^{2m-2} being

$$m(2m-1)\int |y|^2|g|^{2m-2}\,d\mu = m\|y\|_p^2 .$$

Hence Y is a Hilbert space.

To prove the corollary it is now left to apply Theorem 4.4. □

The case $p > 2$ is especially interesting.

Corollary 4.6. *Suppose $p > 2$ and let X be a closed subspace of L_p. Then the following conditions on X are equivalent*:

(i) *X contains no subspace isomorphic to ℓ_2.*

(ii) *X is isomorphic to a subspace of ℓ_p.*

(iii) *For any $\varepsilon > 0$, there is a subspace X_0 of ℓ_p so that $d(X_0, X) < 1 + \varepsilon$.*

Proof. The only step requiring proof is that (i) implies (iii). By Theorem 4.4 it remains to show that B_X is L_1-compact. If not, there is a sequence (f_n) which is weakly null but does not converge to zero in L_1. By passing to a subsequence we can assume that (f_n) is an unconditional basic sequence, cf. [39], Prop. 1.a.11, and that $\inf\|f_n\|_1 > 0$. We thus have for any finite sequence of scalars $a_1, \ldots, a_n$

$$\begin{aligned}\left\|\sum_{k=1}^n a_k f_k\right\|_p &\geqq c_1 \operatorname{ave}\left\|\sum_{k=1}^n \pm a_k f_k\right\|_p \\ &\geqq c_2\left\|\left(\sum_{k=1}^n |a_k|^2|f_k|^2\right)^{1/2}\right\|_p \\ &\geqq c_2\left\|\left(\sum_{k=1}^n |a_k|^2|f_k|^2\right)^{1/2}\right\|_2 \\ &\geqq c_2 \inf\|f_n\|_2\left(\sum_{k=1}^n |a_k|^2\right)^{1/2} \\ &\geqq c_3\left(\sum_{k=1}^n |a_k|^2\right)^{1/2}\end{aligned}$$

for suitable constants $c_i > 0$. Since L_p has type 2, it follows that conversely

$$\begin{aligned}\left\|\sum_{k=1}^n a_k f_k\right\|_p &\leqq c_4 \operatorname{ave}\left\|\sum_{k=1}^n \pm a_k f_k\right\|_p \\ &\leqq c_5\left(\sum_{k=1}^n |a_k|^2\right)^{1/2} .\end{aligned}$$

Thus $\overline{\operatorname{lin}}\{f_1, f_2, \ldots\}$ is isomorphic to a Hilbert space. □

Remarks. (1) The equivalence of (i) and (ii) above, however with a large isomorphism constant, is due to Johnson and Odell [26]; see also [24].

(2) In the case $1 < p < 2$ the equivalence of (ii) and (iii) breaks down. In fact the space $\ell_p \oplus_p \ell_2$ is isometric to a subspace of L_p and has a subspace isometric to the space ℓ_p equipped with the norm $\|\xi\| = (\|\xi\|_p^p + \|\xi\|_2^p)^{1/p}$. A direct calculation shows that this norm does not have property (m_p) (not even property (M)); so ℓ_p with this norm cannot embed almost isometrically into ℓ_p with its classical norm. (A more roundabout way to arrive at the same conclusion is to observe that the above norm is 1-symmetric and to invoke a result due to Hennefeld [21] (see [20], Prop. VI.4.24, for a proof based on property (M)) according to which the spaces $(\ell_p, \|\,\|_p)$ are the only 1-symmetric Banach spaces for which $\mathscr{K}(X)$ is an M-ideal in $\mathscr{L}(X)$ and thus the only ones with property (M).)

Corollary 4.7. *Let $2 < p < \infty$ and suppose X is a closed infinite-dimensional subspace of L_p with property (M). Then either X embeds almost isometrically into ℓ_p, or X is isometric to a Hilbert space.*

Proof. This follows from Corollaries 4.5 and 4.6. (Note that if X embeds almost isometrically into ℓ_2, then the parallelogram identity holds, and X is isometric to a Hilbert space.) □

As a specific example of a subspace of L_p we now discuss the Bergman space B_p. This is the space of all analytic functions on the open unit disk $\mathbb{D}$ in the complex plane for which $\int_{\mathbb{D}} |f(x+iy)|^p dx\,dy$ is finite. Thus B_p is a linear subspace of $L_p = L_p(\mathbb{D}, dx\,dy)$, and it is closed. It is well known that B_p has the metric approximation property (use Fejér kernels, as in [54], Lemma 3.4); in fact, it is isomorphic to ℓ_p [37]. Let us point out that B_p has (m_p). If $f_n \to 0$ weakly, then clearly $f_n \to 0$ pointwise. Since bounded sets in B_p are equicontinuous on compact subsets of $\mathbb{D}$, we conclude that $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$. From this it follows easily that B_p has (m_p).

Hence Corollary 3.6 and Theorem 4.4 imply:

Corollary 4.8. *The Bergman space B_p embeds almost isometrically into ℓ_p, and $\mathscr{K}(B_p)$ is an M-ideal in $\mathscr{L}(B_p)$ if $1 < p < \infty$.*

We also wish to consider the corresponding space of analytic functions when $p = \infty$. This is the so-called 'little' Bloch space β_0 consisting of those analytic functions f on $\mathbb{D}$ for which $\|f\|_\beta := \sup (1-|z|^2)|f(z)| < \infty$ and $\lim_{|z|\to 1} (1-|z|^2) f(z) = 0$. (The analytic functions satisfying $\|f\|_\beta < \infty$ form the Bloch space β. We remark for completeness that in complex analysis the space of functions whose derivatives belong to β is called the Bloch space.) It is known that β is the bidual of β_0 (e.g., [9]), β_0 is an M-ideal in β [54], β_0 has (m_∞) (the above argument works here as well, see also [36]), and β_0 has the metric approximation property (e.g., [54], Lemma 3.4). Hence we get from Corollary 3.6:

Corollary 4.9. *The little Bloch space β_0 embeds almost isometrically into c_0, and $\mathscr{K}(\beta_0)$ is an M-ideal in $\mathscr{L}(\beta_0)$. In fact, for every Banach space X, $\mathscr{K}(X, \beta_0)$ is an M-ideal in $\mathscr{L}(X, \beta)$.*

This result (the last mentioned assertion follows from [53]) was conjectured in [36]. Note that β_0 is not isometric to a subspace of c_0, because its unit ball has extreme points [12], but the only subspaces of c_0 whose unit balls have extreme points are finite-dimensional.

Actually, to obtain the above corollaries it is not necessary to apply the machinery developed in this paper; all one has to know is that B_p embeds almost isometrically into ℓ_p and β_0 into c_0. This can be proved directly as follows; for simplicity let us give the details for β_0. Let $\varepsilon > 0$ and $r_n = 1 - 2^{-n}$ for $n \geqq 0$. Then for each n there are constants K_n and L_n so that if $\|f\|_\beta = 1$, then $|f(z)| \leqq K_n$ and $|f'(z)| \leqq L_n$ for $|z| \leqq r_n$. Now for each n choose a finite subset F_n of the annulus $A_n = \{z : r_{n-1} \leqq |z| \leqq r_n\}$ so that if $z \in A_n$, then there exists $z' \in F_n$ with $|z - z'| \leqq \delta_n$ where $(L_n + 2K_n)\delta_n \leqq \varepsilon$.

Suppose $\|f\|_\beta = 1$. If $z \in A_n$, pick $z' \in F_n$ as above. Hence $|f(z)| \leqq |f(z')| + L_n \delta_n$ and so

$$(1 - |z|^2)|f(z)| \leqq (|z'|^2 - |z|^2)|f(z)| + (1 - |z'|^2)(|f(z')| + L_n \delta_n).$$

Hence

$$\sup_{z \in A_n} (1 - |z|^2)|f(z)| \leqq (L_n + 2K_n)\delta_n + \max_{z \in F_n} (1 - |z|^2)|f(z)|.$$

Thus if $F = \bigcup F_n$,

$$1 \leqq \sup_{z \in F} (1 - |z|^2)|f(z)| + \varepsilon.$$

If we arrange the set F into a sequence (z_n), then the operator T defined by

$$Tf = ((1 - |z_n|^2) f(z_n))$$

maps β_0 into a subspace of c_0 and $(1 - \varepsilon)\|f\|_\beta \leqq \|Tf\| \leqq \|f\|_\beta$.

Let us remark that our arguments are not restricted to the weight $1 - |z|^2$ appearing in the Bloch norm; as a matter of fact one may consider any radial weight of the form $v(|z|)$ where $v : [0,1] \to [0,1]$ is a continuous decreasing function with $v(r) = 0$ if and only if $r = 1$. For all these weighted spaces of analytic functions Corollary 4.9 is valid. Lusky ([41], [42]) has recently investigated those spaces and shown that they always embed into c_0 isomorphically, and he has determined for which weights one actually gets spaces isomorphic to c_0.

5. Property (M) for subspaces of the Schatten classes

At this point we also consider the Schatten classes c_p of operators on a separable Hilbert space $\mathscr{H}$ and prove similar results like for L_p by essentially the same technique. The reader is cautioned that x^* will now denote the Hilbert space adjoint of an operator $x \in c_p$. We will need the following analogue of Lemma 4.3 above. We use $\|\ \|_\infty$ to denote the operator norm.

Lemma 5.1. *Suppose $1 < p < \infty$ and $p \neq 2$. Suppose $x \in c_p$ and (y_n) is a bounded sequence in c_p. The following conditions are equivalent*:

(i) *For any scalars α and β we have*

$$\lim_{n \to \infty} (\|\alpha x + \beta y_n\|_p^p - |\alpha|^p \|x\|_p^p - |\beta|^p \|y_n\|_p^p) = 0 .$$

(ii) $\lim_{n \to \infty} \|y_n^* x\|_\infty = \lim_{n \to \infty} \|y_n x^*\|_\infty = 0 .$

Proof. (ii) $\Rightarrow$ (i): Given $\varepsilon > 0$ we can choose finite-rank orthogonal projections π, π' so that $\|\pi' x \pi - x\|_p < \varepsilon$ and $\operatorname{ran}(\pi) \subset \operatorname{ran}(x^*)$, $\operatorname{ran}(\pi') \subset \operatorname{ran}(x)$. Now $y_n(\xi) \to 0$ for all $\xi \in \operatorname{ran}(x^*)$ and so $\lim_{n \to \infty} \|y_n \pi\|_\infty = 0$. Since π has finite rank, $\lim_{n \to \infty} \|y_n \pi\|_p = 0$. Similarly $\lim_{n \to \infty} \|\pi' y_n\|_p = 0$.

Now it follows that for any α and β,

$$\lim_{n \to \infty} \big(\|\alpha \pi' x \pi + \beta y_n\|_p - \|\alpha \pi' x \pi + \beta (\mathrm{Id} - \pi') y_n (\mathrm{Id} - \pi)\|_p\big) = 0 .$$

Since

$$\|\alpha \pi' x \pi + \beta (\mathrm{Id} - \pi') y_n (\mathrm{Id} - \pi)\|_p^p = |\alpha|^p \|\pi' x \pi\|_p^p + |\beta|^p \|(\mathrm{Id} - \pi') y_n (\mathrm{Id} - \pi)\|_p^p$$

we quickly obtain that

$$\limsup_{n \to \infty} |\, \|\alpha x + \beta y_n\|_p - (|\alpha|^p \|x\|^p + |\beta|^p \|y_n\|_p^p)^{1/p} | \leqq |\alpha| \varepsilon .$$

Since $\varepsilon > 0$ is arbitrary we have the result.

(i) $\Rightarrow$ (ii): Let us start by calling a sequence (y_n) in c_p *tight* if the induced sequences of singular values $\sigma_n = (s_k(y_n))_{k=1}^\infty$ form a relatively compact subset of ℓ_p. This is easily seen to be equivalent to the requirement that given any $\varepsilon > 0$ there exists a natural number $t = t(\varepsilon)$ so that for each $n \in \mathbb{N}$ there is an orthogonal projection π_n of rank at most t so that $\|y_n - y_n \pi_n\|_p < \varepsilon$.

Let us first assume that (y_n) is a tight sequence. Let $\mathscr{U}$ be any non-principal ultrafilter on the natural numbers and consider the ultraproduct $\mathscr{H}_{\mathscr{U}}$ defined by $\ell_\infty(\mathscr{H})$ quotiented by the subspace of all sequences (ξ_n) so that $\lim_{n \in \mathscr{U}} \|\xi_n\| = 0$.

We will consider the maps $\theta \in \mathscr{L}(\mathscr{H}_{\mathscr{U}})$ defined by $\theta((\xi_n)) = (x(\xi_n))$ and ϕ defined by $\phi((\xi_n)) = (y_n(\xi_n))$. Suppose α and β are any scalars. It is clear that the singular values of $\alpha\theta + \beta\phi$ are given by

$$s_k(\alpha\theta + \beta\phi) = \lim_{n \in \mathscr{U}} s_k(\alpha x + \beta y_n) .$$

We will argue that $(\alpha x + \beta y_n)$ is a tight sequence. In fact, given $\varepsilon > 0$ we can find $t \in \mathbb{N}$ so that for each n there is an orthogonal projection π_n of rank at most t so that

$$\|\beta (y_n - y_n \pi_n)\|_p < \varepsilon / 2 .$$

We also fix a finite-rank projection π_0 of rank t_0 so that $\|\alpha(x - x\pi_0)\|_p < \varepsilon/2$. Let π_n' be the orthogonal projection onto $\operatorname{ran}(\pi_0) + \operatorname{ran}(\pi_n)$. Then π_n' has rank at most $t + t_0$ and $\|(\alpha x + \beta y_n)(\mathrm{Id} - \pi_n')\|_p < \varepsilon$.

It thus follows that $\alpha\theta + \beta\phi \in c_p(\mathscr{H}_{\mathscr{U}})$ and that

$$\|\alpha\theta + \beta\phi\|_p = \lim_{n \in \mathscr{U}} \|\alpha x + \beta y_n\|_p .$$

Hence we have that

$$\|\alpha\theta + \beta\phi\|_p^p = |\alpha|^p \|\theta\|_p^p + |\beta|^p \|\phi\|_p^p .$$

A classical result of McCarthy [43] gives that in the case of equality in the generalized Clarkson inequalities in c_p we have $|\theta||\phi| = 0$ or $\phi\theta^* = \phi^*\theta = 0$. Translating this back we obtain that

$$\lim_{n \to \infty} \|y_n x^*\|_\infty = \lim_{n \to \infty} \|y_n^* x\|_\infty = 0 .$$

This completes the proof when (y_n) is tight.

For the general case we may reduce the problem by passing to a subsequence so that $\lim_{n \to \infty} s_k(y_n) = s_k$ exists for all k. Then if y_n is written in its Schmidt series

$$y_n = \sum_{k=1}^{\infty} s_k(y_n)\, \xi_k \otimes \eta_k$$

where (ξ_k) and (η_k) are orthonormal sequences, we will put $y_n' = \sum_{k=1}^{\infty} s_k\, \xi_k \otimes \eta_k$. Then (y_n') is tight and $\lim \|y_n - y_n'\|_\infty = 0$. To see the latter statement notice that $s_k(y_n) \leqq Mk^{-1/p}$ where $M = \sup \|y_n\|_p$ and so $|s_k(y_n) - s_k| \leqq 2Mk^{-1/p}$ but $\lim_{n \to \infty} s_k(y_n) - s_k = 0$ for each k.

Now suppose α and β are given. Then $(\alpha x + \beta y_n')$ is tight as before and hence we can find isometries u_n and v_n so that the set $\big(u_n(\alpha x + \beta y_n')\, v_n\big)$ is relatively compact. However $\|u_n(y_n - y_n')\|_\infty \to 0$ and by (ii) $\Rightarrow$ (i) and a standard compactness argument we get

$$\|u_n(\alpha x + \beta y_n)\, v_n\|_p^p = \|u_n(\alpha x + \beta y_n')\, v_n\|_p^p + |\beta|^p \|u_n(y_n - y_n')\, v_n\|_p^p + \varepsilon_n$$

where $\lim \varepsilon_n = 0$. Thus

$$\|\alpha x + \beta y_n\|_p^p = \|\alpha x + \beta y_n'\|_p^p + |\beta|^p \|y_n - y_n'\|_p^p + \varepsilon_n .$$

We can apply the same equation with $\alpha = 0$ to obtain

$$\lim (\|y_n\|_p^p - \|y_n'\|_p^p - \|y_n - y_n'\|_p^p) = 0 .$$

Putting this together we see that

$$\lim (\|\alpha x + \beta y_n'\|_p^p - |\alpha|^p \|x\|_p^p - |\beta|^p \|y_n'\|_p^p) = 0 .$$

Since (y_n') is tight this means that $\lim \|y_n' x^*\|_\infty = \lim \|(y_n')^* x\|_\infty = 0$ and hence

$$\lim \|y_n x^*\|_\infty = \lim \|y_n^* x\|_\infty = 0 .$$

This completes the proof. □

To state our main theorem let τ be the topology on c_p given by the seminorms $x \mapsto \|x(\xi)\|$ and $x \mapsto \|x^*(\xi)\|$ for $\xi \in \mathscr{H}$. Denote by c_p^n the space of $n \times n$-matrices with the c_p-norm.

Theorem 5.2. *Suppose $1 < p < \infty$ with $p \neq 2$, and X is a closed infinite-dimensional subspace of c_p. Then the following conditions are equivalent*:

(i) *B_X is compact for the topology τ.*

(ii) *X has property (m_p).*

(iii) *For any $\varepsilon > 0$ there is a subspace X_0 of $\ell_p(c_p^n)$ so that $d(X, X_0) < 1 + \varepsilon$.*

Proof. We note first that c_p has nontrivial cotype ([43] and [52]). Let (ξ_n) be any fixed orthonormal basis of $\mathscr{H}$ and let π_n be the orthogonal projection onto $\operatorname{lin}\{\xi_1, \ldots, \xi_n\}$. Then c_p has an unconditional (FDD) with partial sum operators $S_n(x) = \pi_n x \pi_n$, cf. [5].

We claim that c_p has $(m_p(\tau))$. In fact if $\|x\|_p = 1$ and $\|y_n\|_p = 1$ with $y_n \to 0$ for τ, then since x is compact $\|y_n x^*\|_\infty \to 0$ and $\|y_n^* x\|_\infty \to 0$. Hence

$$\lim \|\alpha x + \beta y_n\|_p = (|\alpha|^p + |\beta|^p)^{1/p}$$

by Lemma 5.1.

Now we give the proof of the theorem:

(iii) $\Rightarrow$ (ii): Obvious.

(ii) $\Rightarrow$ (i): Suppose (y_n) converges to 0 weakly in X. Then for any $x \in X$ we have by Lemma 5.1 that $\lim \|y_n^* x\|_\infty = 0$. In particular it follows that $\lim \|y_n^*(\xi)\| = 0$ whenever ξ is in the closed linear span of $\bigcup_{x \in X} \operatorname{ran}(x)$. From this it follows easily that (y_n^*) converges to 0 for the strong operator topology. Similarly (y_n) converges to 0 for the strong operator topology and so $y_n \to 0$ for τ which implies that B_X is τ-compact.

(i) $\Rightarrow$ (iii): We only need to show that $S_n x \to x$ τ-uniformly for $x \in B_X$. In fact if $\xi \in \mathscr{H}$, then $\{x(\xi) : x \in B_X\}$ is compact and so

$$\lim_{n \to \infty} \sup_{x \in B_X} \|\pi_n x(\xi) - x(\xi)\| = 0 .$$

On the other hand, if $x \in B_X$, then

$$\|\pi_n x \pi_n(\xi) - \pi_n x(\xi)\| \leqq \|\xi - \pi_n(\xi)\| .$$

Combining we have that

$$\lim_{n\to\infty} \sup_{x\in B_X} \|\pi_n x \pi_n(\xi) - x(\xi)\| = 0 .$$

This with a similar statement for x^* allows us to apply Theorem 4.2 to obtain the result. □

The following result is analogous to Corollary 4.5.

Corollary 5.3. *Suppose $1 < p < \infty$ and let X be a closed infinite-dimensional subspace of c_p with property (M). Then there exists $r \in \{2, p\}$ so that X has (m_r). If X does not contain a subspace isomorphic to ℓ_2, for any $\varepsilon > 0$ there exists a closed subspace X_0 of $\ell_p(c_p^n)$ with $d(X, X_0) < 1 + \varepsilon$.*

Proof. We argue as before that X is stable, which was proved by Arazy [4]. Therefore, for some r, X has (m_r) and contains a copy of ℓ_r. This implies that $r = 2$ or $r = p$ by [5]. It remains to apply Theorem 5.2. □

Again, we specialize to $p > 2$.

Corollary 5.4. *Suppose $2 < p < \infty$ and that X is a subspace of c_p. Then the following conditions on X are equivalent:*

(i) *X contains no subspace isomorphic to ℓ_2.*

(ii) *X is isomorphic to a subspace of $\ell_p(c_p^n)$.*

(iii) *For any $\varepsilon > 0$ there exists a closed subspace X_0 of $\ell_p(c_p^n)$ so that $d(X, X_0) < 1 + \varepsilon$.*

Proof. The argument is very similar to that of Corollary 4.6. We only have to show that (i) implies (iii), that is, by Theorem 5.2, that B_X is τ-compact. If not, there would be a weakly null sequence (x_n) in B_X which is an unconditional basic sequence and satisfies $\inf\|x_n(\xi)\| > 0$ or $\inf\|x_n^*(\xi)\| > 0$ for some $\xi \in \mathscr{K}$. (One can extract an unconditional basic sequence since c_p, having an unconditional (FDD), embeds into a space with an unconditional basis [39], Th. 1.g.5.) Let us assume without loss of generality that $\inf\|x_n(\xi)\| > 0$. We then have for suitable positive constants c_i and all finite sequences of scalars $a_1, \dots, a_n$

$$\begin{aligned}\left\| \sum_{k=1}^{n} a_k x_k \right\|_p &\geqq c_1 \operatorname{ave} \left\| \sum_{k=1}^{n} \pm a_k x_k \right\|_p \\ &\geqq c_1 \operatorname{ave} \left\| \sum_{k=1}^{n} \pm a_k x_k(\xi) \right\| \\ &\geqq c_2 \left(\sum_{k=1}^{n} \|a_k x_k(\xi)\|^2 \right)^{1/2} \\ &\geqq c_3 \left(\sum_{k=1}^{n} |a_k|^2 \right)^{1/2} .\end{aligned}$$

On the other hand, c_p has type 2 ([43] and [52]), and we get the reverse estimate

$$\left\| \sum_{k=1}^{n} a_k x_k \right\|_p \leqq c_4 \left(\sum_{k=1}^{n} |a_k|^2 \right)^{1/2} .$$

Therefore $\overline{\text{lin}}\{x_1, x_2, \ldots\}$ is isomorphic to a Hilbert space. □

The equivalence of (i) and (ii) in the above corollary was first proved by Arazy and Lindenstrauss [5].

6. An operator version of property (M)

In this section we study M-ideals of compact operators between distinct Banach spaces X and Y. Our investigations will be based on an operator version of property (M).

Definition 6.1. An operator T with $\|T\| \leqq 1$ between Banach spaces X and Y is said to have *property* (M) if for all $x \in X$, $y \in Y$, $\|y\| \leqq \|x\|$, and weakly null sequences $(x_n) \subset X$,

$$\limsup \|y + Tx_n\| \leqq \limsup \|x + x_n\| . \tag{4}$$

Note that a Banach space has property (M) if the identity operator has property (M) of Definition 6.1. We also mention that Lemma 2.2 of [30] implies that every contractive operator in $\mathscr{L}(X)$ has (M) if and only if the identity operator has (M).

We shall need the following lemma.

Lemma 6.2. *If $T: X \to Y$ is a contractive operator with (M), $(u_n) \subset X$ and $(v_n) \subset Y$ are relatively compact sequences with $\|v_n\| \leqq \|u_n\|$ for all n, and if $(x_n) \subset X$ is weakly null, then*

$$\limsup \|v_n + Tx_n\| \leqq \limsup \|u_n + x_n\| .$$

Proof. Otherwise we would have

$$\limsup \|v_n + Tx_n\| > \limsup \|u_n + x_n\| .$$

After passing to subsequences we could deal with convergent sequences, say $\lim u_n = u$ and $\lim v_n = v$, and obtain

$$\limsup \|v + Tx_n\| > \limsup \|u + x_n\|$$

contradicting that $\|v\| \leqq \|u\|$ and that T has (M). □

The next result is a variant of Theorem 2.4 in [30], and it has a very similar proof.

Theorem 6.3. *Suppose that the Banach space X admits a sequence of compact operators $K_n \in \mathscr{K}(X)$ satisfying*

(a) $K_n x \to x \quad \forall x \in X$,

(b) $K_n^* x^* \to x^* \quad \forall x^* \in X^*$,

(c) $\|\mathrm{Id} - 2K_n\| \to 1$.

Let Y be a Banach space. Then $\mathscr{K}(X,Y)$ is an M-ideal in $\mathscr{L}(X,Y)$ if and only if every contraction $T: X \to Y$ has (M).

Proof. Suppose first that every contraction has property (M). To show that $\mathscr{K}(X,Y)$ is an M-ideal in $\mathscr{L}(X,Y)$ we establish the 3-ball property of [33], Th. 6.17 (see also [20], Th. I.2.2), that is, given $S_1, S_2, S_3 \in \mathscr{K}(X,Y)$ with $\|S_i\| \leqq 1$, $T \in \mathscr{L}(X,Y)$ with $\|T\| \leqq 1$, and $\varepsilon > 0$, there is some compact operator S satisfying

$$\|S_i + T - S\| \leqq 1 + \varepsilon, \quad i = 1, 2, 3.$$

In fact, we shall show that $S = TK_n$ will work for large n.

Since $K_n^* \to \mathrm{Id}_{X^*}$ strongly and (K_n) is uniformly bounded, (K_n^*) converges uniformly on relatively compact sets and thus $\|S_i K_m - S_i\| \leqq \varepsilon/2$, $i = 1, 2, 3$, for large m. We also assume that $\|\mathrm{Id} - 2K_m\| \leqq 1 + \varepsilon/2$. Consider a sequence (x_n) in the unit ball of X such that

$$\limsup_n \|S_1 K_m x_n + (T - TK_n) x_n\| = \limsup_n \|S_1 K_m + T - TK_n\|.$$

Note that $(\mathrm{Id} - K_n) x_n \to 0$ weakly. By Lemma 6.2 and the fact that T has (M) we get the inequality

$$\begin{aligned} \limsup_n \|S_1 K_m x_n + T(\mathrm{Id} - K_n) x_n\| &\leqq \limsup_n \|K_m x_n + (\mathrm{Id} - K_n) x_n\| \\ &\leqq \limsup_n \|K_m + (\mathrm{Id} - K_n)\|, \end{aligned}$$

and the last quantity is dominated by $\|\mathrm{Id} - 2K_m\|$ (see [30], p. 153). Therefore,

$$\limsup_n \|S_1 + T - TK_n\| < 1 + \varepsilon,$$

and

$$\lim_r \|S_1 + T - TK_{n_r}\| < 1 + \varepsilon$$

for some subsequence (K_{n_r}). After switching to further subsequences one obtains the same type of inequality for S_2 and S_3 which proves the desired 3-ball property.

On the other hand suppose that $\mathscr{K}(X,Y)$ is an M-ideal in $\mathscr{L}(X,Y)$. Let $x \in X$, $y \in Y$, $\|y\| \leqq \|x\|$. Fix a compact operator S with $\|S\| \leqq 1$ and $Sx = y$. Let $T: X \to Y$ be a contraction and suppose that $x_n \to 0$ weakly. For each $\eta > 0$ there is a compact operator $U: X \to Y$ such that $\|Ux - Tx\| \leqq \eta$ and $\|S + T - U\| \leqq 1 + \eta$ (see [55], Theorem 3.1, Remark). Thus

$$\begin{aligned}\limsup \|y + Tx_n\| &= \limsup \|Sx + Tx_n\| \\ &= \limsup \|S(x + x_n) + (T - U)x_n\| \\ &= \limsup \|S(x + x_n) + (T - U)(x + x_n)\| + \eta \\ &\leqq (1 + \eta) \limsup \|x + x_n\| + \eta .\end{aligned}$$

Since η was arbitrary, this proves that T has property (M). □

It is easy to check that (4) of Definition 6.1 holds if X and Y both have property (M). The validity of Theorem 6.3 under these extra assumptions has already been observed in [45] (cf. [20], Cor. VI.4.18); but we will later encounter situations where Theorem 6.3 applies, but Y fails (M) (cf. Propositions 6.5 and 6.6).

We also mention that the assumptions on X in Theorem 6.3 enforce X (and X^*) to be separable. Nonseparable versions (as in [45] and [20], Chap. VI.4) are easily supplied, and we leave it to the reader to spell these out. Likewise, we don't formulate a corresponding result based on the notion of (M^*) of [30], which can be done without difficulty.

Note that the assumptions on X above hold in particular if X has a 1-unconditional shrinking basis; e.g., $X = \ell_p$ for $1 < p < \infty$. In this case there is an embellished version of Theorem 6.3 which we state next. As usual, (e_n) denotes the unit vector basis in ℓ_p.

Corollary 6.4. *Suppose $1 < p < \infty$. Then $\mathscr{K}(\ell_p, Y)$ is an M-ideal in $\mathscr{L}(\ell_p, Y)$ if and only if for all $y \in Y$ and all operators $T: \ell_p \to Y$ the condition*

$$\limsup \|y + Te_n\| \leqq (\|y\|^p + \|T\|^p)^{1/p} \tag{5}$$

holds.

Proof. The condition is necessary by Theorem 6.3 since

$$\limsup \|x + e_n\| = (\|x\|^p + 1)^{1/p} .$$

Next suppose that (5) holds. If (x_n) is a weakly null sequence in ℓ_p and if (4) fails for some contractive $T: \ell_p \to Y$, $x \in \ell_p$ and $y \in Y$ with $\|y\| \leqq \|x\|$, then we may, after passing to subsequences, assume that

$$\lim \|y + Tx_n\| > \lim \|x + x_n\|$$

and that $\gamma = \lim \|x_n\|$ exists. We may also assume without loss of generality that $\gamma = 1$; thus

$$\lim \|y + Tx_n\| > (\|x\|^p + 1)^{1/p} .$$

On the other hand, some subsequence (ξ_n) of the sequence (x_n) is almost isometrically equivalent to the unit vector basis of ℓ_p. Let $\Phi: \ell_p \to \overline{\operatorname{lin}}\{\xi_1, \xi_2, \ldots\}$ be a $(1 + \eta)$-isomorphism mapping e_n onto ξ_n. Then (5) implies that

$$\limsup \|y + T\xi_n\| = \limsup \|y + (T\Phi)e_n\| \leqq (\|y\|^p + \|T\Phi\|^p)^{1/p} ,$$

thus

$$\limsup \|y + T\xi_n\| \leqq (\|x\|^p + (1+\eta)^p)^{1/p}$$

since $\|T\Phi\| \leqq 1 + \eta$, which is a contradiction if η is small enough. □

Let us mention some situations where this corollary applies, thus providing new examples of M-ideals of compact operators.

Proposition 6.5. *Let $2 \leqq p < \infty$. Then $\mathscr{K}(\ell_p, L_p[0,1])$ is an M-ideal in the space $\mathscr{L}(\ell_p, L_p[0,1])$.*

Proof. Since the case $p = 2$ is well-known, we only have to take care of the case $2 < p < \infty$. Let $T: \ell_p \to L_p[0,1]$ be an operator. We claim that the sequence $(f_n) = (Te_n)$ tends to 0 in measure which will prove that

$$\begin{aligned} \limsup \|y + Te_n\| &= \limsup (\|y\|^p + \|Te_n\|^p)^{1/p} \\ &\leqq (\|y\|^p + \|T\|^p)^{1/p} \end{aligned} \tag{6}$$

whenever $y \in L_p[0,1]$, and hence our assertion by Corollary 6.4. (In (6) we are using the fact that L_p has $(m_p(\tau))$ for the topology of convergence in measure.)

Suppose that (f_n) does not tend to 0 in measure. Then there are $\varepsilon > 0$, $\delta > 0$ and Borel sets A_n of measure $\geqq \delta$ such that $|f_n| \geqq \varepsilon \chi_{A_n}$ a.e. for infinitely many n. There is clearly no loss in generality in assuming this to hold for all n. Then we have for some positive constants c_i (the existence of c_1 below follows from the Khintchin inequalities, cf. [40], p. 50) and all $N \in \mathbb{N}$

$$\begin{aligned} \|T\| N^{1/p} &= \|T\| \left\| \sum_{n=1}^{N} \pm e_n \right\| \\ &\geqq \operatorname{ave} \left\| \sum_{n=1}^{N} \pm Te_n \right\| \\ &= \operatorname{ave} \left\| \sum_{n=1}^{N} \pm f_n \right\| \\ &\geqq c_1 \left\| \left(\sum_{n=1}^{N} |f_n|^2 \right)^{1/2} \right\| \\ &\geqq c_1 \varepsilon \left\| \left(\sum_{n=1}^{N} \chi_{A_n} \right)^{1/2} \right\| \\ &\geqq c_2 N^{1/2} \end{aligned}$$

(where the average extends over all choices of signs), which enforces $p \leqq 2$.

Thus the proof of the proposition is completed. □

Note that $L_p[0,1]$ fails property (M) unless $p=2$. Proposition 6.5 does not extend to $1<p<2$ because in this case $\mathscr{K}(\ell_p, L_p[0,1])$ is not even proximinal in $\mathscr{L}(\ell_p, L_p[0,1])$ [6]; proximinality is a well-known property of M-ideals, see e.g. [20], Prop. II.1.1. On the other hand, Proposition 6.5 implies the proximinality of $\mathscr{K}(\ell_p, L_p[0,1])$ in $\mathscr{L}(\ell_p, L_p[0,1])$ for $2<p<\infty$ which was first proved by Bang and Odell [6].

The well-known result that $\mathscr{K}(\ell_p, \ell_q)$ is an M-ideal in $\mathscr{L}(\ell_p, \ell_q)$ for $1<p\leqq q<\infty$ can clearly also be deduced from Corollary 6.4.

Next we consider the Schatten class c_p of all compact operators on ℓ_2 with p-summable singular values, and we denote its subspace consisting of those operators u whose matrix representations with respect to the unit vector basis in ℓ_2 are upper triagonal (i.e., $u(e_k)\in \operatorname{lin}\{e_1,\dots,e_k\}$ for all k) by UT_p.

Proposition 6.6. (a) *Let $1<p\leqq 2$. Then $\mathscr{K}(\ell_p, UT_p)$ is an M-ideal in $\mathscr{L}(\ell_p, UT_p)$.*

(b) *Let $2\leqq p<\infty$. Then $\mathscr{K}(\ell_p, c_p)$ is an M-ideal in $\mathscr{L}(\ell_p, c_p)$.*

(c) *The space $\mathscr{K}(c_0, \mathscr{K}(\ell_2))$ is an M-ideal in $\mathscr{L}(c_0, \mathscr{K}(\ell_2))$.*

Proof. (a) In the case $p=2$ we are dealing with Hilbert spaces, so the result is clear. Let $1<p<2$. For a fixed $m\in\mathbb{N}$ consider the orthonormal projection $\pi:\ell_2\to\ell_2$ mapping onto the first m coordinates. We first claim that the operator

$$\Phi_p: c_p\oplus_p c_p \to c_p, \quad (y,z)\mapsto y\pi+(z-z\pi),$$

is contractive. Indeed, this is trivially true for $p=1$ by the triangle inequality, and it holds for $p=2$ since the Hilbert-Schmidt norm of an operator on ℓ_2 is just the square sum norm of the corresponding infinite matrix. Applying the complex interpolation method we deduce the same result in the interval $1<p<2$; note that the complex interpolation space $[c_1,c_2]_\theta$ for $\theta=2-\dfrac{2}{p}$ is c_p [49] and that $[c_1\oplus_1 c_1, c_2\oplus_2 c_2]_\theta = c_p\oplus_p c_p$ [8], Th. 5.63.

That Φ_p is a contraction can be rephrased in the following way (denote the matrix entries of an operator $y\in c_p$ by $y(k,l)$): If, for some m, $y(k,l)=0$ for $l>m$ and $z(k,l)=0$ for $l\leqq m$, then

$$\|y+z\|_p \leqq (\|y\|_p^p+\|z\|_p^p)^{1/p}.$$

Let us now check that (5) of Corollary 6.4 holds. Since the operators with finite matrix representations are dense in UT_p, we may assume that $y(k,l)=0$ for $l>m$ for some m. Also, $z_n := Te_n\to 0$ weakly so that $z_n(k,l)\to 0$ for all $k,l\leqq m$. Therefore, with π as above, $\|z_n\pi\|_p\to 0$ (recall that the z_n are upper triagonal) and thus

$$\begin{aligned}\limsup\|y+Te_n\|_p &= \limsup\|y+(z_n-z_n\pi)\|_p\\ &\leqq \limsup(\|y\|_p^p+\|z_n-z_n\pi\|_p^p)^{1/p}\\ &\leqq (\|y\|_p^p+\|T\|_p^p)^{1/p},\end{aligned}$$

since $(z_n-z_n\pi)(k,l)=0$ for $l\leqq m$.

(b) Again, we will check (5) of Corollary 6.4, and we may and shall assume that $y(k,l)=0$ if $k>m$ or $l>m$, for some m. Let $z_n = Te_n$, and let π be as above.

If $z \in c_p$, then πz is a finite-rank operator mapping ℓ_2 into ℓ_2^m. Since $c_p(\ell_2, \ell_2^m)$ is isomorphic to ℓ_2, we can think of the operator $\Psi: \ell_p \to c_p$ mapping $x \in \ell_p$ to $\pi \circ Tx$ as an operator from ℓ_p to ℓ_2, and since $p > 2$, Ψ is compact [39], Prop. 2.c.3. Consequently, $\|\pi \circ Te_n\|_p \to 0$. Likewise, $x \mapsto Tx \circ \pi$ is compact, and $\|Te_n \circ \pi\|_p \to 0$. Therefore $y + Te_n$ becomes essentially block-diagonal, and we have

$$\limsup \|y + Te_n\|_p = \limsup \|\pi y \pi + (\mathrm{Id} - \pi)(Te_n)(\mathrm{Id} - \pi)\|_p$$
$$\leqq (\|y\|_p^p + \|T\|_p^p)^{1/p}.$$

(c) The proof here is similar to that of part (b). □

The M-ideal property of $\mathscr{K}(\ell_p, V)$ for some subspaces V of c_p has been investigated by P. Harmand (unpublished). Since by Corollary 6.4 the class of Banach spaces Y for which $\mathscr{K}(\ell_p, Y)$ is an M-ideal in $\mathscr{L}(\ell_p, Y)$ is hereditary, those results are contained in the above proposition.

The assertion in part (c) above corresponds to the limiting case $p = \infty$ in (b). By contrast, we have the following negative result which was prompted by a question of T.S.S. Rao.

Proposition 6.7. *If $\mathscr{H}$ is an infinite dimensional Hilbert space, then the three-fold injective tensor product $\mathscr{H} \hat{\otimes}_\varepsilon \mathscr{H} \hat{\otimes}_\varepsilon \mathscr{H}$ is not an M-ideal in its bidual. In fact,*

$$\mathscr{H} \hat{\otimes}_\varepsilon \mathscr{H} \hat{\otimes}_\varepsilon \mathscr{H} = \mathscr{K}(\mathscr{H}, \mathscr{K}(\mathscr{H}))$$

is not an M-ideal in $\mathscr{L}(\mathscr{H}, \mathscr{K}(\mathscr{H}))$

Proof. Since the bidual of $\mathscr{H} \hat{\otimes}_\varepsilon \mathscr{H} \hat{\otimes}_\varepsilon \mathscr{H}$ is isometric with $\mathscr{L}(\mathscr{H}, \mathscr{L}(\mathscr{H}))$, it suffices to prove the latter assertion. It is also enough to assume that $\mathscr{H}$ is separable and thus represented as ℓ_2. We shall show that (5) of Corollary 6.4 fails.

Let $T: \ell_2 \to \mathscr{K}(\ell_2)$ be the operator that maps the sequence $(\alpha_n) \in \ell_2$ to the matrix

$$\begin{pmatrix} 0 & \alpha_1 & \alpha_2 & \alpha_3 & \cdots \\ \alpha_1 & 0 & 0 & 0 & \cdots \\ \alpha_2 & 0 & 0 & 0 & \cdots \\ \alpha_3 & 0 & 0 & & \\ \vdots & \vdots & \vdots & & \ddots \end{pmatrix}.$$

Then $\|T\| = 1$, but since the norm of the matrix

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

is $\frac{1}{2}(1+\sqrt{5})$, we have that $\|e_1 \otimes e_1 + Te_n\| = \frac{1}{2}(1+\sqrt{5}) > \sqrt{2}$. □

However, for the space UT_∞ of upper triangular compact operators it follows easily from Corollary 6.4 that $\mathscr{K}(\mathscr{H}, UT_\infty)$ is an M-ideal in $\mathscr{L}(\mathscr{H}, UT_\infty)$.

A more complete picture of those n-fold tensor products of ℓ_p-spaces which are M-ideals in their biduals will be given in [16].

7. Open questions

Here we gather some problems suggested by our work.

Problem 7.1. The method of proof employed in Proposition 6.5 suggests considering a version of property (M) with respect to convergence in measure, meaning that $\limsup \|f + g_n\|$ depends only on $\|f\|$ whenever $g_n \to 0$ in measure. It might be interesting to pursue this idea further.

Let us indicate why such a notion might be useful. Suppose X is a function space on $[0,1]$ that can be renormed to have this property. Then every subspace generated by a sequence of pairwise disjoint functions has the usual property (M). Consequently, by [30], Th. 4.3, if an order continuous nonatomic Banach lattice Y different from L_2 embeds isomorphically into X, then the Haar system in Y cannot embed on disjoint functions in X. This is an important property in the investigation of rearrangement invariant spaces, see [25], [29] and [40], Sect. 2.e.

By way of illustration we observe that this line of reasoning together with [29], Th. 7.5, implies the following result.

Proposition 7.2. *If $X \neq L_2$ is a separable rearrangement invariant space of functions on $[0,1]$ that can be renormed to have the above property (M) with respect to convergence in measure and if Y is a rearrangement invariant space on $[0,1]$ isomorphic to X, then $X = Y$, and the norms of X and Y are equivalent (for short, the space X has unique rearrangement invariant structure).*

One can check that the Lorentz spaces $L_{p,q}[0,1]$ can be so renormed and thus Proposition 7.2 can be applied to them. In fact, the canonical norm works in the case $1 \leqq q < p < \infty$, but for $1 < p < q < \infty$ the canonical expression is only a quasinorm, and one should consider the norm

$$\|f\|_{(p,q)} = \left(\int_0^1 (t^{1/p} f^{**}(t))^q \frac{dt}{t} \right)^{1/q}.$$

Let us remark that $L_{p,q}[0,1]$ even has $(m_q(\tau))$ for the topology τ of convergence in measure. For $q = \infty$ one obtains this conclusion for the span of the bounded functions in $L_{p,\infty}[0,1]$; note that this span is an M-ideal in its bidual [54].

We conjecture that Orlicz spaces allow such a renorming, too. Anyway, in the case of Orlicz spaces one can argue that a sequence of pairwise disjoint functions generates a modular sequence space which can be renormed to have property (M) by [30], Prop. 4.1. This argument is a variant of that given in [25], p. 168, for the reflexive case.

Problem 7.3. Is $\mathscr{K}(\ell_p, c_p)$ an M-ideal in $\mathscr{L}(\ell_p, c_p)$ for $1 < p < 2$ (cf. Proposition 6.6(a))? Note that c_p is isomorphic to UT_p [5].

Problem 7.4. What about the commutative version of Proposition 6.6(a), i.e., is $\mathscr{K}(\ell_p, H_p)$ an M-ideal in $\mathscr{L}(\ell_p, H_p)$ for $1 < p < 2$? Recall that the Hardy space H_p must not be replaced by L_p, by the results of [6].

Problem 7.5. By Corollary 4.9 and well-known results on M-ideals (see [30] or [20], Th. V. 5.4) there is a sequence (K_n) of compact operators on the little Bloch space β_0 such that $K_n \to \mathrm{Id}$ strongly and $\|\mathrm{Id} - 2K_n\| \to 1$. Such operators can actually be obtained by taking suitable convex combinations of Fejér operators. Is it possible to find such K_n explicitly among the classical kernel operators? This would give a direct proof, which does not rely on the embedding of β_0 into c_0, that $\mathscr{K}(\beta_0)$ is an M-ideal in $\mathscr{L}(\beta_0)$ and, moreover, that $\mathscr{K}(X, \beta_0)$ is an M-ideal in $\mathscr{L}(X, \beta_0)$ for all Banach spaces X, cf. [36]. In fact, it suffices to find K_n satisfying $\|\mathrm{Id} - K_n\| \to 1$ [45]. Also, it would be interesting to check property (M^*) for β_0 directly.

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Eingegangen 3. November 1993, in revidierter Fassung 2. November 1994