A REMARK ON A PROBLEM OF KLEE

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This paper treats a property of topological vector spaces first studied by Klee [6]. \( X \) is said to have the Klee property if there are two (not necessarily Hausdorff) vector topologies on \( X \), say \( \tau_1 \) and \( \tau_2 \), such that the quasi-norm topology is the supremum of \( \tau_1 \) and \( \tau_2 \) and such that \( (X, \tau_1) \) has trivial dual while the Hausdorff quotient of \( (X, \tau_2) \) is nearly convex, i.e. has a separating dual. Klee raised the question of whether every topological vector space has the Klee property.

In this paper we will only consider the case when \( X \) is a separable quasi-Banach space. In this context the problem has recently been considered in [2] and [7]. In [7], the problem was considered for the special case when \( X \) is a twisted sum of a one-dimensional space and a Banach space, so that there is subspace \( L \) of \( X \) with \( \dim L = 1 \) and \( X/L \) locally convex; it was shown that \( X \) then has the Klee property if the quotient map is not strictly singular. Then in [2] a twisted sum \( X \) of a one-dimensional space and \( \ell_1 \) was constructed so that the quotient map is strictly singular and \( X \) fails to have the Klee property. Thus Klee’s question has a negative answer. The aim of this paper is to completely characterize the class of separable quasi-Banach spaces with the Klee property. Using this characterization we give a much more elementary counter-example to Klee’s question.

Given a quasi-Banach space \( X \), the dual of \( X \) is denoted by \( X^* \). We define the kernel of \( X \) to be the linear subspace \( \{ x : x^*(x) = 0 \ \forall x^* \in X^* \} \).

Now we state our theorem:

**Theorem.** Let \( X \) be a separable quasi-Banach space, with kernel \( E \). Then \( X \) fails to have the Klee property if and only if \( E \) has infinite codimension and the quotient map \( \pi : X \to X/E \) is strictly singular.

**Proof.** For the “if” part, suppose that \( E \) has infinite codimension, \( \pi \) is strictly singular and that \( X \) has the Klee property. In this case the closure 1991 Mathematics Subject Classification: Primary 46A16.

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of \{0\} for the topology \(\tau_2\) must include the kernel \(E\), so that on \(E\) the topology \(\tau_1\) must coincide with the quasi-norm topology. By a standard construction, there is a vector topology \(\tau_3 \leq \tau_1\) which is pseudo-metrizable and coincides with the quasi-norm topology on \(E\). Let \(F\) be the closure of \(\{0\}\) for this topology.

Note that \(F \cap E = \{0\}\) and that \(E + F\) is \(\tau_3\)-closed and hence also closed for the original topology. This implies that \(\pi\) restricted to \(F\) is an isomorphism, and so if \(F\) is of infinite dimension, we have a contradiction. If \(F\) is of finite dimension, then since \(X^*\) separates the points of \(F\) we can write \(X = X_0 \oplus F\), and \(\tau_3\) is Hausdorff on \(X_0\). Suppose \(\tau_3\) coincides with the original topology on \(X_0\). Then \(X_0\) has trivial dual and so \(X_0 \subset E\). This implies that \(E\) is of finite codimension, and so contradicts our assumption. Thus \(\tau_3\) is strictly weaker than the original topology on \(X_0\).

By a theorem of Aoki–Rolewicz [4], we can assume that the quasi-norm is \(p\)-subadditive for some \(0 < p < 1\). Now let \(|\cdot|\) denote an \(E\)-norm defining \(\tau_3\) on \(X\). There exists \(\delta > 0\) so that \(|x| \leq \delta\) and \(x \in E\) imply \(|x|^p \leq 1/2\). We can also choose \(0 < \eta < 2^{-1/p}\) so that \(x \in X\) and \(|x| \leq \eta\) together imply that \(|x| < \delta/2\). Also, there exists a sequence \((x_n)\) in \(X_0\) so that \(|x_n| \leq 2^{-n}\) and \(|x_n| = 1\). Now by [4] (p. 69, Theorem 4.7) we can pass to a subsequence, also labelled \((x_n)\), which is strongly regular and \(M\)-basic in \(X\), i.e. so that for some \(M\) we have \(\max|a_k| \leq M\|\sum_{k=1}^{\infty} a_k x_k\|\) for all finitely nonzero sequences \((a_k)_{k=1}^{\infty}\). Now pick \(n_0\) so large that \((M + 1)2^{-n_0} < \delta/2\).

Let \(F_0\) be the linear span of \((x_n)_{n>n_0}\); we show that \(\pi\) is an isomorphism on \(F_0\). Indeed, suppose \(e \in E\) and \((a_n)_{n>n_0}\) is finitely nonzero with \(\|e + \sum_{k>n_0} a_k x_k\| < \eta\) but \(\|\sum_{k>n_0} a_k x_k\| = 1\). Then \(\|e + \sum_{k>n_0} a_k x_k\| \leq \delta/2\). Further, \(1 + \max_{n>n_0} |a_k| \leq M + 1\). Hence \(\|e\| \leq \delta\) and \(\|e\|^p \leq 1/2\); this implies \(\|e + \sum_{k>n_0} a_k x_k\| \geq 1/2\), which gives a contradiction. Hence the map \(\pi\) is an isomorphism on \(F_0\), and this contradicts our hypothesis.

Now we turn to the converse. By the theorem of Aoki–Rolewicz we can assume that the quasi-norm is \(p\)-subadditive for some \(0 < p < 1\). Suppose \(\pi\) is an isomorphism on some infinite-dimensional closed subspace \(F\). Then \(F\) has separating dual and hence contains a subspace with a basis. We therefore assume that \(F\) has a normalized basis \((f_n)_{n=1}^{\infty}\), and that \(K\) is a constant so large that \(e \in E\) and \(f \in F\) imply that \(\max\{\|e\|, \|f\|\} \leq K\|e + f\|\) and so that if \((a_n)\) is finitely nonzero then \(\max |a_n| \leq K\|\sum_{n=1}^{\infty} a_n f_n\|\).

Now let \((x_n)_{n=1}^{\infty}\) be a sequence whose linear span is dense in \(X\), chosen in such a way that \(M_n^{n+1}\|x_n\|^p = 2^{-(n+4)}\), where each \(M_n\) is a positive integer. We also require that for each positive integer \(m\) and each \(x_n\), \(m x_n = ax j\) for some \(0 < a < 1\) and some positive integer \(j\). Let \(N_n = M_n - M_{n-1}, M_0 = 0\). Let \(a_n = 2^{(n+4)/p} N_n K^2\). Define \(V\) as the absolutely \(p\)-convex hull of the set \(\{a_n f_k + x_n : M_{n-1} + 1 \leq k \leq M_n, 1 \leq n < \infty\}\). We let \(L\) be the closed linear span of the vectors \(\sum_{k=M_{n-1}+1}^{M_n} f_k\).
We now consider the set $L + V + U_X$, where $U_X$ is the open unit ball of $X$. This is an open absolutely $p$-convex set and generates a $p$-convex semi-quasi-norm $|\cdot|$ on $X$. We will show that $|\cdot|$ generates the original topology on $E$; more precisely, we will show that if $e \in E$ and $e \in L + V + U_X$, then $\|e\| \leq 2^{1/p}K$.

Indeed, assume $e \in E \cap (L + V + U_X)$. Then there exists $y \in U_X$ so that $e - y \in L + V$. It follows that there exist finitely nonzero sequences $(b_k)_{k=1}^\infty$ and $(c_n)_{n=1}^\infty$ so that $\sum_{k=1}^\infty |b_k|^p \leq 1$ and $e - y = z_1 + z_2$, where

$$z_1 = \sum_{n=1}^\infty \sum_{k=N_{n-1}+1}^{N_n} (b_k - c_n)a_n f_k, \quad z_2 = \sum_{n=1}^\infty \left( \sum_{k=N_{n-1}+1}^{N_n} b_k \right) x_n.$$  

Let $\beta_n = \sum_{k=N_{n-1}+1}^{N_n} b_k$. Then

$$\|y + z_2\|^p \leq 1 + \sum_{n=1}^\infty |\beta_n|^p \|x_n\|^p = A^p,$$

say. It follows that

$$\|z_1\| \leq K\|e - z_1\| \leq KA.$$  

Thus

$$\max_n \max_{M_{n-1}+1 \leq k \leq M_n} |b_k - c_n| a_n \leq K^2 A,$$

so

$$c_n^p \leq K^{2p} A^p a_n^{-p} + |b_k|^p \quad (M_{n-1} + 1 \leq k \leq M_n).$$

Adding, and using $\sum |b_k|^p \leq 1$, we obtain

$$N_n c_n^p \leq \sum_{k=N_{n-1}+1}^{N_n} |b_k|^p + a_n^{-p} K^{2p} A^p \leq 1 + N_n a_n^{-p} K^{2p} A^p.$$  

Hence,

$$c_n^p \leq K^{2p} A^p a_n^{-p} + N_n^{-1} \leq 2N_n^{-1} \max(N_n K^{2p} A^p a_n^{-p}, 1).$$

Taking $p$th roots,

$$c_n \leq 2^{1/p} N_n^{-1/p} \max(1, N_n^{1/p} a_n^{-1} K^2 A).$$

It now follows that

$$|\beta_n| \leq 2^{1/p} N_n^{-1/p} \max(1, N_n^{1/p} a_n^{-1} K^2 A) + N_n K^2 a_n^{-1} A.$$  

This implies that

$$|\beta_n|^p \leq 3 N_n^{p} K^{2p} a_n^{-p} A^p + 3 N_n^{p-1}.$$  

We finally arrive at the inequality

$$A^p \leq 1 + A^p \sum_{n=1}^\infty (3 N_n^{p} K^{2p} a_n^{-p} + 3 N_n^{p-1}) \|x_n\|^p \leq 1 + \frac{1}{2} A^p,$$
which implies $A \leq 2^{1/p}$ and hence $\|e\| \leq K\|e - z_1\| \leq KA \leq 2^{1/p}K$, as desired.

This shows that $L + V + U_X$ intersects $E$ in a bounded set and $\|\cdot\|$ induces the original topology on $E$. However, for each $n$,

$$x_n = \frac{1}{N_n} \sum_{k=M_{n-1}+1}^{M_n} (a_n f_k + x_n) - \frac{a_n}{N_n} \sum_{k=M_{n-1}+1}^{M_n} f_k$$

is in the convex hull of $L + V$. Hence, by assumption on $(x_n)$, $mx_n$ is in the convex hull of $L + V$ as well for all $m$ in $N$, and hence $(X, \|\cdot\|)$ has trivial dual.

**Remark.** The main theorem of [7] is an immediate consequence of our theorem. Indeed, assume $X = R \oplus_F Y$ is a twisted sum and that the quasi-linear map $F$ on the separable normed space $Y$ splits on an infinite-dimensional subspace. Then it is bounded on a further infinite-dimensional subspace, so the quotient map $\pi : X \to X/E = X/R$ is not strictly singular.

**Remark.** One special case, which is sometimes applicable, is that $X$ has the Klee property if it has an infinite-dimensional locally convex subspace with the Hahn–Banach Extension Property (cf. [4]). Indeed, in these circumstances, there is a locally convex subspace $Z$ with $\dim Z = \infty$, so the Banach envelope seminorm is equivalent to the original quasi-norm on $Z$; it then follows rapidly that the quotient map $\pi : X \to X/E$ is an isomorphism on $Z$.

For a particular case of this, let $(A_n)$ be a sequence of pairwise disjoint measurable subsets of $(0, 1)$, of positive measure. Let $(f_n)$ be a sequence of measurable functions, with $f_n$ supported on $A_n$ and each $f_n$ having the distribution of $t \to 1/t$, for small $t$. Let $F$ be the closed linear span of $(f_n)$ in weak $L_1$. Then $F$ has the Hahn–Banach Extension Property in weak $L_1$ and so if $X$ is any separable subspace of weak $L_1$ containing $F$ then $X$ has the Klee property.

**Example.** Finally, we construct an elementary counter-example to Klee’s problem, using much less technical arguments than [2]. We use the twisted sum of Hilbert spaces, $Z_2$, introduced in [3] (see alternative treatments in [1] and [5]). To define this it will be convenient to consider the space $c_{00}$ of all finitely nonzero sequences as a dense subspace of $\ell_2$ and consider the map $\Omega : c_{00} \to \ell_2$ given by

$$\Omega(\xi)(k) = \xi(k) \log(\|\xi\|_2/|\xi(k)|),$$

where $\|\cdot\|_2$ is the usual $L_2$-norm.
where as usual the right-hand side is interpreted as zero if $\xi(k) = 0$. Then $\Omega(\alpha\xi) = \alpha\Omega(\xi)$ for $\alpha \in \mathbb{R}$ and

$$\|\Omega(\xi + \eta) - \Omega(\xi) - \Omega(\eta)\|_2 \leq C(\|\xi\|_2 + \|\eta\|_2)$$

for a suitable absolute constant $C$. Now $Z_2 = \ell_2 \oplus \ell_2$ is the completion of $c_{00} \oplus \ell_2$ under the quasi-norm

$$\| (\xi, \eta) \| = \|\xi - \Omega(\eta)\|_2 + \|\eta\|_2.$$ 

Now (cf. [3]) the map $(\xi, \eta) \rightarrow \eta$ extends to a quotient map from $Z_2$ onto $\ell_2$ which is strictly singular. More precisely, if $F$ is any infinite-dimensional subspace of $c_{00}$ then the completion of $\ell_2 \oplus \Omega F$ contains an isometric copy of $Z_2$ (this is essentially Theorem 6.5 of [3], or see [1]). In particular, this subspace is never of cotype 2.

Now to construct our example, embed $\ell_2$ into $L_p$, where $p < 1$. Then $L_p \oplus \ell_2 = X$ has its kernel $E$ isomorphic to $L_p$ and $X/E \sim \ell_2$. If the quotient map is not strictly singular then there is an infinite-dimensional subspace $F$ of $c_{00}$ such that the completion of $L_p \oplus \Omega F$ is linearly isomorphic to $L_p \oplus \ell_2$ and hence has cotype 2. Then $\ell_2 \oplus \Omega F$ is also cotype 2, and this is impossible as we have seen.

It follows from our main theorem that the space we have constructed fails the Klee property.

REFERENCES


