Gap-interpolation theorems for entire functions

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I. Introduction

There are many theorems in classical analysis where gaps play a role. We mention only the Müntz-Szasz Theorem [1], the Fabry Gap Theorem [10], the Banach Lacunary Theorem [19] and the High Indices Theorem [8] as samples. In another direction, there are many interpolation theorems for analytic functions—we mention only the Germay Theorem [17], the Buck-Carleson-Hayman-Newman Theorem [2], and the Leontiev Theorem [13]. The idea of this paper is to mix the two ideas—gaps and interpolation.

To be specific, the Germay Theorem (not usually called by that name since it is such a direct corollary of the Mittag-Leffler Theorem) says that given any sequence $\mathscr{Z} = (z_n)$ of distinct complex numbers with no finite limit point, and any sequence $\mathscr{W} = (w_n)$ of complex numbers, then there is an entire function f such that $f(z_n) = w_n$ for all n. (If some of the z_n repeat a finite number of times, then we look at corresponding derivatives of f at the z_n.) But suppose we ask that f have the form $f(z) = \sum_{\lambda \in \Lambda} a_\lambda z^\lambda$, where Λ is a given set of positive integers (and we do not allow any $z_n = 0$)? For certain Λ (like $\Lambda = \mathbb{N}$, all positive integers), this unrestricted interpolation is always possible, while for other Λ (like all the even non-negative integers) it is not always possible. This question was asked in [11] and later in [14], and some simple cases treated.

We first give some relevant definitions and notation. Then we state the main results of this paper. The principal one provides a necessary and sufficient condition that Λ works for the given sequence $\mathscr{Z}$. This condition is at first rather opaque, so we devote the rest of the paper to some conclusions, of a more concrete and manageable nature, that may be drawn from it. The last sections of the paper are devoted to the proofs of the results.

A surprising feature of this gap-interpolation problem is that in certain circumstances (where we assume, say, that there are at most two z_j of any given modulus) the main consideration is diophantine approximation, i.e. how well certain real numbers can be approximated by rational numbers. The corresponding gaps in the circumstance

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mentioned above, are roughly of the size connected with $\lambda_n = e^{e^{\cdot^{\cdot^{e}}}}$ (with n exponents in the chain). These are huge gaps that surpass any we have seen that naturally occur.

A word about our method of proof. In proving interpolation theorems (say the Germay Theorem), there are three main approaches: The first is via the Mittag-Leffler Theorem, which is in turn usually proved [18] by writing an explicit formula. The second is via solving infinitely many linear equations in infinitely many unknowns (the Taylor coefficients)—see [3], [1]. The third is via functional analysis—specifically the Banach-Dieudonné Theorem (see [15], [4]). We take the third route, and adapt the main result of [4] to the situation where gaps are introduced.

Let us mention that the arithmetic nature of Λ is important. For if Λ consisted only, say, of odd integers, then we would have $f(-1) = -f(+1)$ for all f of the required form, which certainly prevents free interpolation if $\mathscr{Z}$ contains both 1 and -1. Similar considerations lead one to the natural necessary condition that Λ intersects every arithmetic progression $\{qn+p\}$. Our later results show that this necessary condition is far from sufficient in many cases.

Many questions related to those we treat here can be raised. For example, suppose we consider not *entire* functions, but functions that are only supposed analytic on some specific domain G in the plane, or on some open Riemann surface G (see [5]). Or one could pass to several complex variables. Or one could restrict the growth of the functions involved, like considering gap-interpolation by bounded analytic functions in the unit disc or by entire functions of exponential type. These are interesting and challenging questions that we leave untouched.

II. Definitions and notation

Throughout this paper, we consider sequences $\mathscr{Z} = (z_n)$ of non-zero complex numbers with no finite limit point. We say that $\mathscr{Z}$ is terminating if it is indexed by $n = 1, 2, \ldots, N$ instead of by $n = 1, 2, \ldots$. We always index $\mathscr{Z}$ so that $|z_j| \leqq |z_{j+1}|$ for all j. For simplicity, we will suppose that $z_i \neq z_j$ for $i \neq j$. We consider sequences $\Lambda = (\lambda_n)$ of positive integers, with $\lambda_j < \lambda_{j+1}$ for all j.

Definition. We consider certain classes Ω of sequences $\mathscr{Z}$:

i) Ω_∞ is the class of all $\mathscr{Z}$.

ii) Ω_f is the class of all terminating (i.e. finite) $\mathscr{Z}$.

iii) Ω_k is the class of all $\mathscr{Z}$ so that if $|z_j| = |z_{j+1}| = \cdots = |z_{j+r-1}|$, then $r \leqq k$.

In what follows, we shall define "Λ is interpolating for $\mathscr{Z}$" (or for Ω) instead of the more cumbersome, but more descriptive "Λ is an interpolating sequence of exponents for $\mathscr{Z}$" (resp. for Ω).

Definition. Λ is interpolating for $\mathscr{Z}$ (or $\mathscr{Z}$ interpolating) if for every sequence (w_n) of complex numbers, there exists an entire function f of the form

$$f(z) = \sum_{\lambda \in \Lambda} a_\lambda z^\lambda \tag{1}$$

such that $f(z_n) = w_n$ for all n.

Definition. Λ is interpolating for Ω if Λ is $\mathscr{X}$ interpolating for every $\mathscr{X}$ in Ω.

Definition. Λ is $\mathscr{X}$ linearly independent if the condition

$$\sum_{n=1}^{N} a_n z_n^{\lambda} = 0 \quad \text{for all} \quad \lambda \in \Lambda$$

implies that all $a_j = 0$.

Definition. Λ is asymptotically $\mathscr{X}$ linearly independent if for every $\rho > 0$, there exists an $N(\rho)$ so that if $K \geqq 0$, and if

$$\left| \sum_{n=1}^{N} a_n z_n^{\lambda} \right| \leqq K \rho^{\lambda} \quad \text{for all} \quad \lambda \in \Lambda$$

then $a_n = 0$ for all $n \geqq N(\rho)$.

Definition. Λ is totally $\mathscr{X}$ linearly independent if it is both $\mathscr{X}$ linearly independent and asymptotically $\mathscr{X}$ linearly independent.

Definition. By an arithmetic progression A, we mean a sequence of the form $A = (qn + p)$, $n = 1, 2, \ldots$, where $q > 0$ and $p \geqq 0$ are integers.

Definition. A sequence $\Lambda = (\lambda_n)$ is said to have positive upper density if

$$\limsup_{n \to \infty} \frac{\Lambda(n)}{n} > 0,$$

where $\Lambda(n)$ is the number of λ_j that do not exceed n.

III. Statement of results

Theorem 1. *Λ is interpolating for $\mathscr{X}$ if and only if it is totally $\mathscr{X}$ linearly independent.*

Theorem 2. [11] *Λ is Ω_1 interpolating if and only if Λ is infinite.*

Theorem 3. *Λ is Ω_f interpolating if and only if Λ has a non-empty intersection with every arithmetic progression A.*

The next result is a structural one about those Λ that are Ω_∞ interpolating.

Theorem 4. *Let Λ be Ω_∞ interpolating and let Δ be a sequence of positive integers that satisfies:*

2) *Given any finite sequence $F \subseteqq \Lambda$, there exists a non-negative integer t so that $F + t \subseteqq \Delta$, where $F + t = \{(f + t), f \in F\}$. Then Δ is also Ω_∞ interpolating.*

Theorem 5. *If $\Lambda \cap A$ has positive upper density for every arithmetic progression A, then Λ is Ω_∞ interpolating.*

Theorem 6. *If*

3) $$\lim_{k \to \infty} \frac{\log \lambda_{k+1}}{\lambda_k} = \infty,$$

then Λ is not Ω_2 interpolating.

Theorem 7. *If*

4) $$\lim_{k\to\infty} \frac{\log \lambda_{k+2}}{\lambda_k} = 0$$

and $\Lambda \cap A$ is not empty, for every arithmetic progression A, then Λ is Ω_2 interpolating.

We conclude with some auxiliary results. The first is a reformulation of Theorem 1.

Theorem 1′. *Λ is not Ω interpolating if and only if either:*

5) *There exists a $\mathscr{Z}$ in Ω, and constants $a_1, \ldots, a_n$ not all zero such that*

$$a_1 z_1^\lambda + \cdots + a_n z_n^\lambda = 0 \quad \textit{for all} \quad \lambda \in \Lambda$$

or

6) *There exist a sequence $\mathscr{Z}$ in Ω, a number $\rho \geqq 0$, and a constant K so that for any positive integer N, there is an integer $m \geqq N$ and complex numbers $a_1, \ldots, a_m$ exist, with $a_m \neq 0$, so that*

$$|a_1 z_1^\lambda + \cdots + a_m z_m^\lambda| \leqq K\rho^\lambda.$$

Lemma 1. *If Λ is not Ω_∞ interpolating, then either* 5) *holds or, for every ε with $0<\varepsilon<1$, there exist complex numbers $w_1, \ldots, w_m$ with $|w_j| \leqq 1$ for $j=1, \ldots, m$ and $|w_m|=1$, so that*

7) $$|b_1 w_1^\lambda + \cdots + b_{m-1} w_{m-1}^\lambda + w_m^\lambda| \leqq K_\varepsilon \varepsilon^\lambda \quad \textit{for all} \quad \lambda \in \Lambda,$$

where the numbers $b_1, \ldots, b_{m-1}$ and $w_1, \ldots, w_m$ depend on ε, and where K_ε is independent of λ.

Lemma 2. *If Λ is not Ω_∞ interpolating, then either* 5) *holds, or there exists an ε with $0<\varepsilon<1$ and non-zero numbers $\xi_1, \ldots, \xi_n$ with $|\xi_j|=1$ for all $j=1, \ldots, n$, so that*

8) $$|a_1 \xi_1^\lambda + \cdots + a_n \xi_n^\lambda| \leqq K\varepsilon^\lambda \quad \textit{for all} \quad \lambda \in \Lambda,$$

where the a_j are not all zero.

Lemma 3. *If we replace Ω_∞ by Ω_k in Lemma* 2, *then we may choose $n \leqq k$ in* (8).

Lemma 4. *For a given sequence Λ and integers $q>0$, $r \geqq 0$, let $\Lambda_{q,r} = \{m : mq + r \in \Lambda\}$. If Λ is Ω_∞ interpolating then so is each $\Lambda_{q,r}$.*

IV. Proof of Theorem 1

We will fix $\mathscr{Z}$ and Λ. Let E be the space of all entire functions, in the topology of uniform convergence on compact sets, and let E_Λ be the closed span in E of $\{z^\lambda\}_{\lambda\in\Lambda}$. Let $\pi = \pi_\Lambda$ be the projection into E_Λ, that is

$$\pi\left(\sum_{n=0}^{\infty} a_n z^n\right) = \sum_{\lambda\in\Lambda} a_\lambda z^\lambda.$$

Let L_n be the linear functional of projection into E_Λ followed by evaluation at z_n, that is

$$L_n(f) = (\pi f)(z_n).$$

We shall prove that, in the terminology of [4], the sequence $\{L_n\}$ is interpolating iff Λ is totally linearly independent. (To say that $\{L_n\}$ is interpolating means that for any given sequence $\{w_n\}$ of complex numbers, there exists an $F \in E$ such that $L_n(F) = w_n$ for each $n = 1, 2, 3, \ldots$.) Notice that this will prove our theorem since then $f = \pi(F)$ has the desired form $f = \sum a_\lambda z^\lambda$, and $f(z_n) = w_n$ for all n. With the Cauchy transform $\hat{L_n}(z) = L_n\left(\frac{1}{z-w}\right)$, we see that

$$\hat{L_n}(z) = \sum_{\lambda \in \Lambda} \frac{z_n^\lambda}{z^{\lambda+1}}.$$

Letting V_n be the linear span of $L_1, L_2, \ldots, L_n$, we see from Corollary 1 of [4] that Λ is interpolating iff i) $\{L_n\}$ is linearly independent and ii) for every compact disk K, there exists an integer $N(K)$ such that if $T_n \in V_n$ and $\hat{T_n}$ has an analytic continuation to the complement of K, then $T_n \in V_{N(K)}$. But we see that a typical element of V_n has the form

$$g(z) = \sum_{k=0}^{\infty} \frac{d_k}{z^{k+1}}$$

where

$$d_k = \sum_{m=0}^{n} a_m z_m^k,$$

and of course, the radius of convergence of the series for $g(z)$ is

$$\rho = [\limsup_{k \to \infty} |d_k|^{1/k}]^{-1}.$$

It now follows easily that Λ is interpolating iff it is totally linearly independent, and the theorem is proved.

V. Proofs of Theorems 2 and 3

The proof of Theorem 2 is an immediate consequence of Theorem 1. See [11] for a different proof.

Proof of Theorem 3. Since the $\mathscr{X}$ asymptotic linear independence of Λ is vacuously satisfied for terminating sequences $\mathscr{X}$, we need only consider Ω linear independence. It is easy, as remarked earlier, to see that a necessary condition for this is that Λ intersects every arithmetic progression. To see the sufficiency, let

$$R(\mu) = \sum_{n=1}^{N} a_n z_n^\mu$$

where the (a_n) and (z_n) are as in the definition of $\mathscr{X}$ linear independence. Now

$$\sum_{n=1}^{N} \frac{a_n}{z - z_n} = \sum_{\mu=0}^{\infty} \frac{R(\mu)}{z^{\mu+1}}$$

as remarked in the proof of Theorem 1. By the Skolem-Lech Theorem [12], the set $\mathscr{M} = \{\mu : R(\mu) = 0\}$ is, modulo a finite set, the union $\mathscr{M}^*$ of finitely many arithmetic progressions $(qn+p)$. Since $R(\lambda) = 0$ for each $\lambda \in \Lambda$, we see that $\mathscr{M}$ intersects every arithmetic progression. It follows easily that $\mathcal{M}^*$, as the finite union of arithmetic progressions, must contain all sufficiently large positive integers, and hence so must $\mathcal{M}$. But then, say, by Vandermonde determinants, all the a_j must vanish, and the result is proved.

We thank D. J. Newman for pointing out the relevance of the Skolem-Lech Theorem. An interesting feature of this theorem is that the only known proofs depend on p-adic analysis, even though both the hypothesis and the conclusion are quite elementary.

VI. Proof of Theorem 4

We use Theorem 1′, on assuming that Δ is not Ω_∞ interpolating, so that either 5) or 6) holds for Δ. If 5) holds, then on choosing $F=(\lambda_1, \ldots, \lambda_N)$ for any given positive integer N, we have, for some $t=t(N)\geqq 0$, and for some fixed n,

$$a_1 z_1^{\lambda+t}+\cdots+a_n z_n^{\lambda+t}=0 \quad \text{for} \quad \lambda\in\Lambda,\ \lambda\leqq N,$$

where, say, $a_n \neq 0$. Dividing through by $|z_n|^{\lambda+t}$, we may suppose that $|z_i|\leqq 1$ for $i=1, \ldots, n$, and $|z_n|=1$. We may further suppose that $|a_i|\leqq 1$ for $i=1, \ldots, n$. Now we look at the vector

$$W_N=(a_1 z_1^{t(N)}, \ldots, a_n z_n^{t(N)})$$

in the unit polydisc $P=\{w=(w_1, \ldots, w_n): |w_i|\leqq 1,\ i=1, \ldots, n\}$.

By the compactness of P, there exists a cluster point $W=(b_1, \ldots, b_n)$, with $|b_n|\neq 0$, of $\{W_N\}$. Then

$$b_1 z_1^{\lambda}+\cdots+b_n z_n^{\lambda}=0 \quad \text{for } all \quad \lambda\in\Lambda$$

so that 5) fails for Λ, which contradicts Λ being Ω_∞ linearly independent.

On the other hand, if 6) holds for Δ, then given any positive integer N, we see that there exists suitable $\mathscr{X}$, ρ, and (a_j) with $a_m \neq 0$ so that

$$|a_1 z_1^{\delta}+\cdots+a_m z_m^{\delta}|\leqq A\rho^{\delta} \quad \text{for all} \quad \delta\in\Delta.$$

Consequently, given any positive integer M, there exists a non-negative integer t so that

$$|a_1 z_1^{\lambda+t}+\cdots+a_m z_m^{\lambda+t}|\leqq K\rho^{\lambda+t} \quad \text{for all} \quad \lambda\in\Lambda \quad \text{with} \quad \lambda\leqq M.$$

Hence

$$\left|\frac{a_1 z_1^t}{|z_m|^t} z_1^{\lambda}+\cdots+\frac{a_m z_m^t}{|z_m|^t} z_n^{\lambda}\right|\leqq K\rho^{\lambda}\left(\frac{\rho}{|z_m|}\right)^t.$$

Now since we may, for large enough N, suppose that $|z_m|\geqq\rho$, we may again use a compactness argument to obtain

$$|b_1 z_1^{\lambda}+\cdots+b_m z_m^{\lambda}|\leqq K\rho^{\lambda} \quad \text{for } all \quad \lambda\in\Lambda,$$

which by Theorem 1′ (for Λ) implies that Λ is not $\mathscr{X}$ interpolating—a contradiction that proves the theorem.

VII. Proofs of the Lemmas

Proof of Lemma 1. Let $\mathscr{Z}$ be the sequence that satisfies 6) and then choose N so that $|z_m| \geqq \rho/\varepsilon$ for all $m \geqq N$. Now in 6), divide through by $|z_m^\lambda|$ in each equation.

Proof of Lemma 2. Just write 7) with $\varepsilon = \varepsilon_0 = 1/2$, say, and let $w_1, \ldots, w_k$ be those w_i with $|w_i| < 1$ so that $|w_{k+1}| = \cdots = |w_m| = 1$. Say $|w_i| \leqq \delta < 1$ for $i = 1, \ldots, k$. Let $\xi_j = w_{k+j}$ for $j = 1, \ldots, n$, where we set $n = m - k$. By the triangle inequality, with $B = \max(|b_1|, \ldots, |b_k|)$ and $a_i = b_{k+i}$ for $i = 1, \ldots, n$, we get

$$|a_1 \xi_1^\lambda + \cdots + a_n \xi_n^\lambda| \leqq B\delta^\lambda + K_{\varepsilon_0} \varepsilon_0^\lambda \quad \text{for all} \quad \lambda \in \Lambda,$$

and the result follows on letting $\varepsilon = \max(\delta, \varepsilon_0)$.

Proof of Lemma 3. Obvious.

Proof of Lemma 4. First of all, let us suppose $q = 1$. We show that if Λ is Ω_∞ interpolating, then so is $\Lambda_{1,r}$. But this follows directly from the fact that $\sum a_n z_k^{n+r} = w_k$ if and only if $\sum a_n z_k^n = w_k / z_k^r$. So we may suppose $r = 0$, and we write $\Lambda_q = \Lambda_{q,0}$. Given $\mathscr{Z} = \{z_k\}$, let $\mathscr{Z}_q = \{z_k^*\}$ be any sequence where each z_k^* is some q-th root of the corresponding z_k. Now if Λ interpolates at $\mathscr{Z}_q$ then Λ_q interpolates at $\mathscr{Z}$, because

$$\sum a_n (z_k^{1/q})^{nq} = \sum a_n z_k^n.$$

VIII. Proof of Theorem 5

Our proof uses ideas and results from the theory of locally compact abelian groups — see [17] for basic information on this subject.

We shall suppose, by way of contradiction, that Λ is not Ω_∞ interpolating, and apply Lemma 2. By the proof of Theorem 3, we can suppose that Λ is Ω_f asymptotically linearly independent, and by Lemma 2, this implies that there exists some $\rho < 1$ and constants $a_1, \ldots, a_n$ with no $a_j = 0$, and a constant C and numbers $z_1, \ldots, z_n$ with all $|z_j| = 1$ so that

$$\text{10)} \qquad |a_1 z_1^\lambda + \cdots + a_n z_n^\lambda| \leqq C \rho^\lambda \quad \text{for all} \quad \lambda \in \Lambda.$$

We will suppose that we have chosen the least positive integer K for which there are $a_k \neq 0$ and $z_k \in \mathbb{C} \setminus \{0\}$, $k = 1, 2, \ldots, K$ such that

$$\text{10')} \qquad \sum_{k=1}^{K} a_k z_k^\mu \to 0$$

as μ runs through a set M that intersects each arithmetic progression in a set of positive upper density. By 10), there is such a K. We now fix the $\{a_k\}$ and the (new) $\{z_k\}$. We shall derive a contradiction to the minimality of K.

Let T^n be the torus $\{w = (w_1, \ldots, w_n)$: all $|w_j| = 1\}$, and set $z = (z_1, \ldots, z_n)$. Now T^n is a compact abelian group that we shall write multiplicatively. We define

$$\varphi(w) = \sum_{j=1}^{n} a_j w_j \quad \text{for} \quad w \in T^n.$$

Consider the positive measure

$$\mu_N = \frac{1}{N}(\delta_z + \delta_{z^2} + \cdots + \delta_{z^N}), \quad N = 1, 2, \ldots,$$

where δ_w is the unit point mass at w. By compactness of the unit ball in the weak star topology of the space of bounded measures as the dual of the space of continuous complex-valued functions on T^n, we see that $\{\mu_N\}$ has a weak-star cluster point σ. A simple argument shows that σ must be Haar measure on G, the closed subgroup generated by z in T^n. (Usually, G will be either the whole torus, or a "spiral" on it.) Since σ is the only possible cluster point, we see that μ_N converges weak-star to σ. We now choose $\varepsilon > 0$ and let

$$\psi(w) = \psi_\varepsilon(w) = \min\left(1, \frac{1}{\varepsilon}|\varphi(w)|\right).$$

We know that

$$\int \psi(w)\, d\mu_N(w) \to \int \psi(w)\, d\sigma(w) \quad \text{as} \quad N \to \infty.$$

But for a sequence of N in M, with $N \to \infty$,

$$\int \psi(x)\, d\mu_N(x) \leqq (1-\alpha) + \frac{1}{2}\alpha = 1 - \frac{1}{2}\alpha < 1,$$

where α is any positive number smaller than the upper density of Λ. Hence

$$\sigma(\{w \in G : |\varphi(w)| < \varepsilon\}) > \frac{1}{2}\alpha.$$

It follows that $\sigma(E) > 0$, where

$$E = \{w \in G : \varphi(w) = 0\}.$$

We now define, for $w = (w_1, \ldots, w_n)$ in G,

$$\chi_j(w) = w_j \qquad j = 1, 2, \ldots, n.$$

We will now use a strengthened form of the Steinhaus Theorem for G, which says that there is a neighborhood V of 1 (the identity element of G) such that for every $y \in V$, there exists a set E_y, with $\sigma(E_y) > 0$, so that if $x \in E_y$ then $x/y \in E$. The usual form of the Steinhaus Theorem just asserts that $E_y \neq \emptyset$—a proof can be found for the group $\mathbb{R}$ (where quotients become differences) in [7]. Here is how to prove what we need. Let

$$u(s) = \int_G \chi_E(st)\, \chi_E(t)\, d\sigma(t).$$

Then u is a continuous function of $s \in G$, with $u(1) > 0$, and if one chooses for V a neighborhood of 1 throughout which $u > 0$, then the result follows.

Now we have

$$\text{11)} \qquad a_1\chi_1(x) + a_2\chi_2(x) + \cdots + a_K\chi_K(x) = 0 \quad \text{all} \quad x \in E,$$

so that

$$a_1\bar{\chi}_1(y)\,\chi_1(xy) + a_2\bar{\chi}_2(y)\,\chi_2(xy) + \cdots + a_k\bar{\chi}_K(y)\,\chi_K(xy) = 0, \quad \text{all} \quad y \in G, \quad \text{all} \quad x \in E.$$

On changing notation, this becomes

12) $a_1\bar{\chi}_1(y)\,\chi_1(x)+a_2\bar{\chi}_2(y)\,\chi_2(x)+\cdots+a_n\bar{\chi}_n(y)\,\chi_n(x)=0$, all $y\in G$, all $x\in yE$.

On multiplying 11) by $\bar{\chi}_1(y)$ and subtracting from 12), we get

13) $$a_2[\bar{\chi}_1(y)-\bar{\chi}_2(y)]\,\chi_2(x)+\cdots+a_K[\bar{\chi}_1(y)-\bar{\chi}_K(y)]\,\chi_K(x)=0,$$ all $y\in G$, all $x\in E\cap yE$.

In particular, 13) holds for all $y\in V$ and all $x\in E_y$, with $\sigma(E_y)>0$.

If now $\bar{\chi}_1(y)=\bar{\chi}_k(y)\neq 0$ for at least one $y\in V$ and at least one $k=2,3,\ldots,N$, then we have a formula of the form 11) with K replaced by $K-1$, and after a little relabelling, where the new a_j are $a_k[\bar{\chi}_1(y)-\bar{\chi}_k(y)]$, and so on. We could then repeat the argument until we would arrive at an absurdity, unless (and this is therefore forced on us) we have indices k, l with $k\neq l$ in the range $1,2,\ldots,K$ and a neighborhood V of 1 in G, so that

$$\bar{\chi}_k(y)-\bar{\chi}_l(y)=0, \quad \text{all} \quad y\in V.$$

Let us write $\chi^*=\chi_k\chi_l^{-1}$. It follows that $\chi^*=1$ on a closed and open subgroup H of G, namely the closure of the group generated by V. The quotient G/H, being both compact and discrete, must be finite. Hence χ^* takes on only finitely many values, so that $(\chi^*)^q=1$ for some positive integer q. Hence z_k/z_l is a q-th root of unity, since $z\in G$.

Let

14) $$S_r(m)=\sum_{k=1}^{K} a_k z_k^{mq+r}; \quad r=0,1,\ldots,q-1.$$

For each fixed r, $S_r(m)\to 0$ as m runs through a set $M_r=\{m: mq+r\in M\}$ that still, like M, intersects every arithmetic progression in a set of positive upper density, as may easily be seen. We may rewrite 14) as

15) $$S_r(m)=\sum_{k=1}^{K} a_k(z_k^q)^m z_k^r.$$

For simplicity, let us suppose that $z_1=1$ (since we could otherwise divide through) and that $z_2,\ldots,z_p$ exhaust those $\{z_k\}$ that are q-th roots of unity. Since all the $\{z_k\}$ are distinct, we know that $p\leqq q$, since there are just q such roots of unity. We write

$$A_1^{(r)}=a_1z_1^r+\cdots+a_pz_p^r,$$

so that 15) becomes

15') $$S_r(m)=A_1^{(r)}w_1^m+A_2^{(r)}w_2^m+\cdots+A_{K(r)}^{(r)}w_{K(r)}^m,$$

where the $w_1,\ldots,w_{K(r)}$ are distinct. Here $w_1=1$ and the other w_j come by grouping the z_k^q for $k=p+1,\ldots,K$. We notice that $K(r)<K$ for $r=0,1,\ldots,q-1$. This will contradict the minimality of K if we can show that $A_1^{(r)}\neq 0$ for some one $r=0,1,\ldots,q-1$. But because $p\leqq q$, this follows from Vandermonde's determinant, and the resulting contradiction establishes the theorem.

IX. Proof of Theorem 6

We proceed directly without invoking Theorem 1. We will construct $\mathscr{Z}=(z_n)$ with $|z_{2n}|=|z_{2n+1}|=n$ for all $n=1, 2, \ldots$, so that if $w_n=z_{2n}/z_{2n+1}$, then for some function $\varphi(n)$, we have

16) $$|w_n^\lambda - 1| < \frac{1}{n^\lambda} \quad \text{if} \quad \lambda \geqq \varphi(n).$$

We will put $z_{2n}=ne^{2\pi i x_n}$, where the numbers x_n will be constructed to have certain properties of rapid approximation by rational numbers. Now suppose 16) holds. Then,

$$|z_{2n}^\lambda - z_{2n+1}^\lambda| = n^\lambda |w_n - 1|^\lambda,$$

so it follows that if $f(z)=\sum_{\lambda\in\Lambda} a_\lambda z^\lambda$, then

$$|f(z_{2n})-f(z_{2n+1})| \leqq \sum_{\lambda\in\Lambda} |a_\lambda|\, n^\lambda |w_n^\lambda - 1| = \sum_{\lambda<\varphi(n)} + \sum_{\lambda\geqq\varphi(n)} \leqq Kn^{\varphi(n)} + K',$$

where $K=\sup\{|a_\lambda| : \lambda\in\Lambda\}$, which is finite since f is entire, and $K'=\sum_{\lambda\in\Lambda}|a_\lambda|$, which is finite for the same reason. Surely, then, there is no such f for which, say, $f(z_{2n})=0$ and $f(z_{2n+1})=n!n^{\varphi(n)}$.

Let us now construct $\mathscr{Z}$. We shall work on the unit circle, but use the notation of real numbers modulo 1 as a convenience. Let us now hold n fixed. Then

17) $$\frac{\log \lambda_{k+1}}{\lambda_k} \geqq \log 2n \quad \text{for} \quad k \geqq \varphi(n),$$

(which we may take as the definition of $\varphi(n)$) since the limit of the left side is infinite as $k\to\infty$. So we start with $k=\varphi(n)$, so that $\lambda_{k+1}\geqq(2n)^{\lambda_k}$. Now we choose $y_1^{(n)}$ with $0<y_1^{(n)}<1$ so that $|\lambda_k y_1^{(n)}|<1/n^{\lambda_k}$, where $|x|$ means the shortest distance from 1 to $e^{2\pi i x}$ around the circle, divided by 2π. Next, take a closed interval $I_1^{(n)}$ with $y_1^{(n)}\in I_1^{(n)}$ and $l(I_1^{(n)})=1/(\lambda_k n^{\lambda_k})$, (where $l(I)$ denotes the length of I), so that

$$\lambda_k I_1^{(n)} \subseteqq \left[-\frac{1}{n^{\lambda_k}}, \frac{1}{n^{\lambda_k}}\right].$$

Now since $\lambda_{k+1}\geqq(2n)^{\lambda_k}$, we have

$$\frac{\lambda_{k+1}}{\lambda_k n^{\lambda_k}} \geqq 1$$

so that $\lambda_{k+1} I_1^{(n)}$ covers the whole circle. We now repeat the argument by choosing $y_2^{(n)}\in I_1^{(n)}$ so that $|\lambda_{k+1} y_2^{(n)}|<1/n^{\lambda_{k+1}}$ and then $I_2^{(n)}\subseteqq I_1^{(n)}$ with $y_2^{(n)}\in I_2^{(n)}$ and

$$l(I_2^{(n)}) \leqq \frac{1}{\lambda_{k+1} n^{\lambda_{k+1}}}.$$

Similarly, we get a $y_3^{(n)}$ in (0, 1) and a closed interval $I_3^{(n)}\subseteqq I_2^{(n)}$ with $y_3^{(n)}\in I_3^{(n)}$, so that

$$l(I_3^{(n)}) \leqq \frac{1}{\lambda_{k+2} n^{\lambda_{k+2}}}, \text{ and so on.}$$

We let x_n be the unique point that lies in all these intervals.

$$\{x_n\} = \bigcap_{k \geqq \varphi(n)} I_k^{(n)}.$$

Note that as soon as $\lambda_k \geqq \varphi(n)$,

$$|e^{2\pi i \lambda_k x_n} - 1| \leqq \frac{1}{n^{\lambda_k}},$$

and on letting the w_n at the beginning of the proof be $w_n = e^{2\pi i x_n}$, we are done.

X. Proof of Theorem 7

By Lemma 3, if Λ satisfies our hypothesis, but is not Ω_2 interpolating, then there exists an ε with $0<\varepsilon<1$ and numbers $K', a_1, a_2; z_1, z_2$ with $z_1 \neq z_2$ so that $|z_1| = |z_2|$ and

$$|a_1 z_1^\lambda + a_2 z_2^\lambda| \leqq K' \varepsilon^\lambda \quad \text{for all} \quad \lambda \in \Lambda.$$

(We can immediately exclude the case $n=1$, i.e. $|a_1 z_1^\lambda| \leqq K'\varepsilon^\lambda$, since $|z_1|=1$.) Dividing through by $|z_2|^\lambda$, we get

$$|w^\lambda - A| < K\varepsilon^\lambda \quad \text{for all} \quad \lambda \in \Lambda,$$

where $w = z_1/z_2$, $A = -a_2/a_1$, and so on. Again reducing the circle to the real numbers modulo 1, we get, for some x with $0<x<1$, and some real α,

$$|\lambda x - \alpha| < K\varepsilon^\lambda \quad \text{for all} \quad \lambda \in \Lambda.$$

Furthermore, x must be irrational, because Λ intersects every arithmetic progression. For if we had $x = p/q$, then we could choose λ large, $\lambda \equiv 0 \bmod q$, so that $\lambda x \equiv 0 \bmod 1$ so that we would have $\alpha \equiv 0 \bmod 1$. Then choosing $\lambda \equiv 1 \bmod q$, we would have $|x| < K\varepsilon^\lambda$, so that x would be 0 (modulo 1). This would imply that $z_1 = z_2$, contrary to hypothesis.

We now appeal to a theorem of Dirichlet (see [9], often proved by the Schubfach-prinzip) that there exist infinitely many positive integers q so that for some $p = p(q)$,

$$\left| x - \frac{p}{q} \right| \leqq \frac{1}{q^2}, \quad (p, q) = 1.$$

For such a large q, pick λ_{n+1} in Λ as large as possible so that $4\lambda_{n+1} \leqq q$. Then $\lambda_{n+2} > q/4$. Let us look at λ_{n+1} and λ_n—we have

$$\left| \lambda_n x - \lambda_n \frac{p}{q} \right| \leqq \frac{\lambda_n}{q^2} \leqq \frac{1}{4q}, \quad \left| \lambda_{n+1} x - \lambda_{n+1} \frac{p}{q} \right| \leqq \frac{\lambda_{n+1}}{q^2} \leqq \frac{1}{4q}.$$

Hence

$$\left| (\lambda_{n+1} x - \lambda_n x) - (\lambda_{n+1} - \lambda_n) \frac{p}{q} \right| \leqq \frac{1}{2q}.$$

But

$$\frac{\lambda_{n+1} - \lambda_n}{q} p \geqq \frac{1}{q}$$

since $(p, q) = 1$ and $\lambda_{n+1} < q$.

Hence

$$|\lambda_{n+1}x - \lambda_n x| \geqq \frac{1}{2q}.$$

But, by the triangle inequality (modulo 1)

$$|\lambda_{n+1}x - \lambda_n x| \leqq |\lambda_{n+1}x - \alpha| + |\lambda_n x - \alpha| \leqq 2K\varepsilon^{\lambda_n}.$$

Hence

$$2K\varepsilon^{\lambda_n} \geqq \frac{1}{8\lambda_{n+2}} \text{ infinitely often,}$$

which implies that

$$\limsup_{n\to\infty} \frac{\log \lambda_{n+2}}{\lambda_n} \geqq \log\frac{1}{\varepsilon},$$

which contradicts the hypothesis, so completing the proof, and ending the paper.

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