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Transitivity and quotients of Orlicz spaces

To Professor Wladyslaw Orlicz
on the occasion of his 75-th birthday

1. Introduction. The initial objective of this paper was to investigate the quotient structure of general non-locally convex Orlicz function spaces. However, our results required the introduction of the notion of transitivity of an F-space, and during the investigation we came across certain ideas connected with transitive F-spaces which seemed worth developing. Thus the paper falls into two distinct parts. In the first part we study transitivity in general and in the second part we relate the notion to Orlicz spaces.

We introduce in Section 3 the notion of a transitive pair of F-spaces and present some general results on this concept. In Section 4 we introduce ultra-transitive F-spaces — an F-space X is ultra-transitive if every finite rank operator $T\colon E\to X$, where E is a closed subspace of X, may be extended to an endomorphism of X. Of course every Banach space is ultra-transitive, but the existence of non-locally convex ultra-transitive spaces is less obvious. For example the space l_p is not ultra-transitive. However, in Section 4 we construct an ultra-transitive p-Banach space ($p < 1$) which is separable and universal for all separable p-Banach spaces. We do not know any examples of non-locally convex ultra-transitive spaces which arise naturally.

In Section 5 we apply these ideas to Orlicz spaces and show, for example, that an Orlicz space L_F is transitive if and only if either $\liminf_{x\to\infty} x^{-1}F(x) > 0$ or F is equivalent to a submultiplicative function. If $0 < \alpha_F^\infty \leqslant \beta_F^\infty < 1$, then, unless F is equivalent to a submultiplicative function, L_F has two non-isomorphic quotients by one-dimensional subspaces (this extends some results in [5]).

In Section 6 we show that L_p has no quotient isomorphic to another Orlicz function space, but that L_p is isomorphic to a quotient of certain Orlicz function spaces.

Our methods in studying operators on Orlicz spaces are closely related, but not identical, to those used by Turpin ([9], [10]). Turpin uses the concept of the galb of the space, whereas we use (essentially) the idea of analyzing the possible behaviour at infinity of F-norms on the space.

2. Notation. An F-space is a complete metric linear topological space, which may be over the real or complex scalars; unless stated we will treat only the real case, and implicitly assume the complex case.

In a locally bounded F-space X the topology may be given by a quasi-norm $x \mapsto \|x\|$ which satisfies

(i) $\|x\| \geqslant 0$ $(x \in X)$,

(ii) $\|x\| = 0$ if and only if $x = 0$,

(iii) $\|tx\| = |t|\,\|x\|$ $(t \in \mathfrak{R},\ x \in X)$,

(iv) $\|x+y\| \leqslant k(\|x\|+\|y\|)$ $(x, y \in X)$,

where k is independent of x and y. If in addition we have for some p, $0 < p \leqslant 1$,

(v) $\|x+y\|^p \leqslant \|x\|^p + \|y\|^p$ $(x, y \in X)$,

then we say that X is a *p-Banach space* (when equipped with this quasi-norm). By the Aoki–Rolewicz theorem ([8], p. 61) every locally bounded F-space is *locally p-convex* for some $p > 0$ and may therefore be renormed equivalently to be a p-Banach space.

If X and Y are F-spaces we denote by $\mathscr{L}(X, Y)$ the space of operators (i.e. continuous linear maps) from X to Y. If X and Y are both quasi-normed, then $\mathscr{L}(X, Y)$ may be quasi-normed by

$$\|T\| = \sup(\|Tx\|\colon \|x\| \leqslant 1).$$

In the case $X = Y$ we denote $\mathscr{L}(X, Y)$ by $\mathscr{L}(X)$.

The space $\mathscr{L}(X, \mathfrak{R})$ (or $\mathscr{L}(X, \mathfrak{C})$ in the complex case) is denoted by X^*.

An Orlicz function F is a continuous, strictly increasing function $F\colon [0, \infty) \to [0, \infty)$ such that $F(0) = 0$. The Orlicz function space $L_F(0, 1) = L_F$ consists of all measurable functions f on $(0, 1)$ such that

$$\int_0^1 F(\varepsilon|f(x)|)\,dx < \infty$$

for some $\varepsilon > 0$. L_F is an F-space if we equip it with the topology induced by the base of neighbourhoods of 0 of the form $rB_F(\varepsilon)$ $(r > 0,\ \varepsilon > 0)$, where $B_F(\varepsilon)$ consists of all f such that

$$\int_0^1 F(|f(x)|)\,dx \leqslant \varepsilon.$$

Two Orlicz functions F and G are equivalent (at ∞) if there exist constants $k_1, k_2, a_1, a_2 > 0$ such that

$$k_1 F(a_1 x) \leqslant G(x) \leqslant k_2 F(a_2 x) \quad (1 \leqslant x < \infty).$$

Then $L_F \equiv L_G$.

We define

$$a_F^\infty = \sup\left\{p: \sup_{t,x\geqslant 1} \frac{t^p F(x)}{F(tx)} < \infty\right\}, \qquad \beta_F^\infty = \inf\left\{p: \sup_{t,x\geqslant 1} \frac{F(tx)}{t^p F(x)} < \infty\right\}.$$

Then L_F is locally bounded if and only if $a_F^\infty > 0$; it is locally p-convex if and only if

$$\sup_{t,x\geqslant 1} \frac{t^p F(x)}{F(tx)} < \infty$$

(see [6], [10], p. 77).

The condition $\beta_F^\infty < \infty$ is equivalent to the Δ_2-condition (at ∞)

$$\sup_{x\geqslant 1} \frac{F(2x)}{F(x)} < \infty,$$

and to the condition that L_F is separable. Under the Δ_2-condition, if $f \in L_F$, then

$$\int_0^1 F(|f(x)|)\,dx < \infty.$$

F is said to be *subadditive* if

$$F(x+y) \leqslant F(x)+F(y) \quad (0 \leqslant x,\ y < \infty)$$

and *submultiplicative* (at ∞) if

$$F(xy) \leqslant F(x)F(y) \quad (1 \leqslant x,\ y < \infty).$$

F is equivalent to a submultiplicative function if there exists a k such that

$$F(xy) \leqslant kF(x)F(y) \quad (1 \leqslant x,\ y < \infty).$$

(Define

$$G(t) = \sup_{x\geqslant 1} \frac{F(tx)}{F(x)}.)$$

For a full discussion of Orlicz spaces and functions see [10].

3. Transitive spaces. An F-space X is *transitive* if given $x, y \in X$, with $x \neq 0$, there exists $T \in \mathscr{L}(X)$ with $Tx = y$. We shall extend this by defining a pair (X, Y) to be transitive if given $0 \neq x \in X$ and $y \in Y$ there exists $T \in \mathscr{L}(X, Y)$ with $Tx = y$.

Some elementary observations are collected together in Propositions 3.1 and 3.2.

PROPOSITION 3.1. *Suppose (X, Y) is a transitive pair. Then*

(a) *if N is a closed subspace of X, (N, Y) is transitive;*

(b) *if M is a closed subspace of Y, $(X, Y/M)$ is transitive;*

(c) *if (Y, Z) is transitive, then (X, Z) is transitive.*

PROPOSITION 3.2. *Let X be a transitive space. Then*

(a) *if N is a complemented subspace of X, N is transitive;*

(b) *if $X^* \neq \{0\}$, X has a separating dual.*

If X is a locally bounded F-space with quasi-norm $x \mapsto \|x\|$, we define for $0 < p < \infty$, $l_p(X)$ to be the space of all sequences (x_n), $x_n \in X$ such that

$$\|(x_n)\|_p = \left(\sum \|x_n\|^p\right)^{1/p} < \infty.$$

The quasi-norm $(x_n) \mapsto \|(x_n)\|_p$ makes $l_p(X)$ a locally bounded F-space. For $p = \infty$, we define $l_\infty(X)$ to be the space of all sequences (x_n) such that

$$\|(x_n)\|_\infty = \sup_n \|x_n\| < \infty.$$

The space $l_\infty(X)$ has been studied in [7].

The following lemma is an immediate consequence of the Open Mapping Theorem.

LEMMA 3.3. *Let (X, Y) be a transitive pair of locally bounded F-spaces, and suppose $0 \neq u \in X$. Then there exists k such that for any $y \in Y$, there exists $T \in \mathscr{L}(X, Y)$ with $\|T\| \leqslant k\|y\|$ and $Tu = y$.*

THEOREM 3.4. *Suppose X is a transitive locally bounded F-space. Then $l_p(X)$ is also transitive for any p, $0 < p \leqslant \infty$.*

Proof. It is trivial to see that $\big(l_p(X), X\big)$ is transitive. Conversely if $u \in X$ and $(v_n) \in l_p(X)$ there exists $T_n \in \mathscr{L}(X, X)$ such that $T_n u = v_n$ and $\|T_n\| \leqslant k\|v_n\|$, where k is a constant independent of n. Define $S\colon X \to l_p(X)$ by $Sx = (T_n x)$. Then $(Su)_n = (v_n)$. Hence $\big(X, l_p(X)\big)$ is transitive and the proof is completed by Proposition 3.1 (c).

Remark. If X is transitive and $X^* = \{0\}$, then $\big(l_\infty(X)\big)^* = \{0\}$ by 3.4. Compare with [7], where it is shown that for some locally bounded spaces X, with $X^* = \{0\}$ we have $\big(l_\infty(X)\big)^* \neq \{0\}$.

THEOREM 3.5. *Let (X, Y) be a transitive pair of separable locally bounded F-spaces. If Y is locally p-convex, then Y is isomorphic to a quotient of $l_p(X)$.*

Conversely if X is transitive, and Y is isomorphic to a quotient of $l_p(X)$ (where $0 < p \leqslant \infty$), then (X, Y) is transitive.

Proof. Suppose (X, Y) is transitive, and that (y_n) is a sequence dense in the unit ball of Y (where we assume the quasi-norm on Y satisfies $\|x+y\|^p \leqslant \|x\|^p + \|y\|^p$). Fix $u \in X$, with $\|u\| = 1$. By Lemma 3.1, there exist k such that we may find $T_n \in \mathscr{L}(X, Y)$ with $T_n u = y_n$ and $\|T_n\| \leqslant k$. Define $S\colon l_p(X) \to Y$ by

$$S\big((x_n)\big) = \sum_{n=1}^{\infty} T_n x_n.$$

S is bounded and $0 \in \operatorname{int} \overline{S(U)}$, where U is the unit ball of $l_p(X)$. Hence S is an open mapping.

The converse is immediate from Proposition 3.1(b) since $\big(X, l_p(X)\big)$ is transitive.

COROLLARY 3.6. *A separable locally bounded F-space X is isomorphic to a quotient of L_p $(0 < p < 1)$, if and only if X is locally p-convex and (L_p, X) is transitive.*

COROLLARY 3.7. *If $0 < q \leqslant p < 1$, then L_q is isomorphic to a quotient of $l_q(L_p)$.*

Corollaries 3.6 and 3.7 follow from the remarks that $l_p(L_p) \cong L_p$ and $\mathscr{L}(L_p, L_q) \neq \{0\}$ if $p \geqslant q$.

We conclude this section by remarking that there exists a non-atomic measure space (Ω, μ) such that $L_p(\Omega, \mu)$ is not transitive. Indeed if (Ω_0, μ_0) denotes an uncountable product of copies of $(0, 1)$ with Lebesgue measure, there is no continuous operator from $L_p(\Omega_0, \mu_0)$ $(0 < p < 1)$ into a separable F-space. This may be proved by the methods of [1]; indeed any operator is an isomorphism on some subspace isomorphic to a non-separable Hilbert space. Hence $L_p(\Omega_0, \mu_0) \oplus L_p$ is not transitive and this is isomorphic to $L_p(\Omega, \mu)$ for some (Ω, μ).

4. Ultra-transitive spaces. In [5] an F-space X was said to be *strongly transitive* if whenever $E \subset X$ is a subspace of finite dimension then every $T_0 \in \mathscr{L}(E, X)$ has an extension $T \in \mathscr{L}(X)$. Clearly if we replace finite dimension by dimension one in this definition we would obtain a characterization of transitive spaces. For complex locally bounded spaces the two concepts coincide ([5]), but it is an open question whether they coincide in general.

Motivated by this definition we define an F-space X to be *ultra-transitive* if whenever $N \subset X$ is a closed subspace and $T_0 \in \mathscr{L}(N, X)$ is an operator of finite rank, then T_0 has an extension $T \in \mathscr{L}(X)$.

PROPOSITION 4.1. *Let X be an F-space.*

(a) *If X is ultra-transitive, then every quotient of X is ultra-transitive.*

(b) *If X is transitive and $(X/M, X)$ is transitive for every closed subspace M of X, then X is ultra-transitive.*

Proof. (a) Let M be a closed subspace of X and N a closed subspace of X/M. Let T_0: $N \to X/M$ be an operator of finite rank. Then if q: $X \to X/M$ is the quotient map, there exists an operator T_1: $N \to X$ of finite rank such that $qT_1 = T_0$. Then consider $T_1 q$: $q^{-1}(N) \to X$; $T_1 q$ has an extension S: $X \to X$, and as $S(M) = 0$, S factors through q, i.e. $S = S_1 q$, where $S_1 \in \mathscr{L}(X/M, X)$. Then $qS_1 = T$ extends T_0 and belongs to $\mathscr{L}(X/M)$.

(b) It suffices to show that if N is a closed subspace of X and $T_0 \in \mathscr{L}(N, X)$ has rank one, then T_0 may be extended to an endomorphism of X. Let $M = T_0^{-1}(0)$; then if q: $X \to X/M$ in the quotient map, $T_0 = S_0 q$, where S_0: $q(N) \to X$. As $\dim q(N) = 1$ and $(X/M, X)$ is transitive, S_0 may be extended to $S \in \mathscr{L}(X/M, X)$. Then $T = Sq$ extends T_0.

We remark now that l_p $(0 < p < 1)$ is not ultra-transitive since L_p is isomorphic to a quotient of l_p, and $\mathscr{L}(L_p, l_p) = \{0\}$. The space L_p $(0 < p < 1)$ is also not ultra-transitive, but the proof of this will appear elsewhere [4]. Also the space L_0 is not ultra-transitive [4]. In view of this it may appear that the only ultra-transitive spaces are locally convex, but this is not the case. We now construct some non-locally convex ultra-transitive spaces.

Lemma 4.2. *Let X be a separable p-Banach space, and let (E_n) be a sequence of closed subspaces of X. Let T_n: $E_n \to X$ be a sequence of operators with $\|T_n\| \leqslant 1$. Then there is a separable p-Banach space Y containing X such that each T_n extends to an operator $\tilde{T}_n$: $X \to Y$, with $\|\tilde{T}_n\| \leqslant 1$.*

Proof. Consider $l_p(X)$ (where for convenience we index sequences $(x_n) \in l_p(X)$ starting at $n = 0$). Let M be the subspace of $l_p(X)$ of all (x_n) such that $x_n \in E_n$ $(n \geqslant 1)$ and

$$x_0 + \sum_{n=1}^{\infty} T_n x_n = 0.$$

Let $Y = l_p(X)/M$, and q: $l_p(X) \to Y$ be the quotient map. Y is clearly separable. Let $(J_n: n \geqslant 0)$ be the map J_n: $X \to l_p(X)$ defined by $J_n x = (0, \ldots, 0, x, 0, \ldots)$ (x in the n-th place).

First observe that qJ_0 is an isometric embedding of X into Y. For if $(u_n) \in M$ and $x \in X$, then

$$\|x + u_0\|^p + \sum_{n=1}^{\infty} \|u_n\|^p \geqslant \|x + u_0\|^p + \sum_{n=1}^{\infty} \|T_n u_n\|^p \geqslant \|x\|^p.$$

Thus we may identify X with the subspace $qJ_0(X)$. If $x \in E_n$, then $(T_n x, 0, \ldots, -x, 0, \ldots) \in M$ and hence $qJ_n x = qJ_0 T_n x$, so that the map $qJ_n(qJ_0)^{-1}$: $qJ_0(X) \to Y$ extends the map $qJ_0 T_n (qJ_0)^{-1}$: $qJ_0(E_n) \to Y$.

Theorem 4.3. *There is a separable p-Banach space, universal for all separable p-Banach spaces, which is transitive.*

Proof. In [2] Theorem 4.1 it is shown that there is a universal separable p-Banach space, X_0 say. Let D_0 be a dense countable subset of the unit sphere of X_0. We shall construct inductively a sequence (X_n) of p-Banach spaces and a dense countable subset D_n of the unit sphere of X_n such that

(1) $$X_{n+1} \supset X_n \quad (n \geqslant 0),$$

(2) $$D_{n+1} \supset D_n \quad (n \geqslant 0).$$

(3) If $u, v \in D_n$, there is an operator $T^{(n)}_{u,v}: X_n \to X_{n+1}$ with $T^{(n)}_{u,v}(u) = v$, $n \geqslant 0$ and $\|T^{(n)}_{u,v}\| = 1$.

(4) If $u, v \in D_n$, then $T^{(n+1)}_{u,v}(x) = T^{(n)}_{u,v}(x)$, $x \in X_n$, $n \geqslant 0$.

Suppose X_n, D_n and (for $n > 1$) $T^{n-1}_{u,v}$, $u, v \in D_{n-1}$ are given. Then by Lemma 4.2 we may construct X_{n+1} containing X_n such that each operator

$$T^{(n-1)}_{u,v}: X_{n-1} \to X_n, \quad (u, v) \in D_{n-1} \times D_{n-1},$$
$$R_{u,v}: \operatorname{lin}(u) \to X_n, \quad (u, v) \in (D_n \times D_n) \setminus (D_{n-1} \times D_{n-1})$$

(where $R_{u,v}(u) = v$) has a norm one extension, denoted by $T^{(n)}_{u,v}$ for all $(u, v) \in D_n \times D_n$

$$T^{(n)}_{u,v}: X_n \to X_{n+1}.$$

Then choose D_{n+1} a dense countable subset of X_{n+1}, containing D_n. Repeating this construction inductively we obtained the required sequences.

Now let Z be the completion of $\bigcup_{n=0}^{\infty} X_n$. Clearly $D = \bigcup_{n=0}^{\infty} D_n$ is dense in the unit sphere of Z, and if $(u, v) \in D \times D$, there is a unique operator $T_{u,v}: Z \to Z$ such that $T_{u,v}(x) = T^{(n)}_{u,v}(x)$ for $x \in X_n$, $(u, v) \in D_n \times D_n$; $T_{u,v}(u) = v$, and $\|T_{u,v}\| = 1$.

For fixed $z \in Z$, with $\|z\| = 1$, the map $\Phi: \mathscr{L}(Z) \to Z$, defined by $\Phi(T) = Tz$ has the property that the image of the unit ball of $\mathscr{L}(Z)$ is dense in the unit ball of Z. Hence Φ is an open map and so Z is transitive. As $Z \supset X_0$, Z is also universal for all separable p-Banach spaces.

THEOREM 4.4. *A transitive universal separable p-Banach space is ultra-transitive. Hence for each $0 < p < 1$, there is a separable ultra-transitive p-Banach space which is not locally q-convex for any $q > p$.*

Proof. Let W_p be such a space, and let X be a quotient of W_p. Then (X, W_p) is transitive (since X embeds in W_p). Apply Proposition 4.1.

In fact, we observe that by Theorem 4.4 and Theorem 3.4,

COROLLARY 4.5. *For $0 < p < 1$, there is a separable p-Banach space W for which $l_p(W)$ is ultra-transitive.*

THEOREM 4.6. *Let W be a separable p-Banach space such that $l_p(W)$ is ultra-transitive. Let X be any separable p-Banach space; the following conditions are equivalent:*

(a) *$l_p(X)$ is ultra-transitive;*

(b) *X is isomorphic to a quotient of $l_p(W)$;*

(c) *(Y, X) is transitive for any separable p-Banach space Y;*

(d) *X is isomorphic to a quotient of $l_p(Y)$ for any separable p-Banach space Y.*

Proof. (c) $\Rightarrow$ (d) Theorem 3.5.

(d) $\Rightarrow$ (b) Trivial.

(b) $\Rightarrow$ (a) Since $l_p(X)$ is also isomorphic to a quotient of $l_p(W)$, this is immediate from Proposition 4.1.

(a) $\Rightarrow$ (c) Since $l_p(X)$ contains a copy of l_p, it has a quotient Z which is universal for all separable p-Banach spaces. Thus Y embeds in Z, and as Z is transitive so is the pair (Y, Z). However, $\big(Z, l_p(X)\big)$ is also transitive, hence so is $\big(Y, l_p(X)\big)$ and hence (Y, X).

Remark. L_p has an ultra-transitive quotient (see part (d)).

5. Transitivity of Orlicz spaces. In this section we turn to the question of transitivity of pairs of Orlicz function spaces. First we observe that for any Orlicz function F and $f \in L_F$, there exists $T \in \mathscr{L}(L_F)$ with $Tf = 1$ (by 1 we mean the characteristic function of the unit interval). Hence to show that a pair (L_F, L_G) is transitive it suffices to show that for $g \in L_G$ there exists $T \in \mathscr{L}(L_F, L_G)$ with $T1 = g$.

Our first lemma is a positive result.

LEMMA 5.1. *Let F and G be Orlicz functions and suppose $u \in L_G$. Suppose there exists k such that*

$$(*) \qquad \int_0^1 G\big(t|u(x)|\big)dx \leqslant kF(t) \quad (1 \leqslant t < \infty).$$

Then there exists $T \in \mathscr{L}(L_F, L_G)$ with $T1 = u$. If, in addition, u is countably simple and $(*)$ holds for $0 \leqslant t < \infty$, then T may be chosen so that

$$\int_0^1 G\big(|Tf(x)|\big)dx \leqslant k \int_0^1 F\big(|Tf(x)|\big)dx \qquad (f \in L_F).$$

Proof. In fact, we prove the latter assertion first. Suppose

$$u = \sum_{n=1}^{\infty} c_n \chi_n,$$

where χ_n is the characteristic function of the set A_n, where $\lambda(A_n) > 0$. Since the measure spaces A_n and $A_n \times (0, 1)$ are isomorphic, we may define an operator $S\colon L_G((0,1)\times(0,1))\to L_G(0,1)$ such that $S(\chi_{\Delta_n}\times \times(0,1)) = \chi_n$ and

$$\int_0^1 G(|Sf(x)|)\,dx = \int_0^1\int_0^1 G(|f(x,y)|)\,dx\,dy$$

for $f\in L_G((0,1)\times(0,1))$.

Now define $T_0\colon L_F\to L_G((0,1)\times(0,1))$ by

$$T_0 f(x,y) = u(x)f(y).$$

Then ST_0 satisfies the conditions.

For the first part let

$$v = \sum_{n=-\infty}^{\infty} 2^n\chi\{x\colon 2^n\leqslant |u(x)|\leqslant 2^{n+1}\}.$$

Then $v\in L_G$ is countably simple and there exists $A\in\mathscr{L}(L_G)$ such that $Av = u$. Clearly

$$\int_0^1 G(t|v(x)|)\,dx\leqslant kF(t)\qquad (1\leqslant t<\infty).$$

If we define $F^*(t) = \max\{F(t),\, k^{-1}\int_0^1 G(t|v(x)|)\,dx\}$ $(0\leqslant t<\infty)$, then F^* is equivalent to F and

$$\int_0^1 G(t|v(x)|)\,dx\leqslant kF^*(t)\qquad (0\leqslant t<\infty).$$

By the second part of the lemma, there exists $R\in\mathscr{L}(L_F, L_G)$ with $R1 = v$; then $AR1 = u$.

THEOREM 5.2. *Let F and G be two Orlicz functions. In order that the pair (L_F, L_G) be transitive it is necessary and sufficient that either*

(a) $\liminf\limits_{x\to\infty} x^{-1}F(x) > 0$;

or

(b) *there is a constant k such that*

$$G(tx)\leqslant kG(t)F(x),\qquad 1\leqslant t,\ x<\infty.$$

In particular L_F is transitive if and only if either (a) *holds or F is equivalent to a submultiplicative function.*

Proof. If (a) holds, L_F has a separating dual and so (L_F, X) is transitive for any X. If (b) holds, then G satisfies the Δ_2-condition. If $u\in L_G$, let $v(x) = \max(1, |u(x)|)$. Then $v\in L_G$ and hence

$$\int_0^1 G(|v(x)|)\,dx<\infty.$$

However,

$$\int_0^1 G\big(t|v(x)|\big)dx \leqslant kF(t)\int_0^1 G\big(|v(x)|\big)dx, \qquad t \geqslant 1$$

and hence by 5.1, there exists $A \in \mathscr{L}(L_F, L_G)$ with $A1 = v$. Hence there exists T such that $T1 = u$, and (L_F, L_G) is transitive.

Now suppose (L_F, L_G) is transitive. We may suppose that for $0 \leqslant x \leqslant 1$ we have $F(x) = G(x) = x$; suppose also that condition (a) does not hold.

Let

$$h(0) = 1,$$

$$h(x) = \min_{0<t\leqslant x} \frac{G(t)}{t} \qquad (x > 0),$$

and define

$$H(x) = xh(x) \qquad (x \geqslant 0).$$

Then it is easy to check that H is an Orlicz function and $H(x) \leqslant G(x)$ $(x \geqslant 0)$. Also as $H(x)/x$ decreases monotonically, H is subadditive.

As the set $B_F(1)$ is additively bounded ([10], p. 35), we may define for $T \in \mathscr{L}(L_F, L_G)$

$$\gamma(T) = \sup_{f\in B_F(1)} \int_0^1 H\big(|Tf(x)|\big)dx$$

and $\gamma(T) < \infty$. Clearly

$$\gamma(S+T) \leqslant \gamma(S)+\gamma(T), \qquad S, T \in \mathscr{L}(L_E, L_G),$$

and

$$\gamma(aS) \leqslant |a|\gamma(S), \qquad |a| \leqslant 1,\ S \in \mathscr{L}(L_F, L_G).$$

For $n \geqslant 1$, let

$$V_n = \{g \in L_G\colon \exists T \in \mathscr{L}(L_F, L_G),\ \gamma(T) \leqslant n,\ T1 = g\}.$$

Then $V_{m+n} \supset V_m - V_n$, $m, n \in \boldsymbol{N}$. As $\bigcup V_n = L_G$, we may apply the Baire category theorem to deduce $0 \in \operatorname{int} \overline{V}_m$, some m. As $B_G(1)$ is additively bounded, there exists N such that $B_G(1) \subset \overline{V}_N$.

Fix $x \geqslant 1$, and let A be a set of measure $G(x)^{-1}$. Let $u = x\chi_A$; then $u \in B_G(1)$. Hence there is a sequence $T_n \in \mathscr{L}(L_F, L_G)$ with $T_n 1 \to u$ and $\gamma(T_n) \leqslant N$.

For $t \geqslant 1$, let $m = [F(t)]+1 \leqslant 2F(t)$. Partition $(0, 1)$ into m disjoint subsets of measure m^{-1}, with characteristic functions $\chi_1, \ldots, \chi_m$. Then $t\chi_i \in B_F(1)$ and hence

$$\int_0^1 H\big(t|T_n\chi_i(\tau)|\big)d\tau \leqslant N \qquad (i = 1, 2, \ldots, m).$$

Since H is subadditive,

$$\int_0^1 H\big(t|T_n 1(\tau)|\big)\,d\tau \leqslant Nm \leqslant 2NF(t).$$

As $T_n 1 \to u$ is measure, we have, by Fatou's lemma

$$\int_0^1 H\big(t|u(\tau)|\big)\,d\tau \leqslant 2NF(t),$$

i.e.

$$H(tx) \leqslant 2NF(t)G(x) \qquad (1 \leqslant t, x).$$

Now pick θ so that

$$F(\theta) < \theta/2N.$$

For $x \geqslant 1$, we define

$$y = \max\{t\colon H(t) = G(t),\ 0 \leqslant t \leqslant x\}.$$

Then $y \geqslant 1$. If $y < z \leqslant x$, then $h(z) = h(y)$ and

$$H(z) = zh(y) = zy^{-1}G(y).$$

Thus

$$\frac{z}{y}G(y) \leqslant 2NF\left(\frac{z}{y}\right)G(y)$$

and hence $z/y \neq \theta$. Thus $x/y < \theta$, and

$$G(x) \geqslant H(x) = \frac{x}{y}G(y) \geqslant G(\theta^{-1}x) \qquad (x \geqslant 1).$$

H and G are therefore equivalent, and so G satisfies the Δ_2-condition. Hence

$$G(x) \leqslant KH(x) \qquad (x \geqslant 1)$$

so that

$$G(tx) \leqslant 2NKF(t)G(x) \qquad (x, t \geqslant 1).$$

COROLLARY 5.3. *Under condition* (b) *of the theorem, if L_F and L_G are locally bounded there exists $p > 0$ such that L_G is isomorphic to a quotient of $l_p(L_F)$.*

THEOREM 5.4. *Suppose $0 < \alpha_F^\infty \leqslant \beta_F^\infty < 1$. Then the quotient of L_F by a one-dimensional subspace is not isomorphic to L_F; and any two such quotients are isomorphic if and only if F is equivalent to a submultiplicative function.*

Proof. L_F is a $\mathscr{K}$-space ([3], Theorem 4.3), i.e. it has the property that if X is any F-space and $N \subset X$ is a subspace of dimension one then an operator $T\colon L_F \to X/N$ may be lifted to an operator $\overline{T}\colon L_F \to X$ such that $q\overline{T} = T$, where q is the quotient map.

Suppose V and W are two subspaces of L_F of dimension one or zero such that $L_F/V \cong L_F/W$. Let q_v and q_w be the quotient maps and let $S\colon L_F/V \to L_F/W$ be the isomorphism. Let $R\colon L_F \to L_F$ be the 'lift' of Sq_v and T the 'lift' of $S^{-1}q_w$. Then $q_wRT = Sq_vT = q_w$ so that $RT-I$ has rank at most one. As $L_F^* = \{0\}$, $RT = I$ and similarly $TR = I$. Clearly $R(V) = W$, and hence $\dim V = \dim W$. We deduce at once that L_F is not isomorphic to its quotient by a subspace of dimension one, and that if any two such quotients are isomorphic, then L_F is transitive.

Conversely if L_F is transitive, the same arguments as in [5] show that any two such quotients are isomorphic.

Remark. This result may be extended to subspaces of finite dimension as in [5].

6. Orlicz spaces as quotients of Orlicz spaces. The class of F-spaces which are isomorphic to quotients of L_p $(0 < p < 1)$ is rather restricted (see [1] or [10], p. 94, 3.4.3). Our first result shows that no other Orlicz function space is such a quotient.

THEOREM 6.1. *Let L_F be isomorphic to a quotient of L_p. Then $F(x)$ is equivalent to x^p, i.e. $L_F = L_p$.*

Proof. L_F is locally p-convex and F satisfies (for some $k > 0$)

$$F(tx) \geqslant kt^pF(x) \qquad (t, x \geqslant 1).$$

By Theorem 4.2, since (L_p, L_F) is transitive, there exists $K < \infty$ such that

$$F(tx) \leqslant Kt^pF(x).$$

Hence $kF(1)t^p \leqslant F(t) \leqslant KF(1)t^p$ $(1 \leqslant t < \infty)$.

THEOREM 6.2. *Suppose F is an Orlicz function satisfying*

$$0 < \liminf_{t\to\infty} \frac{F(t)}{t^p} < \infty,$$

and

$$a_F^\infty > 0.$$

Then L_p is isomorphic to a quotient of L_F.

Proof. We may suppose without loss of generality that $F(t) = t^p$ $(0 \leqslant t \leqslant 1)$. Then there is a constant $a > 0$ such that

$$F(t) \geqslant at^p \qquad (0 \leqslant t < \infty).$$

Let $\tau_n \to \infty$ be chosen so that for some $k < \infty$

$$F(\tau_n) \leqslant k\tau_n^p, \qquad \sum_{n=1}^{\infty} \tau_n^{-p} \leqslant 1.$$

Let (A_n) be a sequence of disjoint Borel subsets of $(0,1)$ of measure τ_n^{-p}.

Let $\sigma_n\colon (0,1)\to A_n$ be a measurable map such that $\lambda[\sigma_n^{-1}(B)] = \tau_n^p\lambda(B)$ for $B\subset A_n$, B measurable. Define $S_n\colon L_F(A_n)\to L_F$ by

$$S_nf(x) = f(\sigma_n x).$$

Let (u_n) be a sequence in L_p dense in the unit ball of L_p. Then

$$\int_0^1 |t\tau_n^{-1}u_n(x)|^p\,dx \leqslant \left(\frac{t}{\tau_n}\right)^p \leqslant \frac{1}{a\tau_n^p}F(t) \qquad (0\leqslant t<\infty).$$

By Lemma 4.1, there exists $T_n\colon L_F\to L_p$ such that $T_n1 = \tau_n^{-1}u_n$, and

$$\|T_nf\|^p \leqslant \frac{1}{a\tau_n^p}\int_0^1 F\big(|f(x)|\big)\,dx \qquad (f\in L_F).$$

Denoting by P_n the natural projection of L_F onto $L_F(A_n)$ we consider $T\colon L_F\to L_p$ defined by

$$Tf = \sum_{n=1}^{\infty} T_nS_nP_nf.$$

For $f\in L_F$

$$\|Tf\|^p \leqslant \sum_{n=1}^{\infty}\frac{1}{a\tau_n^p}\int_0^1 F\big(|S_nP_nf(x)|\big)\,dx \leqslant a^{-1}\sum_{n=1}^{\infty}\int_{A_n} F\big(|P_nf(x)|\big)\,dx$$
$$\leqslant a^{-1}\int_0^1 F\big(|f(x)|\big)\,dx.$$

Hence T is continuous. Clearly

$$T(\tau_n\chi_{A_n}) = u_n$$

and

$$\int_0^1 F\big(|\tau_n\chi_{A_n}(x)|\big)\,dx = \frac{F(\tau_n)}{\tau_n^p}\leqslant k.$$

Hence $T\big(B_F(k)\big)$ is dense in the unit ball of L_p. As L_F is locally bounded, this implies that T is an open map.

Remarks. We do not know whether the conditions of Theorem 6.2 are, in any sense, necessary for this result.

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