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Analyse fonctionnelle/*Functional Analysis*

Banach spaces for which the ideal of compact operators is an M-ideal

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Abstract — We characterize those separable Banach spaces X such that $\mathscr{K}$(X) is an M-ideal in $\mathscr{L}$(X). We use this characterization to give new examples and also to produce examples of spaces which cannot be renormed so that $\mathscr{K}$(X) is an M-ideal.

Espaces de Banach pour lesquels l'idéal des opérateurs compacts est un M-idéal

Résumé — *Nous caractérisons les espaces de Banach séparable* X *tels que* $\mathscr{K}$(X) *soit un* M-*idéal de* $\mathscr{L}$(X). *Nous utiliserons cette caractérisation pour donner de nouveaux exemples, et aussi pour montrer que certains espaces ne peuvent être renormés pour que* $\mathscr{K}$(X) *soit un* M-*idéal de* $\mathscr{L}$(X).

Version française abrégée — On dira qu'un espace de Banach X a la propriété (M) si pour tous $u, v \in X$ tels que $\|u\|=\|v\|$ et pour toute suite (x_n) tendant faiblement vers zéro, on a

$$\limsup_{n\to\infty}\|u+x_n\|=\limsup_{n\to\infty}\|v+x_n\|.$$

Dualement, on dira que X a la propriété (M*) si, pour tous $u^*, v^* \in X^*$ de même norme et toute suite (x_n^*) préfaiblement convergente vers 0 on a

$$\limsup_{n\to\infty}\|u^*+x_n^*\|=\limsup_{n\to\infty}\|v^*+x_n^*\|.$$

On appellera suite contractante approximante (s. c. a) une suite (K_n) d'opérateurs compacts de X dans X telle que

$$\lim\|K_n x-x\|=\lim\|K_n^* x^*-x^*\|=0$$

pour tout $x \in X$ et tout $x^* \in X^*$. Le résultat principal de ce travail est le théorème suivant :

Théorème 1. — *Soit* X *un espace à dual séparable. Les conditions suivantes sont équivalentes :*

(1) $\mathscr{K}$(X) *est un* M-*idéal de* $\mathscr{L}$(X).

(2) *Pour tout sous-espace séparable* $\mathscr{E}$ *de* $\mathscr{L}$(X) *il existe une s. c. a.* (K_n) *telle que*

$$\limsup_{n\to\infty}\|SK_n+T(I-K_n)\|\leqq\max(\|S\|,\|T\|)$$

pour tous $S, T \in \mathscr{E}$.

(3) *Il existe une s. c. a.* (K_n) *telle que* $\|K_n\|\leqq 1$, *et telle que pour tous* $S \in \mathscr{K}(X)$ *et* $T \in \mathscr{L}(X)$ *on a*

$$\limsup_{n\to\infty}\|S+T(I-K_n)\|\leqq\max(\|S\|,\|T\|).$$

(4) *Il existe une s. c. a.* (K_n) *telle que pour tout opérateur compact* S *tel que* $\|S\|\leqq 1$ *on a*

$$\limsup_{n\to\infty}\|S+I-K_n\|\leqq 1.$$

(5) X *a la propriété* (M) *et une s. c. a.* (K_n) *telle que* $\lim\|I-2K_n\|=1$.

(6) X *a la propriété* (M*) *et une s. c. a.* (K_n) *telle que* $\lim\|I-2K_n\|=1$.

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Ce théorème permet de décrire précisement quels sont les espaces X pour lesquels $\mathscr{K}$(X) est un M-idéal de $\mathscr{L}$(X). X doit avoir une propriété d'approximation appropriée, et la propriété géométrique adéquate (M) ou (M*).

On en déduit plusieurs conséquences, par exemple une extension d'un résultat de Cho-Johnson [3] :

COROLLAIRE 2. — *Soit* X *un espace de Banach séparable tel que* $\mathscr{K}$(X) *soit un* M-*idéal de* $\mathscr{L}$(X). *Soit* E *un sous-espace ou un quotient de* X. *Alors* $\mathscr{K}$(E) *est un* M-*idéal de* $\mathscr{L}$(E) *si et seulement si* E *a la propriété métrique d'approximation.*

Les deux résultats suivants sont de nature isomorphique.

THÉORÈME 3. — *Soit* X *un espace de Banach ayant une base symétrique* (u_n). *L'espace* X *a une norme équivalente telle que* $\mathscr{K}$(X) *soit un* M-*idéal de* $\mathscr{L}$(X) *si et seulement si* X* *est séparable et* X *est isomorphe à un espace de suite d'Orlicz* h_F.

Pour les espaces réticulés sans atomes on a le résultat négatif suivant :

THÉORÈME 4. — *Soit* X *un Banach réticulé séparable sans atomes et avec une norme continue en ordre. Si* X *a une norme équivalente avec la propriété* (M) *alors* X *est isomorphe à* L_2 *pour la norme et pour l'ordre.*

On en déduit une amélioration d'un résultat de Lindenstrauss et Tzafriri [13] :

COROLLAIRE 5. — *Soit* X *un Banach réticulé continu en ordre et non-atomique, isomorphe à un sous-espace d'un espace de suite d'Orlicz* h_F. *Alors* X *est isomorphe à* L_2 *pour la norme et pour l'ordre.*

If X is a Banach space and E is a closed subspace of X then E is called an M-ideal in X if X* admits a direct sum decomposition $X^*=E^\perp\oplus_1 V$ for some subspace V. M-ideals were introduced by Alfsen and Effros [1]. See the forthcoming book [7] for a fuller discussion. It is a classical result of Dixmier [4] that the space $\mathscr{K}(l_2)$ of all compact operators on l_2 is an M-ideal in the space $\mathscr{L}(l_2)$ of all bounded operators, and this was extended by Lima [12] to the spaces l_p for $1<p<\infty$ and c_0. However, relatively few other examples of this phenomenon are known. The purpose of this Note is to announce a complete classification of all Banach spaces X so that $\mathscr{K}$(X) is an M-ideal in $\mathscr{L}$(X). Using this classification we are able to give some new examples and also give examples of spaces X, which admit no equivalent norm for which $\mathscr{K}$(X) is an M-ideal. Full details of our work will appear in [9]. We would like to thank Gilles Godefroy for his help in preparing this Note.

We say that a Banach space X has *property* (M) if whenever $u, v\in X$ with $\|u\|=\|v\|$ and (x_n) is weakly null sequence in X then,

$$\limsup_{n\to\infty}\|u+x_n\|=\limsup_{n\to\infty}\|v+x_n\|.$$

There is a similar dual notion, property (M*). We say that X has *property* (M*) if whenever $u^*, v^*\in X^*$ with $\|u^*\|=\|v^*\|$ and whenever (x_n^*) is a weak*-null sequence then

$$\limsup\|u^*+x_n^*\|=\limsup\|v^*+x_n^*\|.$$

The most typical examples are the spaces l_p which, for $1<p<\infty$, enjoy both properties (M) and (M*). However l_1 has property (M) but not property (M*). In fact property (M*) is formally stronger than property (M). We have:

PROPOSITION 1. — *Let* X *be a separable Banach space with property* (M*). *Then* X *has property* (M) *and* X *is an* M-*ideal in* X**.

Suppose X is a separable Banach space. We say that (K_n) is a compact approximating sequence for X if each $K_n : X \to X$ is a compact operator and $\lim \| K_n x - x \| = 0$ for every $x \in X$. We say (K_n) is shrinking if $\lim \| K_n^* x^* - x^* \| = 0$ for every $x^* \in X^*$. If X has a shrinking compact approximating sequence (K_n) then X* is separable and further X has a shrinking compact approximating sequence (L_n) with $\| L_n \| \leqq 1$, *i. e.* X has the metric shrinking compact approximation property. If $\mathscr{K}(X)$ is an M-ideal in $\mathscr{L}(X)$ then it has a shrinking compact approximating sequence [6], and necessarily X* is separable.

The following results is a characterization of those separable Banach spaces X so that $\mathscr{K}(X)$ is an M-ideal in $\mathscr{L}(X)$.

THEOREM 2. — *Let* X *be a separable Banach space with separable dual. The following conditions are equivalent:*

(1) $\mathscr{K}(X)$ *is an* M-*ideal in* $\mathscr{L}(X)$.

(2) *For any separable subspace* $\mathscr{E} \subset \mathscr{L}(X)$ *there exists a shrinking compact approximating sequence* (K_n) *such that*

$$\limsup_{n \to \infty} \| S K_n + T (I - K_n) \| \leqq \max (\| S \| , \| T \|)$$

whenever S, $T \in \mathscr{E}$.

(3) *There exists a shrinking compact approximating sequence* (K_n) *with* $\| K_n \| \leqq 1$, *such that for any* $S \in \mathscr{K}(X)$, $T \in \mathscr{L}(X)$ *we have*

$$\limsup_{n \to \infty} \| S + T (I - K_n) \| \leqq \max (\| S \| , \| T \|).$$

(4) *The exists a shrinking compact approximating sequence* (K_n) *such that for every compact operator* S *with* $\| S \| \leqq 1$, *we have*

$$\limsup_{n \to \infty} \| S + I - K_n \| \leqq 1.$$

(5) X *has property* (M) *and a shrinking compact approximating sequence* (K_n) *such that* $\lim \| I - 2 K_n \| = 1$.

(6) X *has property* (M*) *and a shrinking compact approximating sequence* (K_n) *such that* $\lim \| I - 2 K_n \| = 1$.

We remark here that the equivalence of (1) and (2) is essentially a theorem of W. Werner [15], with a slightly modified statement.

The condition that X has a shrinking compact approximating sequence (K_n) with $\lim \| I - 2 K_n \| = 1$ is equivalent to requiring, for every $\varepsilon > 0$, the existence of a sequence of compact operators (A_n) so that $\sum_{n=1}^{\infty} A_n x = x$ for $x \in X$, $\sum_{n=1}^{\infty} A_n^* x^* = x^*$ for $x^* \in X^*$ and

$$\left\| \sum_{k=1}^{n} \theta_k A_k \right\| \leqq 1 + \varepsilon$$

whenever $n \in \mathbb{N}$ and $\theta_k = \pm 1$ for $1 \leqq k \leqq n$ (*see* [2]). The fact that if $\mathscr{K}(X)$ is an M-ideal in $\mathscr{L}(X)$ then X has such a compact approximating sequence can be traced to [6]; the corresponding result for unconditional expansions of the identity in the case when X has (MAP) has been exploited in [5] and [11].

Theorem 2 essentially reduces the question of when $\mathscr{K}(X)$ is an M-ideal in $\mathscr{L}(X)$ to two criteria: X must have the right approximation property, and X must have the appropriate geometric property (M) (or (M*)). Since (M) passes to subspaces and (M*) to quotients, it is not difficult to prove an extension of the Cho-Johnson theorem [3] on subspaces of l_p, where $1<p<\infty$:

THEOREM 3. — *Let* X *be a separable Banach space such that* $\mathscr{K}(X)$ *is an* M-*ideal in* $\mathscr{L}(X)$. *Let* E *be either a subspace or a quotient space of* X. *Then* $\mathscr{K}(E)$ *is an* M-*ideal in* $\mathscr{L}(E)$ *if and only if* E *has the metric compact approximation property.*

Here the only part of the proof left is to show that if E has the metric compact approximation property then it actually inherits the stronger approximation property of Theorem 2 (5) or (6) from X. Note also that if E is reflexive then it is enough to suppose that X has the compact approximation property.

The next lemma shows the connection between our work and the theory of types developed in [10] and subsequent work.

LEMMA 4. — *Suppose* X *is a separable Banach space. Then* X *has property* (M) *if and only if every weakly null type* τ *is of the form* $\tau(x)=f(\|x\|)$ *for a suitable function* $f:\mathbf{R}\to\mathbf{R}$.

Let us now construct a family of examples. Let $\mathscr{N}$ be the family of all absolute norms N on $\mathbf{R}^2$ satisfying $N(1,0)=1$. If $N\in\mathscr{N}$, then for any sequence $(\xi_i)_{i=0}^{\infty}$ we define $\Lambda_0(\xi)=|\xi_0|$ and then, inductively,

$$\Lambda_k(\xi)=N(\Lambda_{k-1}(\xi),\ \xi_k).$$

We then define $\tilde{\Lambda}(N)$ to be the space of ξ such that

$$\|\xi\|_{\Lambda(N)}=\sup_k \Lambda_k(\xi)<\infty.$$

We define $\Lambda(N)$ to be the closed linear span of the canonical basis vectors $(e_k)_{k=0}^{\infty}$.

We recall that if F is an Orlicz function then the Orlicz sequence space l_F is defined to be the space of all sequences $(\xi)_{i=0}^{\infty}$ such that $\sum_{k=1}^{\infty} F(\alpha|\xi_k|)<\infty$ for some $\alpha>0$. This can be normed by

$$\|\xi\|_{l(F)}=\inf\left\{\rho>0:\sum_{k=0}^{\infty} F(|\xi_k|/\rho)\leq 1\right\}.$$

The closed linear span of (e_n) in l_F is denoted by h_F. See Lindenstrauss-Tzafriri [14] for details.

PROPOSITION 5. — $\Lambda(N)$ *is isomorphic to the Orlicz sequence space* h_F, *where* $F(t)=N(1,t)-1$, *and* $\Lambda(N)$ *has property* (M).

Referring back to our main theorem, Theorem 2, we then have:

THEOREM 6. — *Let* h_F *be an Orlicz sequence space with separable dual. Then* h_F *can be equivalently normed so that its canonical basis is subsymmetric and* $\mathscr{K}(h_F)$ *is an* M-*ideal in* $\mathscr{L}(h_F)$.

It should be noted here that Hennefeld [8] showed that if X has a symmetric basis and $\mathscr{K}(X)$ is an M-ideal in $\mathscr{L}(X)$ then X is isometric to one of the spaces l_p, $1<p<\infty$ or c_0. However we see that reflexive Orlicz sequence spaces are examples of spaces with symmetric bases which can be renormed so that $\mathscr{K}(X)$ is an M-ideal. It turns out that there is a converse to this result.

THEOREM 7. — *Let* X *be a Banach space with a symmetric basis* (u_n). *In order that* X *can be equivalently normed so that* $\mathcal{K}(X)$ *is an* M-*ideal in* $\mathcal{L}(X)$ *it is necessary and sufficient that* X *is isomorphic to an Orlicz sequence space* h_F *and* X^* *is separable.*

This result gives us a large collection of spaces with symmetric bases which cannot be renormed so that $\mathcal{K}(X)$ is an M-ideal. For nonatomic Banach lattices, we have the following negative result.

THEOREM 8. — *Let* X *be a separable order-continuous nonatomic Banach lattice. If* X *has an equivalent norm with property* (M) *then* X *is lattice-isomorphic to* L_2.

In particular if $p \neq 2$ there is no renorming of L_p so that $\mathcal{K}(L_p)$ is an M-ideal. The fact that this is true for the usual norm is well-known [12].

COROLLARY 9. — *Let* X *be a nonatomic order-continuous Banach lattice which is isomorphic to a subspace of an Orlicz sequence space* h_F. *Then* X *is lattice-isomorphic to* L_2.

This result, which is an immediate deduction from Proposition 5 gives an extension of an old result of Lindenstrauss and Tzafriri [13] that a separable r. i. space on [0, 1] which is not equal to L_2 cannot be embedded in a separable Orlicz sequence space.

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