Sequences of random variables in L_p for $p<1$

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1. Introduction

Let (Ω, Σ, P) be a probability space and suppose $0<p<2$. Suppose $X \in L_p(\Omega)$ and that $(X_n\colon n \geqq 1)$ is a sequence of independent identically distributed random variables on Ω with $\operatorname{dist} X_n = \operatorname{dist} X$; denote by $\Lambda_p(X)$ the closed linear span of $(X_n\colon n \geqq 1)$. If $p \geqq 1$ and $E(X)=0$ then (X_n) is a basis for $\Lambda_p(X)$ equivalent to the unit vector basis of a certain Orlicz sequence space (see [2]). In general if $E(X) \neq 0$, $\Lambda_p(X)$ is still isomorphic to an Orlicz sequence space, as can be proved by considering the basic sequence $(X_n - E(X_n))$.

The purpose of this note is to investigate the situation when $0<p<1$. We shall show that in this case it is possible for $\Lambda_p(X)$ to fail to have a separating dual, and that when this happens $\Lambda_p(X)$ is a twisted sum [5] of the real line and an Orlicz sequence space; more precisely if R is the subspace of constants, then R is an uncomplemented subspace of $\Lambda_p(X)$ and $\Lambda_p(X)/R$ is isomorphic to an Orlicz sequence space.

A special case arises when X has the probability density function f given by:

$$f(x) = \begin{cases} \dfrac{1}{x^2}, & x \geqq 1, \\ 0\,, & x<1 \end{cases}$$

(or one may use $X=|Y|$, where Y has the Cauchy distribution). In this case $\Lambda_p(X)$ for $0<p<1$ turns out to be isomorphic to the *Ribe space* constructed in [6]. The Ribe space is an example of a non-locally convex space whose quotient by a line is isomorphic to the Banach space l_1; thus the Ribe space embeds into L_p for $0<p<1$. This embedding can be further used to show that L_p contains a needle-point which is the critical step in Roberts's proof [7] that L_p contains a compact convex set with no extreme points; indeed any point of R is a needle-point of $\Lambda_p(X)$, as a simple calculation shows.

We note that the ideas of this paper are based on a recent paper of Bourgain and Rosenthal [1].

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2. Notation

Let ϕ be a strictly increasing continuous real-valued function on $[0, \infty)$ with $\phi(0)=0$. Suppose that for some constants α_1, α_2 with $1<\alpha_1 \leqq \alpha_2<\infty$ we have

$$\alpha_1 \phi(x) \leqq \phi(2x) \leqq \alpha_2 \phi(x), \qquad 0 \leqq x \leqq 1. \tag{1}$$

Then the Orlicz sequence space l_ϕ is the space of real sequences (x_n) with

$$\sum \phi(|x_n|) < \infty .$$

This is a complete locally bounded F-space (i.e. a quasi-Banach space) when quasi-normed by

$$\|x\| = \inf\{\theta > 0 : \sum \phi(\theta^{-1}|x_n|) \leqq 1\}$$

for $x=(x_n) \in l_\phi$.

Now suppose M and N are two quasi-Banach spaces. Then a twisted sum L of M and N is a quasi-Banach space with a subspace $M_0 \cong M$ such that $L/M \cong N$. Twisted sums can be constructed by using quasi-linear maps ([3], [5], [6]). Let N_0 be a dense subspace of N and $F\colon N_0 \to M$ be a map satisfying the conditions

$$F(tx) = tF(x), \qquad t \in R,\ x \in N_0, \tag{2}$$

$$\|F(x_1+x_2)-F(x_1)-F(x_2)\| \leqq B(\|x_1\|+\|x_2\|), \qquad x_1, x_2 \in N_0 \tag{3}$$

where B is some constant independent of x_1 and x_2. Then $M \oplus_F N_0$ is the algebraic direct sum $M \oplus N_0$ quasi-normed by

$$\|(y, x)\| = \|y - F(x)\| + \|x\|. \tag{4}$$

The completion $M \oplus_F N$ is a twisted sum of M and N. It will be a direct sum (i.e. there will be a projection onto the subspace $M \oplus \{0\}$) if and only if for some linear map $G\colon N_0 \to R$ and some constant B_1

$$\|F(x) - G(x)\| \leqq B_1 \|x\|, \qquad x \in N_0, \tag{5}$$

If H is any other quasi-linear map $H\colon N_0 \to M$ then $M \oplus_F N$ and $M \oplus_H N$ will be isomorphic provided that for some $c \neq 0$, $B_1 < \infty$ and linear $G\colon N_0 \to M$

$$\|F(x) - cH(x) - G(x)\| \leqq B_1 \|x\|, \qquad x \in N_0. \tag{6}$$

(see [6]).

The Ribe space is an example when $M=\mathbb{R}$ and $N=l_1$. Let $N_0=\mathbb{R}^\infty$, the finitely supported sequences in l_1, and define

$$F(x) = \sum_{i=1}^{\infty} x_i \log|x_i| - \left(\sum_{i=1}^{\infty} x_i\right) \log\left|\sum_{i=1}^{\infty} x_i\right| \tag{7}$$

(where $0 \log 0 = 0$). Other non-trivial twisted sums were constructed in [3] and [8]. In [7] it is shown that the Ribe space is, in certain respects, the "worst" twisted sum of R and l_1.

Now suppose (Ω, Σ, P) is a probability space. If X is a random variable on Ω we denote by $\hat{X}$ its characteristic function i.e.

$$\hat{X}(s) = E(e^{isX}) = \int e^{isX(\omega)}\, dP(\omega).$$

We denote by $X \wedge Y$ the pointwise minimum of two random variables. If $X \in L_p$ then

$$\|X\|_p = (E(|X|^p))^{1/p}$$

is the quasi-norm defining the topology on L_p.

3. Main results

We start with a simple lemma which can be easily deduced from known results (cf. [9] p. 112). However it is quick and easy to prove directly.

Lemma 3. 1. *Let p be fixed with $0<p<2$. Then for every $\varepsilon>0$ there exists $\delta=\delta(\varepsilon)>0$ so that whenever $\{Y_1,\ldots, Y_n\}$ are mutually independent random variables with $|Y_i|\leqq 1$, $1\leqq i\leqq n$ and $c\in \mathbb{R}$ is such that*

$$\|c+Y_1+\cdots+Y_n\|_p \leqq \delta(\varepsilon)$$

then

$$\|c+Y_1+\cdots+Y_n\|_2 \leqq \varepsilon.$$

Proof. If the lemma is false there exists for each $m\in\mathbb{N}$ a finite sequence of mutually independent random variables $(Y_1,\ldots, Y_{N(m)})$ with $|Y_i|\leqq 1$ and scalars c_m so that if

$$Z_m = c_m + Y_1 + \cdots + Y_{N(m)}$$

then

$$\|Z_m\|_2 \geqq \varepsilon \quad \text{but} \quad \|Z_m\|_p \leqq \frac{1}{m}.$$

Let $U_{m,n}=Y_{m,n}-E(Y_{m,n})$. By Taylor's theorem if $|t|\leqq 1$

$$\left|e^{it}-1-it-\frac{1}{2}t^2\right| \leqq \frac{1}{3}t^2$$

and so if $|s|\leqq \frac{1}{2}$

$$|\hat{U}_{m,n}(s)| \leqq 1-\frac{1}{6}s^2 E(U_{m,n}^2).$$

If $V_m = Z_m - E(Z_m)$ then

$$|\hat{V}_m(s)| = \prod_{n=1}^{N(m)} |\hat{U}_{m,n}(s)| \leqq \exp\left(-\frac{1}{6}s^2 \sum_{n=1}^{N(m)} E(U_{m,n}^2)\right), \quad |s|\leqq\frac{1}{2}$$
$$= \exp\left(-\frac{1}{6}s^2 E(V_m^2)\right).$$

Now $Z_m \to 0$ in probability and hence $|\hat{V}_m(s)| \to 1$. Thus $E(V_m^2)\to 0$. As $V_m - Z_m \to 0$ in probability, $E(Z_m)\to 0$ and so $\|Z_m\|_2 \to 0$, contrary to assumption.

From now on we fix p with $0<p<1$ and suppose $X \in L_p$ but $X \notin L_2$. Define for $t>0$

$$U_t(\omega)=\begin{cases} X(\omega) & \text{if } |X(\omega)| \leqq t^{-1}, \\ 0 & \text{otherwise.} \end{cases}$$

For convenience in notation, let $U_0 = X$.

Lemma 2. *There exists $a>0$ so that if $0<t \leqq a$ then*

$$E(|U_t|) \leqq \frac{1}{2} \{E(U_t^2)\}^{1/2}.$$

Proof. $\|U_t\|_2 \to \infty$ and so

$$U_t/\|U_t\|_2 \to 0 \quad \text{as} \quad t \to 0$$

in probability and is equi-integrable in L_1. Hence

$$\|U_t\|_1/\|U_t\|_2 \to 0.$$

Now define

$$\phi(t)=E(t^2X^2 \wedge t^p|X|^p), \qquad t \geqq 0 \tag{9}$$

$$\psi(t)=tE(U_t), \qquad t \geqq 0 \tag{10}$$
$$= \int_{|tX| \leqq 1} tX(\omega)\, dP(\omega).$$

ϕ is an Orlicz function satisfying condition (1). We denote by $\|\cdot\|_\phi$ the quasi-norm on l_ϕ.

Define a map $F: \mathbb{R}^\infty \to \mathbb{R}$ by

$$F(a)=\|a\|_\phi \sum_{n=1}^{\infty} \psi\left(\frac{a_n}{\|a\|_\phi}\right), \qquad a \neq 0. \tag{11}$$

F clearly satisfies (2). We shall deduce later that F satisfies (3) with respect to $\|\cdot\|_\phi$.

Let $(X_n: n \geqq 1)$ be a sequence of independent random variables with $\operatorname{dist} X_n = \operatorname{dist} X$. Let M be the linear span of 1 and $(X_n: n \geqq 1)$.

Theorem 1. *The L_p-quasi-norm as M is equivalent to the quasi-norm*

$$|||c+\sum_{j=1}^{\infty} a_j X_j||| = |c+F(a)| + \|a\|_\phi, \qquad a \in R^\infty, \quad c \in R. \tag{12}$$

Proof. We start by observing that it is not immediately clear that (12) defines a quasi-norm on M; this requires F to satisfy (3). However both facts follow automatically if we can show that there exists constants $0<\beta_1<\beta_2<\infty$ with

$$\beta_1 \|Z\|_p \leqq |||Z||| \leqq \beta_2 \|Z\|_p, \qquad Z \in M.$$

First suppose

$$Z = a_1 X + \cdots + a_n X_n - F(a)$$

and that $\|a\|_\phi = 1$. Let

$$Y_j(\omega) = \begin{cases} a_j X_j(\omega) & \text{if } |a_j X_j(\omega)| \leqq 1, \\ 0 & \text{if } |a_j X_j(\omega)| > 1 \end{cases}$$

and let $Y_j' = a_j X_j - Y_j$. Then

$$\left\| \sum_{j=1}^n Y_j' \right\|_p^p \leqq \sum_{j=1}^n \|Y_j'\|_p^p = \sum_j \int_{|a_j X| > 1} |a_j X|^p \, dP \leqq \sum_j \phi(|a_j|) = 1$$

and

$$\left\| \sum_{j=1}^n Y_j - F(a) \right\|_p = \left\| \sum_{j=1}^n (Y_j - E(Y_j)) \right\|_p \leqq \left\| \sum_{j=1}^n (Y_j - E(Y_j)) \right\|_2 = \left\{ \sum_{j=1}^n \|Y_j - E(Y_j)\|_2^2 \right\}^{1/2}$$

(since $Y_j - E(Y_j)$ are mutually independent)

$$\leqq \left\{ \sum_{j=1}^n \|Y_j\|_2^2 \right\}^{1/2} \leqq \left\{ \sum_{j=1}^n \int_{|a_j X| \leqq 1} |a_j X|^2 \, dP \right\}^{1/2} \leqq \left(\sum_{j=1}^n \phi(|a_j|) \right)^{1/2} = 1.$$

Combining

$$\|Z\|_p \leqq 2^{1/p}.$$

Now suppose $|||Z||| = 1$ where

$$Z = c + \sum_{j=1}^n a_j X_j.$$

Then

$$\left\| \sum_{j=1}^n a_j X_j - F(a) \right\|_p \leqq 2^{1/p} \|a\|_\phi \tag{13}$$

by the preceding argument. Hence

$$\left\| c + \sum_{j=1}^n a_j X_j \right\|_p \leqq 2^{1/p} \left(|c + F(a)| + 2^{1/p} \|a\|_\phi \right) \leqq 2^{2/p}.$$

We conclude that

$$|||Z||| \geqq 2^{-2/p} \|Z\|_p, \qquad Z \in M. \tag{14}$$

For the converse, we shall use Lemma 2 to pick $\theta > 0$ so small that if $\|a\|_\phi \leqq \theta$ ($a \in \mathbb{R}^\infty$) then

$$\sum_{j=1}^n \phi(|a_j|) \leqq \frac{1}{2}$$

and

$$E(|U_t|) \leqq \frac{1}{2} \left\{ E(U_t^2) \right\}^{1/2}$$

whenever $0 < t \leqq \max_j |a_j|$.

We shall show that for $\eta > 0$ there exists $\rho(\eta) > 0$ with $\lim_{\eta \to 0} \rho(\eta) = 0$ so that if $Z \in M$ with $\|Z\|_p \leqq \eta$, $|||Z||| \leqq \theta$ then $|||Z||| \leqq \rho(\eta)$. From this we can conclude that $|||Z||| \leqq \beta_2 \|Z\|_p$ for some $\beta_2 < \infty$.

Suppose $\|Z\|_p \leqq \eta$ and $|||Z||| \leqq \theta$, where

$$Z = c + \sum_{j=1}^{n} a_j X_j .$$

Let $B_j = \{\omega\colon |a_j X_j(\omega)| \leqq 1\}$ and $C_j = \Omega \setminus B_j$. Put $B = \bigcap_{j=1}^{n} B_j$. Clearly

$$P(C_j) \leqq \phi(|a_j|), \qquad j = 1, 2, \ldots, n$$

and so

$$P(B) \geqq \prod_{j=1}^{n} (1 - \phi(|a_j|)) \geqq 1 - \sum_{j=1}^{n} \phi(|a_j|) \geqq \frac{1}{2} .$$

Now let

$$Y_j = a_j X_j 1_{B_j}, \quad Y_j' = a_j X_j - Y_j .$$

Define a new probability measure P_B on Ω by

$$P_B(A) = \frac{P(A \cap B)}{P(B)}, \qquad A \in \Sigma .$$

Now with respect to P_B, $\{Y_j\colon j = 1, 2, \ldots, n\}$ are mutually independent; furthermore if expectation with respect to P_B is denoted by E_B

$$E_B(Y_j) = P(B_j)^{-1} E(Y_j), \quad E_B(Y_j^2) = P(B_j)^{-1} E(Y_j^2).$$

Now $P(B_j) \geqq P(B) \geqq \frac{1}{2}$ and by choice of θ we have

$$\frac{1}{2} \{E(Y_j^2)\}^{1/2} \geqq E(|Y_j|).$$

Hence

$$E_B(Y_j - E_B(Y_j))^2 = E_B(Y_j^2) - (E_B(Y_j))^2 \geqq (P(B_j)^{-1} - \frac{1}{4} P(B_j)^{-2})\, E(Y_j^2) \geqq 3/4\, E(Y_j^2).$$

We conclude from the P_B-independence of $Y_1, \ldots, Y_n$,

$$\sum_{j=1}^{n} E(Y_j^2) \leqq \frac{4}{3} E_B \left(\sum_{j=1}^{n} (Y_j - E_B(Y_j)) \right)^2 \leqq \frac{4}{3} E_B \left[\left(c + \sum_{j=1}^{n} Y_j \right)^2 \right] .$$

Thus

$$\sum_{j=1}^{n} E(Y_j^2) \leqq \frac{4}{3} \int \left| c + \sum_{j=1}^{n} Y_j \right|^2 dP_B . \tag{15}$$

Now

$$\int_{\Omega} \left| c + \sum_{j=1}^{n} Y_j \right|^p dP_B \leqq 2 \int_{\Omega} |Z|^p \, dP \leqq 2\eta^p$$

and so by Lemma 1, (5) yields

$$\sum_{j=1}^{n} E(Y_j^2) \leqq \Delta(\eta) \tag{16}$$

where $\Delta(\eta) \to 0$ when $\eta \to 0$. We conclude

$$\sum_{j=1}^{n} \int_{|a_j X| \leqq 1} |a_j X|^2 \, dP \leqq \Delta(\eta). \tag{17}$$

Now let $\xi = \sum_{j=1}^{n} E(Y_j)$. Then

$$\left\| \sum_{j=1}^{n} Y_j - \xi \right\|_2^2 = \sum_{j=1}^{n} \|Y_j - E(Y_j)\|_2^2 \leqq \sum_{j=1}^{n} \|Y_j\|_2^2 = \Delta(\eta).$$

Hence

$$\left\| 1_B \cdot \left(\sum_{j=1}^{n} Y_j - \xi \right) \right\|_p \leqq \sqrt{\Delta(\eta)}. \tag{18}$$

Now

$$\left\| 1_B \cdot \left(\sum_{j=1}^{n} Y_j + c \right) \right\|_p \leqq \eta \tag{19}$$

and so

$$\|1_B \cdot (c + \xi)\|_p \leqq 2^{1/p} (\eta + \sqrt{\Delta(\eta)})$$

and

$$|c + \xi| \leqq 2^{2/p} (\eta + \sqrt{\Delta(\eta)}).$$

We now turn to $Y_1', \ldots, Y_n'$. We have

$$\sum_{j=1}^{n} Y_j' = Z - (c + \xi) - \left(\sum_{j=1}^{n} Y_j - \xi \right)$$

and so

$$\left\| \sum_{j=1}^{n} Y_j' \right\|_p \leqq 3^{1/p} (\eta + 2^{2/p} (\eta + \sqrt{\Delta(\eta)}) + \sqrt{\Delta(\eta)}) = \gamma(\eta)$$

where $\gamma(\eta) \to 0$ as $\eta \to 0$. However

$$\left\| \sum_{j=1}^{n} Y_j' \right\|_p^p \geqq \sum_j \int_{C_j \cap \bigcap_{k \neq j} B_k} |Y_j'|^p \, dP \geqq \frac{1}{2} \sum_j \int_{C_j} |Y_j'|^p \, dP = \frac{1}{2} \sum_j \int_{|a_j X| > 1} |a_j X|^p \, dP.$$

Combining with (17) we obtain

$$\sum_{j=1}^{n} \phi(|a_j|) \leqq 2(\gamma(\eta))^p + \Delta(\eta) \tag{20}$$

and so

$$\|a\|_\phi \leqq \lambda(\eta)$$

where $\lambda(\eta) \to 0$ as $\eta \to 0$. Thus

$$\|-F(a) + a_1 X_1 + \cdots + a_n X_n\|_p \leqq 2^{1/p}\, \lambda(\eta)$$

by (13). Hence

$$|c + F(a)| \leqq 2^{1/p}(\eta + 2^{1/p}\, \lambda(\eta))$$

and

$$|||Z||| \leqq \lambda(\eta) + 2^{1/p}(\eta + 2^{1/p}\, \lambda(\eta)) = \rho(\eta)$$

where $\rho(\eta) \to 0$ as $\eta \to 0$. The proof is complete.

Remark. The Kolmogoroff Three Series Theorem ([9] p. 113) shows that $\sum a_n X_n$ converges almost surely if $\sum \phi(|a_n|) < \infty$ and $\sum \psi(a_n)$ converges.

We can now state our main results on the linear structure of the closed linear span $\Lambda_p(X)$ of $\{X_n\}$.

Theorem 2. (a) *In order that $1 \notin \Lambda_p(X)$ it is necessary and sufficient that for some constant $C < \infty$*

$$|\psi(t)| \leqq C\phi(t) \qquad 0 \leqq t \leqq 1. \tag{21}$$

(b) *In order that $\Lambda_p(X)$ has a separating dual it is necessary and sufficient that for some constants c, C,*

$$|\psi(t) - ct| \leqq C\phi(t) \qquad 0 \leqq t \leqq 1. \tag{22}$$

If (22) holds then $\Lambda_p(X) \cong l_\phi$; if (22) does not hold then $\Lambda_p(X)$ is isomorphic to a non-trivial twisted sum of R and l_ϕ.

Proof. (a) If $Z_m = \sum_{j=1}^{n(m)} a_{j,m} X_j$ and $a_m = (a_{j,m})_{j=1}^{\infty}$ then $Z_m \to 1$ if and only if $\|a_m\|_\phi \to 0$ and $F(a_m) \to -1$. Thus $1 \in \Lambda_p(X)$ if and only if

$$\sup\left(|F(a)|:\ \|a\|_\phi = 1\right) = \infty$$

i.e. if and only if

$$\sup_{0 < t \leqq 1} \frac{\psi(t)}{\phi(t)} = \infty$$

by appealing to the definition (11) of F.

(b) Theorem 1 shows that if (21) fails then $\Lambda_p(X)$ is isomorphic to the twisted sum $R \oplus_{(-F)} l_\phi$. This is a direct sum if and only if for some linear $G\colon R^\infty \to R$

$$|F(a) - G(a)| \leqq C \|a\|_\phi, \qquad a \in R^\infty.$$

If such a G exists suppose $G(e_n) = c_n$, where e_n is the nth basis vector. As $F(e_n)$ is constant, $G(e_n)$ is bounded. Select a subsequence n_k so that $c_{n_k} \to c$. Then for $a \in R^\infty$,

$$\left| F\left(\sum_{i=1}^{l} a_i e_{n_{k+i}} \right) - \sum_{i=1}^{l} c_{n_{k+i}} a_i \right| \leqq C \|a\|_\phi$$

and so taking limits

$$\left| F(a) - c \sum_{i=1}^{l} a_i \right| \leqq C \|a\|_\phi$$

or

$$\left| \sum_{i=1}^{\infty} (\psi(a_i) - c a_i) \right| \leqq C \|a\|_\phi, \qquad a \in R^\infty.$$

This leads easily to (22), with a possibly modified constant.

Conversely if (22) holds, we may define $G(a) = c \sum_{i=1}^{\infty} a_i$. If (21) holds $\Lambda_p(X) \cong \mathbb{R} \oplus l_\phi \cong l_\phi$. If (22) fails then $\Lambda_p(X)$ is by Theorem 1 isomorphic to the non-trivial twisted sum $\mathbb{R} \oplus_{(-F)} l_\phi$.

Corollary. *Suppose $X \geqq 0$ is such that*

$$\liminf_{t \to 0} \frac{\phi(t)}{t} < \infty. \tag{23}$$

Then $\Lambda_p(X)$ is a non-trivial twisted sum of R and l_ϕ if $E(X) = \infty$.

Remark. (23) is valid if X belongs to weak L^1, i.e.

$$P(X \geqq x) = O(x^{-1}) \quad \text{as} \quad x \to \infty$$

as may be verified by integration; in fact $\phi(t)/t$ is bounded.

The converse is false; we can have $E(X) < \infty$ and $\Lambda_p(X)$ a non-trivial twisted sum. Let X have probability density function $w(x) \sim (x \log x)^{-2}$ for large x.

However if (22) and (23) hold then for some sequence $t_n \to 0$

$$\phi(t_n) \leqq C t_n$$

and hence

$$\frac{\psi(t_n)}{t_n} \leqq C + |c|.$$

Thus

$$E(U_{t_n}) \leqq C + |c| \quad \text{and} \quad E(X) < \infty.$$

4. Examples and remarks

We shall show first how to embed the Ribe space into L_p for $0<p<1$. Let X be the positive random variable with probability density function

$$w(x)=\begin{cases}1/x^2, & 1\leqq x<\infty,\\ 0, & x<1.\end{cases}$$

Then $X\in L_p$ for $0<p<1$, and for $0<t\leqq 1$

$$\phi(t)=t^2\int_1^{t^{-1}} x^2 w(x)\,dx+t^p\int_{t^{-1}}^{\infty} x w(x)\,dx=t-t^2+(1-p)^{-1}t.$$

Thus $l_\phi\equiv l_1$. Now

$$\psi(t)=\begin{cases} t\int_1^{t^{-1}}\frac{1}{x}\,dx, & 0<t\leqq 1,\\ 0, & t\geqq 1 \quad\text{or}\quad t=0\end{cases}$$

i.e.

$$\psi(t)=t\log_+\frac{1}{t},\qquad 0\leqq t<\infty$$

if we define $0\log\infty=0$.

Hence $\Lambda_p(X)\cong \mathbb{R}\oplus_{(-F)} l_1$ where

$$F(a)=\sum a_i\log_+\frac{\|a\|_\phi}{|a_i|}.$$

The Ribe space is $\mathbb{R}\oplus_H l_1$ where

$$H(a)=\sum a_i\log|a_i|-(\sum a_i)\log|\sum a_i|.$$

Hence

$$F(a)+H(a)=\sum a_i\log\|a\|_\phi-\sum a_i\log|\sum a_i|-\sum_{|a_i|\geqq\|a\|_\phi} a_i\log\frac{\|a\|_\phi}{|a_i|}$$

and is clearly bounded on the unit ball of l_1. Thus we have proved that $\Lambda_p(X)$ is isomorphic to the Ribe space.

Theorem 3. *The Ribe space embeds into L_p for $0<p<1$.*

These considerations may be generalized in the spirit of the results of [4], Section 9. There it was proved that if f is an Orlicz function satisfying $l_f\subset l_1$, then there is a non-trivial twisted sum of R and l_f if and only if $\beta_f\geqq 1$ where

(24) $$\beta_f=\inf\{p\,|\,\exists M\colon f(ax)\geqq Ma^p f(x),\quad 0<a,\ x<1\}.$$

An equivalent formulation of this condition is that l_1 is isomorphic to a subspace of l_f.

If we suppose, in addition that l_f embeds into L_q for some $q<1$ then we can actually construct the twisted sum as a subspace of L_p for any $p<q$. In this case (cf. [2]) we may suppose that $x^{-q}f(x)$ is increasing, and that $x^{-2}f(x)$ is decreasing for $0\leqq x\leqq 1$. Then if we define for $t\geqq 1$

$$q(t)=\int_1^t\frac{1}{x^2}\,d\left(x^2 f\left(\frac{1}{x}\right)\right)$$

q is increasing and by a straightforward integration by parts q approaches a finite limit as $t \to \infty$. Choose $\alpha > 0$ so that $\lim_{t \to \infty} \alpha q(t) = 1$ and let X be random variable with distribution function

$$Q(t) = \begin{cases} \alpha q(t), & t \geqq 1, \\ 0, & t < 1. \end{cases}$$

Again it is straightforward to show $X \in L_p$ and

$$\int_{1/t}^{\infty} x^p \, dQ(x) \leqq \alpha \left(\frac{2-q}{q-p} \right) t^{-p} f(t),$$

$$\int_{1}^{1/t} x^2 \, dQ(x) = \alpha (t^{-2} f(t) - f(1)).$$

There ϕ is equivalent to f.

Now

$$\psi(t) = t \int_{1}^{1/t} x \, dQ(x) = \alpha t \int_{1}^{1/t} x^{-1} d\left(x^2 f\left(\frac{1}{x} \right) \right) = \alpha \left(f(t) - t f(1) + t \int_{t}^{1} \frac{f(x)}{x^2} dx \right).$$

It is easy to see that $\Lambda_p(X)$ is a non-trivial twisted sum if and only if $h(t)/f(t)$ is unbounded for $0 \leqq t \leqq 1$ where

$$h(t) = t \int_{t}^{1} \frac{f(x)}{x^2} dx.$$

It is shown in [4] that this happens precisely when $\beta_f \geqq 1$.

We conclude with some problems. We do not know whether the twisted sum of R and l_1 constructed in [3] embeds into L_p for $p < 1$. Equally can a non-trivial twisted sum of l_p with itself for $0 < p < 1$ embedded into some L_q for $0 < q < p$? We remark that the twisted sum of two Hilbert spaces constructed in [5] does not embed into any L_p for $0 < p < \infty$.

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