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Transitivity and quotients of Orlicz spaces

To Professor Władysław Orlicz
on the occasion of his 75-th birthday

1. Introduction. The initial objective of this paper was to investigate the quotient structure of general non-locally convex Orlicz function spaces. However, our results required the introduction of the notion of transitivity of an $F$-space, and during the investigation we came across certain ideas connected with transitive $F$-spaces which seemed worth developing. Thus the paper falls into two distinct parts. In the first part we study transitivity in general and in the second part we relate the notion to Orlicz spaces.

We introduce in Section 3 the notion of a transitive pair of $F$-spaces and present some general results on this concept. In Section 4 we introduce ultra-transitive $F$-spaces — an $F$-space $X$ is ultra-transitive if every finite rank operator $T : E \to X$, where $E$ is a closed subspace of $X$, may be extended to an endomorphism of $X$. Of course every Banach space is ultra-transitive, but the existence of non-locally convex ultra-transitive spaces is less obvious. For example the space $l^p$ is not ultra-transitive. However, in Section 4 we construct an ultra-transitive $p$-Banach space ($p < 1$) which is separable and universal for all separable $p$-Banach spaces. We do not know any examples of non-locally convex ultra-transitive spaces which arise naturally.

In Section 5 we apply these ideas to Orlicz spaces and show, for example, that an Orlicz space $L_F$ is transitive if and only if either $\liminf_{x \to \infty} x^{-1}F(x) > 0$ or $F$ is equivalent to a submultiplicative function. If $0 < \alpha_F^\infty \leq \beta_F^\infty < 1$, then, unless $F$ is equivalent to a submultiplicative function, $L_F$ has two non-isomorphic quotients by one-dimensional subspaces (this extends some results in [5]).

In Section 6 we show that $L_p$ has no quotient isomorphic to another Orlicz function space, but that $L_p$ is isomorphic to a quotient of certain Orlicz function spaces.
Our methods in studying operators on Orlicz spaces are closely related, but not identical, to those used by Turpin ([9], [10]). Turpin uses the concept of the gau of the space, whereas we use (essentially) the idea of analyzing the possible behavior at infinity of $F$-norms on the space.

2. Notation. An $F$-space is a complete metric linear topological space, which may be over the real or complex scalars; unless stated we will treat only the real case, and implicitly assume the complex case.

In a locally bounded $F$-space $X$ the topology may be given by a quasi-norm $x \mapsto \|x\|$ which satisfies

- (i) $\|x\| \geq 0$ ($x \in X$),
- (ii) $\|x\| = 0$ if and only if $x = 0$,
- (iii) $\|tx\| = |t| \|x\|$ ($t \in \mathbb{R}$, $x \in X$),
- (iv) $\|x + y\| \leq k(\|x\| + \|y\|)$ ($x$, $y \in X$),

where $k$ is independent of $x$ and $y$. If in addition we have for some $p$, $0 < p \leq 1$,

- (v) $\|x + y\|^p \leq \|x\|^p + \|y\|^p$ ($x$, $y \in X$),

then we say that $X$ is a $p$-Banach space (when equipped with this quasi-norm). By the Aoki-Robert theorem ([8], p. 61) every locally bounded $F$-space is locally $p$-convex for some $p > 0$ and may therefore be renormed equivalently to be a $p$-Banach space.

If $X$ and $Y$ are $F$-spaces we denote by $\mathcal{L}(X, Y)$ the space of operators (i.e. continuous linear maps) from $X$ to $Y$. If $X$ and $Y$ are both quasi-normed, then $\mathcal{L}(X, Y)$ may be quasi-normed by

$$\|T\| = \sup(\|Tx\| : \|x\| \leq 1).$$

In the case $X = Y$ we denote $\mathcal{L}(X, Y)$ by $\mathcal{L}(X)$.

The space $\mathcal{L}(X, \mathbb{R})$ (or $\mathcal{L}(X, \mathbb{C})$ in the complex case) is denoted by $X^*$. An Orlicz function $F$ is a continuous, strictly increasing function $F: [0, \infty) \rightarrow [0, \infty)$ such that $F(0) = 0$. The Orlicz function space $L_F(0, 1) = L_F$ consists of all measurable functions $f$ on $(0, 1)$ such that

$$\int_0^1 F(\varepsilon |f(x)|) \, dx < \infty$$

for some $\varepsilon > 0$. $L_F$ is an $F$-space if we equip it with the topology induced by the base of neighborhoods of 0 of the form $rB_F(\varepsilon)$ ($r > 0$, $\varepsilon > 0$), where $B_F(\varepsilon)$ consists of all $f$ such that

$$\int_0^1 F(|f(x)|) \, dx \leq \varepsilon.$$
Two Orlicz functions $F$ and $G$ are equivalent (at $\infty$) if there exist constants $k_1, k_2, \alpha_1, \alpha_2 > 0$ such that

$$k_1 F(\alpha_1 x) \leq G(x) \leq k_2 F(\alpha_2 x) \quad (1 \leq x < \infty).$$

Then $L_F = L_G$.

We define

$$\alpha_F^\infty = \sup \left\{ p : \sup_{t, x > 1} t^p \frac{F(x)}{F(tx)} < \infty \right\}, \quad \beta_F^\infty = \inf \left\{ p : \sup_{t, x > 1} \frac{F(tx)}{t^p F(x)} < \infty \right\}.$$

Then $L_F$ is locally bounded if and only if $\alpha_F^\infty > 0$; it is locally $p$-convex if and only if

$$\sup_{t, x > 1} \frac{t^p F(x)}{F(tx)} < \infty$$

(see [6], [10], p. 77).

The condition $\beta_F^\infty < \infty$ is equivalent to the $A_\infty$-condition (at $\infty$)

$$\sup_{x > 1} \frac{F(2x)}{F(x)} < \infty,$$

and to the condition that $L_F$ is separable. Under the $A_\infty$-condition, if $f \in L_F$, then

$$\int_0^1 F(|f(x)|) \, dx < \infty.$$

$F$ is said to be subadditive if

$$F(x+y) \leq F(x) + F(y) \quad (0 \leq x, y < \infty)$$

and submultiplicative (at $\infty$) if

$$F(xy) \leq F(x)F(y) \quad (1 \leq x, y < \infty).$$

$F$ is equivalent to a submultiplicative function if there exists a $k$ such that

$$F(xy) \leq kF(x)F(y) \quad (1 \leq x, y < \infty).$$

(Define

$$G(t) = \sup_{x > 1} \frac{F(tx)}{F(x)}.$$ )

For a full discussion of Orlicz spaces and functions see [10].

3. Transitive spaces. An $F$-space $X$ is transitive if given $x, y \in X$, with $x \neq 0$, there exists $T \in \mathcal{L}(X)$ with $Tx = y$. We shall extend this by defining a pair $(X, Y)$ to be transitive if given $0 \neq x \in X$ and $y \in Y$ there exists $T \in \mathcal{L}(X, Y)$ with $Tx = y$. 
Some elementary observations are collected together in Propositions 3.1 and 3.2.

**Proposition 3.1.** Suppose \((X, Y)\) is a transitive pair. Then
(a) if \(N\) is a closed subspace of \(X\), \((N, Y)\) is transitive;
(b) if \(M\) is a closed subspace of \(Y\), \((X, Y/M)\) is transitive;
(c) if \((Y, Z)\) is transitive, then \((X, Z)\) is transitive.

**Proposition 3.2.** Let \(X\) be a transitive space. Then
(a) if \(N\) is a complemented subspace of \(X\), \(N\) is transitive;
(b) if \(X^* \neq \{0\}\), \(X\) has a separating dual.

If \(X\) is a locally bounded \(F\)-space with quasi-norm \(x \mapsto \|x\|\), we define for \(0 < p < \infty\), \(l_p(X)\) to be the space of all sequences \((x_n)\), \(x_n \in X\) such that
\[
\|(x_n)\|_p = \left(\sum \|x_n\|^p\right)^{1/p} < \infty.
\]
The quasi-norm \((x_n) \mapsto \|(x_n)\|_p\) makes \(l_p(X)\) a locally bounded \(F\)-space. For \(p = \infty\), we define \(l_\infty(X)\) to be the space of all sequences \((x_n)\) such that
\[
\|(x_n)\|_\infty = \sup_n \|x_n\| < \infty.
\]
The space \(l_\infty(X)\) has been studied in [7].

The following lemma is an immediate consequence of the Open Mapping Theorem.

**Lemma 3.3.** Let \((X, Y)\) be a transitive pair of locally bounded \(F\)-spaces, and suppose \(0 \neq u \in X\). Then there exists \(k\) such that for any \(y \in Y\), there exists \(T \in \mathcal{L}(X, Y)\) with \(\|T\| \leq k\|y\|\) and \(Tu = y\).

**Theorem 3.4.** Suppose \(X\) is a transitive locally bounded \(F\)-space. Then \(l_p(X)\) is also transitive for any \(p\), \(0 < p \leq \infty\).

**Proof.** It is trivial to see that \((l_p(X), X)\) is transitive. Conversely if \(u \in X\) and \((v_n) \in l_p(X)\), there exists \(T_n \in \mathcal{L}(X, X)\) such that \(T_n u = v_n\) and \(\|T_n\| \leq k\|v_n\|\), where \(k\) is a constant independent of \(n\). Define \(S: X \rightarrow l_p(X)\) by \(Sx = (T_n x)\). Then \((Su)_n = (v_n)\). Hence \((X, l_p(X))\) is transitive and the proof is completed by Proposition 3.1 (c).

**Remark.** If \(X\) is transitive and \(X^* = \{0\}\), then \((l_\infty(X))^* = \{0\}\) by 3.4. Compare with [7], where it is shown that for some locally bounded spaces \(X\), with \(X^* = \{0\}\) we have \((l_\infty(X))^* \neq \{0\}\).

**Theorem 3.5.** Let \((X, Y)\) be a transitive pair of separable locally bounded \(F\)-spaces. If \(Y\) is locally \(p\)-convex, then \(Y\) is isomorphic to a quotient of \(l_p(X)\).

Conversely if \(X\) is transitive, and \(Y\) is isomorphic to a quotient of \(l_p(X)\) (where \(0 < p \leq \infty\)), then \((X, Y)\) is transitive.
Proof. Suppose \((X, Y)\) is transitive, and that \((y_n)\) is a sequence dense in the unit ball of \(Y\) (where we assume the quasi-norm on \(Y\) satisfies \(||x+y||^p \leq ||x||^p + ||y||^p\)). Fix \(u \in X\), with \(||u|| = 1\). By Lemma 3.1, there exist \(k\) such that we may find \(T_n \in \mathcal{L}(X, Y)\) with \(T_n u = y_n\) and \(||T_n|| \leq k\). Define \(S: l_p(X) \to Y\) by
\[
S((x_n)) = \sum_{n=1}^{\infty} T_n x_n.
\]
\(S\) is bounded and \(0 \in \text{int} \overline{S(U)}\), where \(U\) is the unit ball of \(l_p(X)\). Hence \(S\) is an open mapping.

The converse is immediate from Proposition 3.1(b) since \((X, l_p(X))\) is transitive.

**Corollary 3.6.** A separable locally bounded \(F\)-space \(X\) is isomorphic to a quotient of \(L_p\) \((0 < p < 1)\), if and only if \(X\) is locally \(p\)-convex and \((L_p, X)\) is transitive.

**Corollary 3.7.** If \(0 < q < p < 1\), then \(L_q\) is isomorphic to a quotient of \(l_q(L_p)\).

Corollaries 3.6 and 3.7 follow from the remarks that \(l_p(L_p) \cong L_p\) and \(\mathcal{L}(L_p, L_q) \neq \{0\}\) if \(p \geq q\).

We conclude this section by remarking that there exists a non-atomic measure space \((\Omega, \mu)\) such that \(L_p(\Omega, \mu)\) is not transitive. Indeed if \((\Omega_0, \mu_0)\) denotes an uncountable product of copies of \((0, 1)\) with Lebesgue measure, there is no continuous operator from \(L_p(\Omega_0, \mu_0)\) \((0 < p < 1)\) into a separable \(F\)-space. This may be proved by the methods of [1]; indeed any operator is an isomorphism on some subspace isomorphic to a non-separable Hilbert space. Hence \(L_p(\Omega_0, \mu_0) \oplus L_p\) is not transitive and this is isomorphic to \(L_p(\Omega, \mu)\) for some \((\Omega, \mu)\).

4. **Ultra-transitive spaces.** In [5] an \(F\)-space \(X\) was said to be strongly transitive if whenever \(E \subset X\) is a subspace of finite dimension then every \(T_0 \in \mathcal{L}(E, X)\) has an extension \(T \in \mathcal{L}(X)\). Clearly if we replace finite dimension by dimension one in this definition we would obtain a characterization of transitive spaces. For complex locally bounded spaces the two concepts coincide ([5]), but it is an open question whether they coincide in general.

Motivated by this definition we define an \(F\)-space \(X\) to be ultra-transitive if whenever \(N \subset X\) is a closed subspace and \(T_0 \in \mathcal{L}(N, X)\) is an operator of finite rank, then \(T_0\) has an extension \(T \in \mathcal{L}(X)\).

**Proposition 4.1.** Let \(X\) be an \(F\)-space.

(a) If \(X\) is ultra-transitive, then every quotient of \(X\) is ultra-transitive.

(b) If \(X\) is transitive and \((X/M, X)\) is transitive for every closed subspace \(M\) of \(X\), then \(X\) is ultra-transitive.
Proof. (a) Let $M$ be a closed subspace of $X$ and $N$ a closed subspace of $X/M$. Let $T_0: N \to X/M$ be an operator of finite rank. Then if $q: X \to X/M$ is the quotient map, there exists an operator $T_1: N \to X$ of finite rank such that $qT_1 = T_0$. Then consider $T_1q: q^{-1}(N) \to X$; $T_1q$ has an extension $S: X \to X$, and as $S(M) = 0$, $S$ factors through $q$, i.e. $S = S_1q$, where $S_1 \in \mathcal{L}(X/M, X)$. Then $qS_1 = T$ extends $T_0$ and belongs to $\mathcal{L}(X/M)$.

(b) It suffices to show that if $N$ is a closed subspace of $X$ and $T_0 \in \mathcal{L}(N, X)$ has rank one, then $T_0$ may be extended to an endomorphism of $X$. Let $M = T_0^{-1}(0)$; then if $q: X \to X/M$ in the quotient map, $T_0 = S_0q$, where $S_0: q(N) \to X$. As $\dim q(N) = 1$ and $(X/M, X)$ is transitive, $S_0$ may be extended to $S \in \mathcal{L}(X/M, X)$. Then $T = Sq$ extends $T_0$.

We remark now that $l_p$ ($0 < p < 1$) is not ultra-transitive since $L_p$ is isomorphic to a quotient of $l_p$, and $\mathcal{L}(L_p, l_p) = \{0\}$. The space $L_p$ ($0 < p < 1$) is also not ultra-transitive, but the proof of this will appear elsewhere [4]. Also the space $L_p$ is not ultra-transitive [4]. In view of this it may appear that the only ultra-transitive spaces are locally convex, but this is not the case. We now construct some non-locally convex ultra-transitive spaces.

**Lemma 4.2.** Let $X$ be a separable $p$-Banach space, and let $(E_n)$ be a sequence of closed subspaces of $X$. Let $T_n: E_n \to X$ be a sequence of operators with $\|T_n\| \leq 1$. Then there is a separable $p$-Banach space $Y$ containing $X$ such that each $T_n$ extends to an operator $\hat{T}_n: X \to Y$, with $\|\hat{T}_n\| \leq 1$.

Proof. Consider $l_p(X)$ (where for convenience we index sequences $(x_n) \in l_p(X)$ starting at $n = 0$). Let $M$ be the subspace of $l_p(X)$ of all $(x_n)$ such that $x_n \in E_n$ ($n \geq 1$) and

$$x_0 + \sum_{n=1}^{\infty} T_n x_n = 0.$$ 

Let $Y = l_p(X)/M$, and $q: l_p(X) \to Y$ be the quotient map. $Y$ is clearly separable. Let $(J_n: n \geq 0)$ be the map $J_n: X \to l_p(X)$ defined by $J_n x = (0, \ldots, 0, x, 0, \ldots)$ ($x$ in the $n$-th place).

First observe that $qJ_0$ is an isometric embedding of $X$ into $Y$. For if $(u_n) \in M$ and $x \in X$, then

$$\|x + u_0\|^p + \sum_{n=1}^{\infty} \|u_n\|^p \geq \|x + u_0\|^p + \sum_{n=1}^{\infty} \|T_n u_n\|^p \geq \|x\|^p.$$

Thus we may identify $X$ with the subspace $qJ_0(X)$. If $x \in E_n$, then $(T_n x, 0, \ldots, -x, 0, \ldots) \in M$ and hence $qJ_n x = qJ_0 T_n x$, so that the map $qJ_n(qJ_0)^{-1}: qJ_0(X) \to Y$ extends the map $qJ_0 T_n(qJ_0)^{-1}: qJ_0(E_n) \to Y$.

**Theorem 4.3.** There is a separable $p$-Banach space, universal for all separable $p$-Banach spaces, which is transitive.
Proof. In [2] Theorem 4.1 it is shown that there is a universal separable \( p \)-Banach space, \( X_0 \) say. Let \( D_0 \) be a dense countable subset of the unit sphere of \( X_0 \). We shall construct inductively a sequence \( (X_n) \) of \( p \)-Banach spaces and a dense countable subset \( D_n \) of the unit sphere of \( X_n \) such that

1. \( X_{n+1} \supset X_n \quad (n \geq 0) \),
2. \( D_{n+1} \supset D_n \quad (n \geq 0) \).

3. If \( u, v \in D_n \), there is an operator \( T_{u,v}^{(n)} : X_n \to X_{n+1} \) with \( T_{u,v}^{(n)}(u) = v \), \( n \geq 0 \) and \( \|T_{u,v}^{(n)}\| = 1 \).

4. If \( u, v \in D_n \), then \( T_{u,v}^{(n+1)}(x) = T_{u,v}^{(n)}(x) \), \( x \in X_n \), \( n \geq 0 \).

Suppose \( X_n, D_n \) and (for \( n > 1 \)) \( T_{u,v}^{(n-1)} \), \( u, v \in D_{n-1} \) are given. Then by Lemma 4.2 we may construct \( X_{n+1} \) containing \( X_n \) such that each operator

\[
T_{u,v}^{(n-1)} : X_{n-1} \to X_n, \quad (u, v) \in D_{n-1} \times D_{n-1},
\]

\[
R_{u,v} : \text{lin}(u) \to X_n, \quad (u, v) \in (D_n \times D_n) \setminus (D_{n-1} \times D_{n-1})
\]

(where \( R_{u,v}(u) = v \)) has a norm one extension, denoted by \( T_{u,v}^{(n)} \) for all \( (u, v) \in D_n \times D_n \)

\[
T_{u,v}^{(n)} : X_n \to X_{n+1}.
\]

Then choose \( D_{n+1} \) a dense countable subset of \( X_{n+1} \), containing \( D_n \). Repeating this construction inductively we obtained the required sequences.

Now let \( Z \) be the completion of \( \bigcup_{n=0}^{\infty} X_n \). Clearly \( D = \bigcup_{n=0}^{\infty} D_n \) is dense in the unit sphere of \( Z \), and if \( (u, v) \in D \times D \), there is a unique operator \( T_{u,v} : Z \to Z \) such that \( T_{u,v}(x) = T_{u,v}^{(n)}(x) \) for \( x \in X_n \), \( (u, v) \in D_n \times D_n \); \( T_{u,v}(u) = v \), and \( \|T_{u,v}\| = 1 \).

For fixed \( z \in Z \), with \( \|z\| = 1 \), the map \( \Phi : \mathcal{L}(Z) \to Z \), defined by \( \Phi(T) = Tz \) has the property that the image of the unit ball of \( \mathcal{L}(Z) \) is dense in the unit ball of \( Z \). Hence \( \Phi \) is an open map and so \( Z \) is transitive. As \( Z \supset X_0 \), \( Z \) is also universal for all separable \( p \)-Banach spaces.

**Theorem 4.4.** A transitive universal separable \( p \)-Banach space is ultra-transitive. Hence for each \( 0 < p < 1 \), there is a separable ultra-transitive \( p \)-Banach space which is not locally \( q \)-convex for any \( q > p \).

**Proof.** Let \( W_p \) be such a space, and let \( X \) be a quotient of \( W_p \). Then \( (X, W_p) \) is transitive (since \( X \) embeds in \( W_p \)). Apply Proposition 4.1.

In fact, we observe that by Theorem 4.4 and Theorem 3.4,

**Corollary 4.5.** For \( 0 < p < 1 \), there is a separable \( p \)-Banach space \( W \) for which \( l_p(W) \) is ultra-transitive.
Theorem 4.6. Let $W$ be a separable $p$-Banach space such that $l_p(W)$ is ultra-transitive. Let $X$ be any separable $p$-Banach space; the following conditions are equivalent:

(a) $l_p(X)$ is ultra-transitive;
(b) $X$ is isomorphic to a quotient of $l_p(W);$
(c) $(Y, X)$ is transitive for any separable $p$-Banach space $Y$;
(d) $X$ is isomorphic to a quotient of $l_p(Y)$ for any separable $p$-Banach space $Y$.

Proof. (c) $\Rightarrow$ (d) Theorem 3.5.
(d) $\Rightarrow$ (b) Trivial.
(b) $\Rightarrow$ (a) Since $l_p(X)$ is also isomorphic to a quotient of $l_p(W)$, this is immediate from Proposition 4.1.
(a) $\Rightarrow$ (c) Since $l_p(X)$ contains a copy of $l_p$, it has a quotient $Z$ which is universal for all separable $p$-Banach spaces. Thus $Y$ embeds in $Z$, and as $Z$ is transitive so is the pair $(Y, Z)$. However, $(Z, l_p(X))$ is also transitive, hence so is $(Y, l_p(X))$ and hence $(Y, X)$.

Remark. $L_p$ has an ultra-transitive quotient (see part (d)).

5. Transitivity of Orlicz spaces. In this section we turn to the question of transitivity of pairs of Orlicz function spaces. First we observe that for any Orlicz function $F$ and $f \in L_F$, there exists $T \in L(L_F)$ with $Tf = 1$ (by 1 we mean the characteristic function of the unit interval). Hence to show that a pair $(L_F, L_G)$ is transitive it suffices to show that for $g \in L_G$ there exists $T \in L(L_F, L_G)$ with $T1 = g$.

Our first lemma is a positive result.

Lemma 5.1. Let $F$ and $G$ be Orlicz functions and suppose $u \in L_G$. Suppose there exists $k$ such that

\[(*) \quad \int_0^1 G(t |u(x)|) \, dx \leq k F(t), \quad (1 \leq t < \infty).\]

Then there exists $T \in L(L_F, L_G)$ with $T1 = u$. If, in addition, $u$ is countably simple and (*) holds for $0 \leq t < \infty$, then $T$ may be chosen so that

\[\int_0^1 G(|Tf(x)|) \, dx \leq k \int_0^1 F(|Tf(x)|) \, dx \quad (f \in L_F).\]

Proof. In fact, we prove the latter assertion first. Suppose

\[u = \sum_{n=1}^\infty e_n \chi_n,\]

where $\chi_n$ is the characteristic function of the set $A_n$, where $\lambda(A_n) > 0$. Since the measure spaces $A_n$ and $A_n \times (0, 1)$ are isomorphic, we may
define an operator $S: L_G((0, 1) \times (0, 1)) \to L_G(0, 1)$ such that $S(\chi_{A_n} \times (0, 1)) = \chi_n$ and
\[
\int_0^1 \int_0^1 G(|Sf(x)|) \, dx \, dy = \int_0^1 \int_0^1 G(|f(x, y)|) \, dx \, dy
\]
for $f \in L_G((0, 1) \times (0, 1))$.

Now define $T_0: L_F \to L_G((0, 1) \times (0, 1))$ by
\[
T_0f(x, y) = u(x)f(y).
\]
Then $ST_0$ satisfies the conditions.

For the first part let
\[
v = \sum_{n=-\infty}^{\infty} 2^n \chi \{x: 2^n \leq |u(x)| < 2^{n+1}\}.
\]
Then $v \in L_G$ is countably simple and there exists $A \in \mathcal{L}(L_G)$ such that $Av = u$. Clearly
\[
\int_0^1 \int_0^1 G(t|v(x)|) \, dx \leq kF(t) \quad (1 \leq t < \infty).
\]
If we define $F^*(t) = \max \{F(t), \frac{1}{k} \int_0^1 G(t|v(x)|) \, dx\}$ $(0 \leq t < \infty)$, then $F^*$ is equivalent to $F$ and
\[
\int_0^1 \int_0^1 G(t|v(x)|) \, dx \leq kF^*(t) \quad (0 \leq t < \infty).
\]

By the second part of the lemma, there exists $R \in \mathcal{L}(L_F, L_G)$ with $R1 = v$; then $AR1 = u$.

**Theorem 5.2.** Let $F$ and $G$ be two Orlicz functions. In order that the pair $(L_F, L_G)$ be transitive it is necessary and sufficient that either

(a) $\liminf_{x \to \infty} x^{-1}F(x) > 0$;

or

(b) there is a constant $k$ such that
\[
G(tx) \leq kG(t)F(x), \quad 1 \leq t, \ x < \infty.
\]

In particular $L_F$ is transitive if and only if either (a) holds or $F$ is equivalent to a submultiplicative function.

**Proof.** If (a) holds, $L_F$ has a separating dual and so $(L_F, X)$ is transitive for any $X$. If (b) holds, then $G$ satisfies the $A_2$-condition. If $u \in L_G$, let $v(x) = \max(1, |u(x)|)$. Then $v \in L_G$ and hence
\[
\int_0^1 G(|v(x)|) \, dx < \infty.
\]
However,
\[ \int_0^1 G(t|v(x)||)dx \leq kF(t) \int_0^1 G(|v(x)||)dx, \quad t \geq 1 \]
and hence by 5.1, there exists \( A \in \mathcal{L}(L_F, L_G) \) with \( A1 = v \). Hence there exists \( T \) such that \( T1 = u \), and \( (L_F, L_G) \) is transitive.

Now suppose \( (L_F, L_G) \) is transitive. We may suppose that for \( 0 \leq x \leq 1 \) we have \( F(x) = G(x) = x \); suppose also that condition (a) does not hold.

Let
\[ h(0) = 1, \]
\[ h(x) = \min_{0 < t \leq x} \frac{G(t)}{t}, \quad (x > 0), \]
and define
\[ H(x) = xh(x) \quad (x \geq 0). \]

Then it is easy to check that \( H \) is an Orlicz function and \( H(x) \leq G(x) \) \((x \geq 0)\). Also as \( H(x)/x \) decreases monotonically, \( H \) is subadditive.

As the set \( B_F(1) \) is additively bounded ([10], p. 35), we may define for \( T \in \mathcal{L}(L_F, L_G) \)
\[ \gamma(T) = \sup_{f \in B_F(1)} \int_0^1 H(|Tf(x)||)dx \]
and \( \gamma(T) < \infty \). Clearly
\[ \gamma(S + T) \leq \gamma(S) + \gamma(T), \quad S, T \in \mathcal{L}(L_F, L_G), \]
and
\[ \gamma(aS) = |a| \gamma(S), \quad |a| \leq 1, \ S \in \mathcal{L}(L_F, L_G). \]

For \( n \geq 1 \), let
\[ V_n = \{ g \in L_G : \exists T \in \mathcal{L}(L_F, L_G), \gamma(T) \leq n, T1 = g \}. \]
Then \( V_{m+n} \supseteq V_m - V_n, \ m, n \in \mathbb{N} \). As \( \bigcup V_n = L_G \), we may apply the Baire category theorem to deduce \( 0 \in \text{int} V_m \), some \( m \). As \( B_G(1) \) is additively bounded, there exists \( N \) such that \( B_G(1) \subset V_N \).

Fix \( x \geq 1 \), and let \( A \) be a set of measure \( G(x)^{-1} \). Let \( u = \chi_A; \) then \( u \in B_G(1) \). Hence there is a sequence \( T_n \in \mathcal{L}(L_F, L_G) \) with \( T_n1 \rightarrow u \) and \( \gamma(T_n) \leq N \).

For \( t \geq 1 \), let \( m = [F(t)] + 1 \leq 2F(t) \). Partition \((0, 1)\) into \( m \) disjoint subsets of measure \( m^{-1} \), with characteristic functions \( \chi_1, \ldots, \chi_m \). Then \( t\chi_i \in B_F(1) \) and hence
\[ \int_0^1 H(t|T_n\chi_i(\tau)||)d\tau \leq N \quad (i = 1, 2, \ldots, m). \]
Since $H$ is subadditive,
\[ \int_0^1 H(\{T_n1(\tau)\}) d\tau \leq Nm \leq 2NF(t). \]

As $T_n1 \to u$ is measure, we have, by Fatou's lemma
\[ \int_0^1 H(\{u(\tau)\}) d\tau \leq 2NF(t), \]
i.e.
\[ H(tw) \leq 2NF(t)G(x) \quad (1 \leq t, x). \]

Now pick $\theta$ so that
\[ F(\theta) < \theta/2N. \]

For $x \geq 1$, we define
\[ y = \max \{t : H(t) = G(t), \ 0 \leq t \leq x\}. \]
Then $y \geq 1$. If $y < z \leq x$, then $h(z) = h(y)$ and
\[ H(z) = zh(y) = zy^{-1}G(y). \]
Thus
\[ \frac{z}{y} G(y) \leq 2NF \left( \frac{z}{y} \right) G(y) \]
and hence $z/y \neq \theta$. Thus $x/y < \theta$, and
\[ G(x) \geq H(x) = \frac{x}{y} G(y) \geq G(\theta^{-1}x) \quad (x \geq 1). \]

$H$ and $G$ are therefore equivalent, and so $G$ satisfies the $A_\theta$-condition. Hence
\[ G(x) \leq KH(x) \quad (x \geq 1) \]
so that
\[ G(tx) \leq 2NKF(t)G(x) \quad (x, t \geq 1). \]

**Corollary 5.3.** Under condition (b) of the theorem, if $L_F$ and $L_G$ are locally bounded there exists $p > 0$ such that $L_G$ is isomorphic to a quotient of $l_p(L_F)$.

**Theorem 5.4.** Suppose $0 < \alpha_F^2 \leq \beta_F^2 < 1$. Then the quotient of $L_F$ by a one-dimensional subspace is not isomorphic to $L_F$; and any two such quotients are isomorphic if and only if $F$ is equivalent to a submultiplicative function.

**Proof.** $L_F$ is a $\mathcal{K}$-space ([3], Theorem 4.3), i.e. it has the property that if $X$ is any $F$-space and $N \subset X$ is a subspace of dimension one then an operator $T: L_F \to X/N$ may be lifted to an operator $\overline{T}: L_F \to X$ such that $q\overline{T} = T$, where $q$ is the quotient map.
Suppose $V$ and $W$ are two subspaces of $L_F$ of dimension one or zero such that $L_F/V \cong L_F/W$. Let $q_v$ and $q_w$ be the quotient maps and let $S: L_F/V \to L_F/W$ be the isomorphism. Let $R: L_F \to L_F$ be the ‘lift’ of $S q_v$ and $T$ the ‘lift’ of $S^{-1} q_w$. Then $q_v R T = S q_v T = q_w$ so that $R T - I$ has rank at most one. As $L_F^* = \{0\}$, $R T = I$ and similarly $T R = I$. Clearly $R(V) = W$, and hence $\dim V = \dim W$. We deduce at once that $L_F$ is not isomorphic to its quotient by a subspace of dimension one, and that if any two such quotients are isomorphic, then $L_F$ is transitive.

Conversely if $L_F$ is transitive, the same arguments as in [5] show that any two such quotients are isomorphic.

Remark. This result may be extended to subspaces of finite dimension as in [5].

6. Orlicz spaces as quotients of Orlicz spaces. The class of $F$-spaces which are isomorphic to quotients of $L_p$ ($0 < p < 1$) is rather restricted (see [1] or [10], p. 94, 3.4.3). Our first result shows that no other Orlicz function space is such a quotient.

**Theorem 6.1.** Let $L_F$ be isomorphic to a quotient of $L_p$. Then $F(x)$ is equivalent to $x^p$, i.e. $L_F = L_p$.

**Proof.** $L_F$ is locally $p$-convex and $F$ satisfies (for some $k > 0$)

$$F(t^p) \geq kt^p F(x) \quad (t, x \geq 1).$$

By Theorem 4.2, since $(L_p, L_F)$ is transitive, there exists $K < \infty$ such that

$$F(t^p) \leq K t^p F(x).$$

Hence $k F(1)t^p \leq F(t) \leq K F(1)t^p$ ($1 \leq t < \infty$).

**Theorem 6.2.** Suppose $F$ is an Orlicz function satisfying

$$0 < \liminf_{t \to \infty} \frac{F(t)}{t^p} < \infty,$$

and

$$a_F^\infty > 0.$$

Then $L_p$ is isomorphic to a quotient of $L_F$.

**Proof.** We may suppose without loss of generality that $F(t) = t^p$ ($0 \leq t \leq 1$). Then there is a constant $a > 0$ such that

$$F(t) \geq at^p \quad (0 \leq t < \infty).$$

Let $r_n \to \infty$ be chosen so that for some $k < \infty$

$$F(r_n) \leq k r_n^p, \quad \sum_{n=1}^\infty r_n^{-p} \leq 1.$$
Let \((A_n)\) be a sequence of disjoint Borel subsets of \((0, 1)\) of measure \(\tau_n^{-p}\).

Let \(\sigma_n\colon (0, 1) \to A_n\) be a measurable map such that \(\lambda[\sigma_n^{-1}(B)] = \tau_n^p \lambda(B)\) for \(B \subset A_n\), \(B\) measurable. Define \(S_n\colon L_F(A_n) \to L_F\) by

\[
S_n f(x) = f(\sigma_n x).
\]

Let \((u_n)\) be a sequence in \(L_p\) dense in the unit ball of \(L_p\). Then

\[
\int_0^1 |t \tau_n^{-1} u_n(x)|^p dx \leqslant \left( \frac{1}{\tau_n} \right)^p \leqslant \frac{1}{a \tau_n^p} F(t) \quad (0 \leqslant t < \infty).
\]

By Lemma 4.1, there exists \(T_n\colon L_F \to L_p\) such that \(T_n 1 = \tau_n^{-1} u_n\), and

\[
\|T_n f\|^p \leqslant \frac{1}{a \tau_n^p} \int_0^1 F(|f(x)|) dx \quad (f \in L_F).
\]

Denoting by \(P_n\) the natural projection of \(L_F\) onto \(L_F(A_n)\) we consider \(T\colon L_F \to L_p\) defined by

\[
T f = \sum_{n=1}^{\infty} T_n S_n P_n f.
\]

For \(f \in L_F\)

\[
\|T f\|^p \leqslant \sum_{n=1}^{\infty} \frac{1}{a \tau_n^p} \int_0^1 F(|S_n P_n f(x)|) dx \leqslant a^{-1} \sum_{n=1}^{\infty} \int_{A_n} F(|P_n f(x)|) dx
\]

\[
\leqslant a^{-1} \int_0^1 F(|f(x)|) dx.
\]

Hence \(T\) is continuous. Clearly

\[
T(\tau_n \chi_{A_n}) = u_n
\]

and

\[
\int_0^1 F(|\tau_n \chi_{A_n}(x)|) dx = \frac{F(\tau_n)}{\tau_n^p} \leqslant k.
\]

Hence \(T(B_F(k))\) is dense in the unit ball of \(L_p\). As \(L_F\) is locally bounded, this implies that \(T\) is an open map.

Remarks. We do not know whether the conditions of Theorem 6.2 are, in any sense, necessary for this result.
References


[5] — and N. T. Peck, Quotients of $L_p(0, 1)$ for $0 < p < 1$, Studia Math., to appear.


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