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Perturbations of the H^∞-calculus

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Abstract

Suppose A is a sectorial operator on a Banach space X, which admits an H^∞-calculus. We study conditions on a multiplicative perturbation B of A which ensure that B also has an H^∞-calculus. We identify a class of bounded operators $T : X \to X$, which we call strongly triangular, such that if $B = (1+T)A$ is sectorial then it also has an H^∞-calculus. In the case X is a Hilbert space an operator is strongly triangular if and only if $\sum s_n(T)/n < \infty$ where $(s_n(T))_{n=1}^\infty$ are the singular values of T.

1. Introduction

Let A be a sectorial operator on a Banach space X (see § 3 for the precise definition). We will say that a closed operator B is a *(multiplicative) perturbation* of A if $\mathrm{Dom}(A) = \mathrm{Dom}(B)$ and there is a constant C such that

$$C^{-1}\|Ax\| \le \|Bx\| \le C\|Ax\| \qquad x \in \mathrm{Dom}(A).$$

This implies that we can write B in the form $B = SA$ where $S : X \to X$ is a bounded operator with a lower bound. In order for B to be again sectorial it is necessary but not sufficient that S be invertible. It is well-known, however, that if S is a small enough perturbation of the identity then B is always sectorial.

Now suppose A has an H^∞-calculus, as introduced by McIntosh [26] (see § 3). Then McIntosh and Yagi [27] showed that a perturbation of A can be sectorial but fail to have an H^∞-calculus. We therefore may ask for conditions on S (or more

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appropriately $T = S - I$) which will ensure that B again has an H^∞-calculus. Such a result was recently proved by Arendt and Batty [2] who showed that if A is invertible and if B is a sectorial operator which is a nuclear perturbation of A (i.e. T is nuclear) then B has an H^∞-calculus with angle $\omega_H(B) \le \max(\omega(B), \omega_H(A))$. (See § 3 for precise definitions of these quantities).

The aim of this paper is to show that the class of permissible perturbations that preserve the H^∞-calculus is significantly larger than just the nuclear perturbations. We first discuss our results in the context of Hilbert space where our results are essentially best possible. Let us say that a compact operator T on a Hilbert space is *triangular* if its singular values satisfy the inequality

$$\sum_{n=1}^{\infty} \frac{s_n(T)}{n} < \infty. \tag{1.1}$$

The triangular operators are identifiable with the predual of the Matsaev ideal (see [12]). Then every sectorial operator B which is a triangular perturbation of A automatically has an H^∞-calculus with $\omega_H(B) \le \max(\omega(B), \omega_H(A))$. Conversely if (a_n) is a sequence with

$$\sum_{n=1}^{\infty} \frac{a_n}{n} = \infty$$

then we may construct an example of a sectorial operator A with an H^∞-calculus and a compact operator T with $s_n(T) \le a_n$ so that none of the operators $(1 + 2^{-n}T)A$ has an H^∞-calculus.

Thus the triangular operators form the largest ideal which preserve the H^∞-calculus. Notice that the restriction on the singular values is very mild indeed: one only requires a condition like $s_n(T) = O((\log n)^{-1-\epsilon})$.

For general Banach spaces our results are not quite so clean. We first observe that it is possible to replace nuclear operators in the Arendt-Batty result by absolutely summing operators (which may not even be compact).

Then we define a triangular operator $T : X \to Y$ to be an operator such that for a suitable constant C one has

$$\Big|\sum_{j=1}^{n}\sum_{k=1}^{j}\langle Tx_j, y_k^*\rangle\Big| \le C \sup_{|a_j|=1}\Big\|\sum_{j=1}^{n} a_j x_j\Big\| \sup_{|a_k|=1}\Big\|\sum_{k=1}^{n} a_k y_k^*\Big\|$$

whenever $x_1, \ldots, x_n \in X,\ y_1^*, \ldots, y_n^* \in Y^*$. The least such constant C defines $\Theta(T)$ the triangular norm of T. This definition is equivalent to the definition above for operators on a Hilbert space. The class of triangular operators forms an operator ideal in the sense of Pietsch which includes the ideal of absolutely summing operators. One can also show that if T is compact and its approximation numbers $a_n(T)$ satisfy (in place of (1.1)):

$$\sum_{n=1}^{\infty} \frac{a_n(T)\log n}{n} < \infty \tag{1.2}$$

then T is triangular. This condition is a little more restrictive than the condition for operators on Hilbert spaces, but still is quite mild: one requires an estimate like $a_n(T) = O((\log n)^{-2-\epsilon})$.

It then turns out that the H^∞-calculus is preserved under small triangular perturbations. We say that T is strongly triangular if for every $\epsilon > 0$ there is an absolutely summing S with $\Theta(T-S) < \epsilon$. For example (1.2) suffices to ensure that T is strongly triangular (and one only needs S to be finite-rank); however strongly triangular operators need not be compact. We then have a full extension of the Arendt-Batty result: if B is a strongly triangular perturbation of A and is sectorial then B has an H^∞-calculus with angle $\omega_H(B) \leq \max(\omega(B), \omega_H(A))$. Let us remark here that to prove this result under the hypothesis (1.2) one can avoid the use of absolutely summing perturbations (Theorem 7.5) (using only finite-rank perturbations and somewhat more elementary arguments). However, the use of Grothendieck's theorem and the theory of absolutely summing operators appears to be essential in the characterization of triangular operators by (1.1) or (1.2). Furthermore, it seems to us that it is very natural to consider absolutely summing perturbations as the proof of Theorem 7.5 is quite direct and uses only the definition of absolutely summing operators.

We also give in § 4 some results on the sectoriality of perturbations which imply this result can be strengthened when X has non-trivial type to $\omega_H(B) \leq \omega_H(A)$ unless B has an eigenvalue λ with $|\arg \lambda| > \omega_H(A)$. We also show that in certain special Banach spaces like L_1 or ℓ_1 then the H^∞-calculus is preserved under all compact perturbations. This result depends very much on the special behavior of these spaces. We also show that at least for Hilbert spaces our results are sharp.

Let us make a few comments on the organization of the paper. In § 2 we collect together some well-known results from classical Banach space theory which are used in the paper. § 3 contains the background on sectorial operators. In § 4 we discuss some general results on perturbations of sectorial operators, extending known results in certain cases; however, the results of this section (Proposition 4.1 through Theorem 4.4) are not necessary in order to read the remainder of the paper. § 5 and § 6 collect together the Banach space ideas used in the main results. Finally § 7 contains the main results on when perturbations have an H^∞-calculus and § 8 gives an example to show the results are sharp.

2. Preliminaries from Banach space theory

In this section we review some of the basic facts in general Banach space theory which will be used in the remainder of the paper. We will only consider complex Banach spaces. We refer to [1, 34, 22, 23] for general background.

Let X be a Banach space; we denote the dual of X by X^* and use $\langle , \rangle$ for the natural pairing of X and X^*. If X is a Hilbert space we use $(,)$ for the inner-product on X. If A is a subset of X we denote by $[A]$ the closed linear span of A.

We recall that a sequence $(e_n)_{n=1}^\infty$ in X is called a *(Schauder) basis* if there is a sequence $(e_n^*)_{n=1}^\infty$ in X^* such that $e_k^*(e_k) = 1$, $e_j^*(e_k) = 0$ for $j \neq k$, and $x = \sum_{n=1}^\infty \langle x, e_n^* \rangle e_n$ for each $x \in X$. $(e_n)_{n=1}^\infty$ is called an *unconditional basis* if the series $\sum_{n=1}^\infty \langle x, e_n^* \rangle e_n$ converges unconditionally for every $x \in X$. $(e_n)_{n=1}^\infty$ is called a *basic sequence (respectively, an unconditional basic sequence)* if it is a basis (respectively an unconditional basis) for its closed linear span $[e_n]_{n=1}^\infty$. Two basic sequences $(e_n)_{n=1}^\infty$ and $(x_n)_{n=1}^\infty$ are *equivalent* if there is an invertible operator $T : [e_n]_{n=1}^\infty \to [x_n]_{n=1}^\infty$ with $Te_n = x_n$.

It is often important to know when an arbitrary sequence has a subsequence which is basic and a complete answer can be given (see [17] or [1, p. 22]):

Proposition 2.1

Let $(x_n)_{n=1}^\infty$ be a sequence in a Banach space X. Suppose $0 < \inf_{n\in\mathbb{N}} \|x_n\| \le \sup_{n\in\mathbb{N}} \|x_n\| < \infty$. Suppose $(x_n)_{n=1}^\infty$ fails to have a subsequence which is basic; then the set $\{x_n : n \in \mathbb{N}\}$ is relatively weakly compact in $X \setminus \{0\}$.

We also have:

Proposition 2.2

Let X be a Banach space with an unconditional basis. Then every weakly null sequence $(x_n)_{n=1}^\infty$ with $0 < \inf_{n\in\mathbb{N}} \|x_n\| \le \sup_{n\in\mathbb{N}} \|x_n\| < \infty$ has a subsequence which is an unconditional basic sequence.

This is very well-known and follows from the Bessaga-Pełczyński Selection criterion ([1, p. 14]). Note that L_p has an unconditional basis if $1 < p < \infty$ but L_1 does not embed in any space with an unconditional basis ([34, p. 62]). Indeed the proposition is false if $X = L_1$ [16].

We recall that a sequence $(x_n)_{n=1}^\infty$ in a Banach space X is *weakly Cauchy* if $\lim_{n\to\infty}\langle x_n, x^*\rangle$ exists for all $x^* \in X^*$. The following deep result is due to Rosenthal [32] (for the real case) and Dor [10] (for the complex case). See [1, p. 252] or [22, p. 99].

Theorem 2.3

Let $(x_n)_{n=1}^\infty$ be a sequence in a Banach space X such that $0 < \inf_{n\in\mathbb{N}} \|x_n\| \le \sup_{n\in\mathbb{N}} \|x_n\| < \infty$. If $(x_n)_{n=1}^\infty$ has no weakly Cauchy subsequence then $(x_n)_{n=1}^\infty$ has a subsequence which is basic and equivalent to the canonical basis of ℓ_1.

We will also need the concepts of Rademacher type and cotype. Let $(\epsilon_j)_{j=1}^\infty$ denote a sequence of independent Rademachers (i.e. random variables on some probability space such that $\mathbb{P}(\epsilon_j = 1) = \mathbb{P}(\epsilon_j = -1) = \frac{1}{2}$). We say that a Banach space X has *type* p (where $1 < p \le 2$) if there is a constant C such that

$$\Big(\mathbb{E}\Big\|\sum_{j=1}^n \epsilon_j x_j\Big\|^p\Big)^{1/p} \le C\Big(\sum_{j=1}^n \|x_j\|^p\Big)^{1/p} \qquad x_1, \ldots, x_n \in X.$$

X has *cotype* q (where $2 \le q < \infty$) if there is a constant C so that

$$\Big(\sum_{j=1}^n \|x_j\|^q\Big)^{1/q} \le C\Big(\mathbb{E}\Big\|\sum_{j=1}^n \epsilon_j x_j\Big\|^q\Big)^{1/q} \qquad x_1, \ldots, x_n \in X.$$

X has non-trivial type if it has type $p > 1$ for some p.

We now consider ideals of operators. A good general reference here is [9] or the survey article [8].

First we recall that an operator $T : X \to Y$ is said to be *strictly singular* if for every infinite-dimensional subspace E of X the restriction $T|_E$ fails to be an isomorphism (into Y). Compact operators are strictly singular but the converse is false: however, strictly singular operators have essentially the same Fredholm properties as compact operators. We will need the following result ([22, p. 79]):

Proposition 2.4

Let $T : X \to X$ be a strictly singular operator. Then $1+T$ is a Fredholm operator of index zero.

If X and Y are Banach spaces and $\mathcal{L}(X,Y)$ is the space of all bounded operators then we will say that a subspace $\mathcal{I}$ with an associated norm Φ is an *operator ideal* if:

(i) $(\mathcal{I},\Phi)$ is complete.
(ii) If $R : X \to Y$ has rank one then $R \in \mathcal{I}$ and $\Phi(R) = \|R\|$.
(iii) If $S \in \mathcal{I}$ and $R \in \mathcal{L}(Y) = \mathcal{L}(Y,Y)$, $T \in \mathcal{L}(X)$, then $RST \in \mathcal{I}$ with $\Phi(RST) \le \|R\|\Phi(S)\|T\|$.

We will say that $\mathcal{I}$ is *maximal* if $\{T : \Phi(T) \le 1\}$ is closed for the strong operator topology.

Our definition of an operator ideal is a simplified version of the more general concept of an operator ideal developed by Pietsch [29] (see also [6, 8]); we have elected not to use the more general definition only to avoid complications. Note that both the compact and strictly singular operators form an operator ideal for the operator norm. We now turn to some other examples of operator ideals, which are not closed in $\mathcal{L}(X,Y)$.

An operator $T : X \to Y$ is called *nuclear* if it can be represented in the form $Tx = \sum_{n=1}^\infty \langle x, x_n^*\rangle y_n$ where $(x_n^*)_{n=1}^\infty$ is a sequence in X^*, $(y_n)_{n=1}^\infty$ is a sequence in Y and $\sum_{n=1}^\infty \|x_n^*\|\|y_n\| < \infty$. The nuclear norm $\nu(T)$ is defined to the infimum of $\sum_{n=1}^\infty \|x_n^*\|\|y_n\|$ over all such representations. The nuclear operator operators form an operator ideal.

T is called *absolutely p-summing* if there is a constant C so that we have

$$\Big(\sum_{k=1}^n \|Tx_k\|^p\Big)^{1/p} \le C \max_{\|x^*\|\le 1} \Big(\sum_{k=1}^n |\langle x_k, x^*\rangle|^p\Big)^{1/p} \qquad x_1,\ldots,x_n \in X.$$

The least constant C is called the absolutely p-summing norm and denoted $\pi_p(T)$. It is clear that the absolutely p-summing operators form a maximal operator ideal according to our definition above. Absolutely p-summing operators are always strictly singular. We will need mainly the case $p = 1$ when we use the term *absolutely summing operator*.

We will need the following fact (combine e.g. Corollary 5.8 and Theorem 5.26 of [9]):

Proposition 2.5

Let $T : \ell_\infty^n \to X$ be a linear operator. Then $\pi_1(T) = \nu(T)$.

The central result concerning absolutely summing operators is of course Grothendieck's theorem (see [22, p. 69], [9, p. 60], [31, p. 57]):

Theorem 2.6

Let $X = \ell_1$ or $X = L_1$. Then every bounded operator $T : X \to \ell_2$ is absolutely summing and $\pi_1(T) \le K_G\|T\|$ where K_G is Grothendieck's constant.

Note that Grothendieck's constant in the complex case is different from the real case. A Banach space X such that every operator $T : X \to \ell_2$ is absolutely summing is called a GT-space; see [31, Chapter 6], for a full discussion.

We will have particular need of a dual form of Grothendieck's theorem ([9, p. 64], [22, p. 70]). We state this this for finite-rank operators.

Theorem 2.7

Let X be a Banach space isometric to a subspace of L_1 (e.g. let X be a Hilbert space). Then if $T : \ell_\infty^n \to X$ is a linear operator we have $\pi_2(T) \le K_G\|T\|$.

Theorem 2.7 holds for any X with cotype 2 by use of Pisier's abstract Grothendieck theorem ([31, Chaper 4]).

Combined with the Grothendieck-Pietsch Factorization theorem ([9, pp. 44-49], [31, p. 13]) this gives a factorization result:

Theorem 2.8

Let X be a Banach space isometric to a subspace of L_1. Then if $T : \ell_\infty^n \to X$ is a linear operator we can find a factorization

$$\ell_\infty^n \xrightarrow{D} \ell_2^n \xrightarrow{U} X$$

so that D is a diagonal operator, $T = UD$ and $\|U\|\|D\| \le K_G\|T\|$.

Finally if $T : X \to Y$ is any bounded operator we define

$$a_n(T) = \inf\{\|T - F\| : \operatorname{rank} F < n\}$$

be the *nth-approximation number* of T, so that $a_1(T) = \|T\|$. If $X = Y$ is a Hilbert space then $a_n(T) = s_n(T)$ is the *nth-singular value* of T.

3. Sectorial operators

Let X be a complex Banach space. A sector of angle $0 < \phi < \pi$ in the complex plane is the open set defined by

$$\Sigma_\phi = \{\lambda \in \mathbb{C} \setminus \{0\} \, : \, |\arg \lambda| < \phi\}.$$

A closed operator A on X is called sectorial if:

(i) A is one-to-one.

(ii) The domain $\mathrm{Dom}(A)$ and range $\mathrm{Ran}(A)$ are dense in X.

(iii) There exists $0 < \phi < \pi$ so that the spectrum $\sigma(A)$ is contained in $\overline{\Sigma}_\phi$ and one has the resolvent estimate:.

$$\|\lambda R(\lambda, A)\| \le C, \quad \lambda \in \mathbb{C} \setminus \overline{\Sigma}_\phi \tag{3.1}$$

Notice that this definition does not require A to be invertible. It follows from this definition that A^{-1} is also a sectorial operator. We define the angle of sectoriality of A by letting $\omega(A)$ be the infimum of all ϕ so that (3.1) holds.

We denote by $H^\infty(\Sigma_\phi)$ the space of all bounded analytic functions on the sector Σ_ϕ where $0 < \phi < \pi$. We define $H_0^\infty(\Sigma_\phi)$ to be the space of all $f \in H^\infty(\Sigma_\phi)$ which obey the estimate of the form $|f(z)| \le C|z|^\delta(1+|z|)^{-2\delta}$ for $z \in \Sigma_\phi$ with $\delta > 0$.

For any $\phi > \omega$ suppose $f \in H_0^\infty(\Sigma_\phi)$. Then we can define $f(A)$ as a bounded operator by a contour integral i.e.

$$f(A) = \frac{1}{2\pi i}\int_{\Gamma_\nu} f(\zeta)R(\zeta, A)\, d\zeta, \tag{3.2}$$

where $\Gamma_\nu = \{|t|e^{-i(\operatorname{sgn} t)\nu} : -\infty < t < \infty\}$ and $\omega < \nu < \phi$. Notice that we get an estimate:

$$\|f(tA)\| \le C\int_{-\infty}^{\infty} |f(se^{i(\operatorname{sgn} s)\nu})|\,\frac{ds}{|s|} \tag{3.3}$$

where $C = C(\nu, A)$. The map $f \to f(A)$ is an algebra homomorphism from $H_0^\infty(\Sigma_\phi)$ into $\mathcal{L}(X)$.

If $f \in H^\infty(\Sigma_\phi)$ then (3.2) does not necessarily converge as a Bochner integral. However for every $g \in H_0^\infty(\Sigma_\phi)$ we can define the operator $(fg)(A)$. Thus if $x \in \operatorname{Dom}(A)\cap\operatorname{Ran}(A)$ since then $x = A(1+A)^{-2}y$ for some y we can define $f(A)x = g(A)y$ where $g(z) = z(1+z)^{-2}f(z)$.

If we define

$$v_n(z) = \frac{n}{n+z} - \frac{1}{1+nz}$$

then $v_n(A)$ maps X into $\operatorname{Dom}(A)\cap\operatorname{Ran}(A)$ and it may be shown that $(fv_n)(A)x = f(A)v_n(A)x$. If $\sup_n \|(v_nf)(A)\| < \infty$ then we can define

$$f(A)x = \lim_{n\to\infty}(v_nf)(A)x \qquad x \in X$$

as a bounded operator. This is equivalent to the fact that $f(A)$ satisfies an estimate

$$\|f(A)x\| \le C\|x\| \qquad x \in \operatorname{Dom}(A)\cap\operatorname{Ran}(A).$$

As an alternative way to view this procedure, one can densely define $f(A)$ by

$$f(A)x = \frac{1}{2\pi i}\int_{\Gamma_\nu} f(\zeta)\zeta^{1/2}A^{1/2}R(\zeta, A)x\, d\zeta, \qquad x \in \operatorname{Dom}(A)\cap\operatorname{Ran}(A) \tag{3.4}$$

where $A^{1/2}R(\zeta, A)$ is a well-defined operator since $z^{1/2}(\zeta - z)^{1/2}$ belongs to $H_0^\infty(\Sigma_{\phi'})$ where $\omega < \phi' < \nu$. Then $f(A)$ extends to a bounded operator if one has a norm estimate

$$\|f(A)x\| \le C\|x\| \qquad x \in \operatorname{Dom}(A)\cap\operatorname{Ran}(A).$$

If $f(A)$ is bounded for for all $f \in H^\infty(\Sigma_\phi)$ we say that A has $H^\infty(\Sigma_\phi)$-calculus. We then have an estimate

$$\|f(A)\| \le C\|f\|_{H^\infty(\Sigma_\phi)} \qquad f \in H^\infty(\Sigma_\phi).$$

We can define the corresponding angle of H^∞-calculus by letting $\omega_H(A)$ be the infimum of all ϕ so that A has an $H^\infty(\Sigma_\phi)$-calculus. See [19, 5] for details. A good reference for the functional calculus described above is [15].

4. Perturbations of sectorial operators

Let A be a sectorial operator; a closed operator B is called a *perturbation* of A if $\mathrm{Dom}(A) = \mathrm{Dom}(B)$ and for a suitable constant C we have

$$C^{-1}\|Ax\| \le \|Bx\| \le C\|Ax\| \qquad x \in \mathrm{Dom}(A).$$

This implies that $B = SA$ where $S : X \to X$ is a bounded operator, with a lower bound $\|Sx\| \ge c\|x\|$. For B to be also sectorial it is necessary that S also has dense range and therefore is invertible.

If $T := S - I$ is compact we say that B is a *compact perturbation* of A. More generally if T belongs to a specific operator ideal $\mathcal{I}$ we say that B is an *$\mathcal{I}$-perturbation* of A.

Suppose $\omega(A) < \phi < \pi$. In order for B to be sectorial with $\omega(B) < \phi$ it is necessary and sufficient that the operator $(\lambda - B)R(\lambda, A)$ defines a bounded invertible operator on X for $\lambda \notin \overline{\Sigma}_\phi$ and that we have a bound

$$\|((\lambda - B)R(\lambda, A))^{-1}\| \le C \qquad \lambda \notin \overline{\Sigma}_\phi.$$

Now

$$(\lambda - B)R(\lambda, A) = I - TAR(\lambda, A) \tag{4.1}$$

where $T = S - I$.

We therefore recover immediately the well-known perturbation result that if

$$\|T\| \sup_{\lambda \notin \overline{\Sigma}_\phi} \|AR(\lambda, A)\| < 1 \tag{4.2}$$

then B is sectorial and $\omega(B) \le \phi$. See for example [7, 24, 3].

Let us also note the following easy consequence (which again is essentially contained in the result of [7]).

Proposition 4.1

Suppose B is a compact perturbation of A and $\phi > \omega(A)$. If B fails to be sectorial with $\omega(B) \le \phi$ then there exists an eigenvalue λ of B with $\lambda \in \mathbb{C}\setminus\{0\}$ with $|\arg\lambda| \ge \phi$.

Proof. Let

$$M = \sup\left\{\|\lambda R(\lambda, A)\| : \ \lambda \ne 0, \ |\arg\lambda| \ge \phi\right\}.$$

Suppose B has no such eigenvalue. Then $1 - TAR(\lambda, A)$ is injective by (4.1) above. Since T is compact the operator $I - TAR(\lambda, A)$ is Fredholm and hence invertible. Thus the map $\lambda \to (I - TAR(\lambda, A))^{-1}$ is well-defined for $|\arg\lambda| \ge \phi$ and $\lambda \ne 0$. It is clearly also continuous and analytic on the interior of the region. It is enough to show the existence of $0 < r_1 < r_2 < \infty$ and C so that

$$\|(I - TAR(\lambda, A))^{-1}\| \le C \qquad 0 < |\lambda| < r_1 \text{ or } r_2 < |\lambda| < \infty, \ |\arg\lambda| \ge \phi.$$

By the compactness of T there exists r_2 so that

$$\|AR(\lambda,A)T\| < \frac{1}{2} \qquad r_2 < |\lambda| < \infty,\ |\arg\lambda| \geq \phi.$$

Since

$$(I - TAR(\lambda,A))^{-1} = I + \sum_{n=1}^{\infty} T(AR(\lambda,A)T)^{n-1}AR(\lambda,A)$$

we have

$$\|(I - TAR(\lambda,A))^{-1}\| \leq 1 + 2(M+1)\|T\| \qquad r_2 < |\lambda| < \infty,\ |\arg\lambda| \geq \phi.$$

Let $T_0 = S^{-1}T$. There similarly exists r_1 so that

$$\|\lambda R(\lambda,A)T_0\| < \frac{1}{2} \qquad 0 < |\lambda| < r_1,\ |\arg\lambda| \geq \phi.$$

Now

$$I - TAR(\lambda,A) = S - \lambda TR(\lambda,A) = S(1 - \lambda T_0R(\lambda,A)).$$

Then

$$(1 - \lambda T_0R(\lambda,A))^{-1} = I + \sum_{n=1}^{\infty} T_0(\lambda R(\lambda,A)T_0)^{n-1}(\lambda R(\lambda,A)).$$

Thus

$$\|(I - TAR(\lambda,A))^{-1}\| \leq \|S^{-1}\|(1 + 2M\|S^{-1}\|\|T\|) \qquad 0 < |\lambda| < r_1,\ |\arg\lambda| \geq \phi.$$

Combining the two estimates the proof is complete. □

We will now investigate some extensions of this result for strictly singular perturbations. We will need the following proposition:

Proposition 4.2

Let A be a sectorial operator on X and let $T : X \to X$ be a strictly singular operator. Then (i), (ii) and (iii) are equivalent.

(i) *Suppose $(x_n)_{n=1}^\infty$ in X, and $\mu \in \mathbb{C}, (\lambda_n)_{n=1}^\infty \subset \mathbb{C}$ are such that $|\arg\lambda_n| \geq \phi$, $\lim_{n\to\infty}|\lambda_n| = \infty$, and*

$$\lim_{n\to\infty} \|\mu TAR(\lambda_n,A)x_n - x_n\| = 0.$$

Then $\lim_{n\to\infty}\|x_n\| = 0$.

(ii) *For any $r > 0$, there are constants $M, s > 0$ so that $(1 - \zeta TAR(\lambda,A))$ is invertible for (ζ,λ) such that $|\zeta| \leq r$, $|\lambda| \geq s$ and $|\arg\lambda| \geq \phi$ and*

$$\|(1 - \zeta TAR(\lambda,A))^{-1}\| \leq M \qquad |\zeta| \leq r, |\lambda| \geq s, |\arg\lambda| \geq \phi. \tag{4.3}$$

(iii) *For any $r > 0$, there are constants $M, s > 0$ so that*

$$\|(TAR(\lambda,A))^n\| \leq Mr^{-n}, \qquad |\lambda| \geq s, |\arg\lambda| \geq \phi. \tag{4.4}$$

Similarly, (iv), (v) and (vi) are equivalent.

(iv) *Suppose* $(x_n)_{n=1}^\infty$ *in* X, *and* $\mu \in \mathbb{C}, (\lambda_n)_{n=1}^\infty \subset \mathbb{C}$ *are such that* $|\arg \lambda_n| \geq \phi$, $\lim_{n\to\infty} |\lambda_n| = 0$, *and*

$$\lim_{n\to\infty} \|\mu\lambda_n TR(\lambda_n, A)x_n - x_n\| = 0.$$

Then $\lim_{n\to\infty} \|x_n\| = 0$.

(v) *For any* $r > 0$, *there are constants* $M, s > 0$ *so that* $(1-\zeta\lambda TR(\lambda, A))$ *is invertible for* (ζ, λ) *such that* $|\zeta| \leq r$, $|\lambda| \leq s$ *and* $|\arg \lambda| \geq \phi$ *and*

$$\|(1 - \zeta\lambda TR(\lambda, A))^{-1}\| \leq M \qquad |\zeta| \leq r,\ |\lambda| \leq s, |\arg \lambda| \geq \phi. \tag{4.5}$$

(vi) *For any* $r > 0$, *there are constants* $M, s > 0$ *so that*

$$\|(\lambda TR(\lambda, A))^n\| \leq Mr^{-n}, \qquad |\lambda| \leq s, |\arg \lambda| \geq \phi. \tag{4.6}$$

Proof. $(i) \implies (ii)$: Since T is strictly singular, $1 - \zeta TAR(\lambda, A)$ is invertible if and only if it is one-one by Proposition 2.4. Thus if the conclusion of (ii) fails for every choice of s, M it is clear that for every $n \in \mathbb{N}$ we can find λ_n with $|\arg \lambda_n| \geq \phi$, $|\lambda_n| \geq n$, ζ_n with $|\zeta_n| \leq r^{-1}$ and $x_n \in X$ with $\|x_n\| = 1$ and

$$\|(1 - \zeta_n TAR(\lambda_n, A))x_n\| < n^{-1}.$$

It follows that that (ζ_n) does not have zero as a cluster point; if we let μ be a cluster point of the sequence we have the failure of (i).

$(ii) \implies (iii)$ follows from the Cauchy estimates. $(iii) \implies (i)$ is trivial.

The other implications are similar. ∎

Lemma 4.3

Let A *be a sectorial operator on* X *with an* H^∞*-calculus and suppose that* T *is a bounded operator on* X. *Suppose either that*

(i) X *embeds into a Banach space with an unconditional basis and* T *is strictly singular, or*
(ii) T *belongs to some maximal operator ideal* $\mathcal{I}$ *contained in the strictly singular operators.*

Then for every $r > 0$ *and* $\omega_H(A) < \phi < \pi$ *there exists a constant* M *and* $0 < s_1 < s_2 < \infty$ *such that if* $|\zeta| \leq r$, $(1 - \zeta TAR(\lambda, A))$ *is invertible for* $|\lambda| \geq s_2$ *and* $(1 - \zeta\lambda TR(\lambda, A))$ *is invertible for* $|\lambda| \leq s_1$ *and we have estimates*

$$\|(1 - \zeta TAR(\lambda, A))^{-1}\| \leq M \qquad |\zeta| \leq r,\ |\lambda| \geq s_2,\ |\arg \lambda| \geq \phi,$$

$$\|(1 - \zeta\lambda TR(\lambda, A))^{-1}\| \leq M \qquad |\zeta| \leq r,\ |\lambda| \leq s_1,\ |\arg \lambda| \geq \phi.$$

Proof. We prove the existence of $s_1 > 0$ as the other proof is similar. By Proposition 4.2 we assume by way of contradiction that exists $\mu \in \mathbb{C}$ and a sequence $(\lambda_n)_{n=1}^\infty$ in $\mathbb{C}$ with $|\arg \lambda_n| \geq \phi$ and $\lim_{n\to\infty} |\lambda_n| = 0$ so that for some normalized sequence $(x_n)_{n=1}^\infty$ in X we have

$$\lim_{n\to\infty} \|\mu\lambda_n TR(\lambda_n, A)x_n - x_n\| = 0.$$

This implies that the sequence $y_n = \lambda_n R(\lambda_n, A)x_n$ is bounded above and below. Next observe that $\lambda_n A(1+A)^{-2}R(\lambda_n, A) = f_n(A)$ where the functions f_n converge uniformly to zero on the sector Σ_ψ, where $\omega_H(A) < \psi < \phi$. Hence $\lim_{n\to\infty} \|A(1+A)^{-2}y_n\| = 0$ so that the sequence $(y_n)_{n=1}^\infty$ can have no weak cluster point other than zero. This implies, by Proposition 2.1 that, passing to a subsequence, we can assume that $(y_n)_{n=1}^\infty$ is basic.

Now passing to a subsequence we further suppose that

$$\begin{aligned} \|\mu\lambda_n TR(\lambda_n, A)x_n - x_n\| &< 2^{-n}, && n = 1, 2, \dots, \\ \|\mu\lambda_{n+1}R(\lambda_{n+1}, A)x_n\| &< 2^{-n} && n = 1, 2, \dots. \\ |\lambda_{n+1}| &< 4^{-n}|\lambda_n| && n = 1, 2, \dots. \end{aligned}$$

Let us define operators $(P_n)_{n=0}^\infty$ by

$$P_0 = 1 - \lambda_1 R(\lambda_1, A)$$

and then

$$P_n = \lambda_n R(\lambda_n, A) - \lambda_{n+1}R(\lambda_{n+1}, A) \qquad n = 1, 2, \dots.$$

Thus $P_n = g_n(A)$ where

$$g_n(z) = \frac{(\lambda_{n+1} - \lambda_n)z}{(\lambda_n - z)(\lambda_{n+1} - z)}.$$

If $\omega_H(A) < \psi < \phi$ then it is clear that we have an estimate

$$|g_n(z)| \le C\frac{|\lambda_n||z|}{(|\lambda_n| + |z|)(|\lambda_{n+1}| + |z|)} \qquad z \in \Sigma_\psi.$$

Thus, for a suitable constant C'

$$|g_n(z)| \le \begin{cases} C'|\lambda_{n+1}|^{-1}|z| & |z| < |\lambda_{n+1}|,\ z \in \Sigma_\psi \\ C' & |\lambda_{n+1}| \le |z| \le |\lambda_n|,\ z \in \Sigma_\psi \\ C'|\lambda_n||z|^{-1} & |z| > |\lambda_n|,\ z \in \Sigma_\psi \end{cases}$$

Hence $\sum_{n=1}^\infty |g_n(z)|$ is uniformly bounded on Σ_ψ. Now it follows from the fact that A has an H^∞-calculus with $\omega_H(A) < \phi$ that there is a constant K so that for every bounded sequence $(\alpha_n)_{n=0}^\infty$ the series $\sum_{n=0}^\infty \alpha_n P_n x$ converges for every $x \in X$ and

$$\Big\|\sum_{n=0}^\infty \alpha_n P_n x\Big\| \le K \sup_n |\alpha_n|\|x\| \qquad x \in X. \tag{4.7}$$

The other property of (P_n) that we need is that

$$\|\mu TP_n x_n - x_n\| < 2^{-n}(1 + \|T\|)$$

and so

$$\sum_{n=1}^\infty \|\mu TP_n x_n - x_n\| < \infty. \tag{4.8}$$

Let us first prove (i). Let $(\epsilon_j)_{j=1}^\infty$ be a sequence of independent Rademachers. Then for any finitely nonzero sequence $(\alpha_j)_{j=1}^\infty$,

$$\begin{aligned}\Big\|\sum_{j=1}^\infty \alpha_j P_j x_j\Big\| &= \Big\|\mathbb{E}\Big(\sum_{j=1}^\infty \epsilon_j P_j\Big)\Big(\sum_{k=1}^\infty \epsilon_k\alpha_k x_k\Big)\Big\| \\ &\le \mathbb{E}\Big\|\Big(\sum_{j=1}^\infty \epsilon_j P_j\Big)\Big(\sum_{k=1}^\infty \epsilon_k\alpha_k x_k\Big)\Big\| \\ &\le K\mathbb{E}\Big\|\Big(\sum_{k=1}^\infty \epsilon_k\alpha_k x_k\Big)\Big\|.\end{aligned}$$

It follows that if any infinite subsequence $(x_n)_{n\in\mathbb{M}}$ is an unconditional basic sequence then there is a bounded operator $U : [x_n]_{n\in\mathbb{M}} \to X$ such that $Ux_n = P_n x_n$. But then $\lim_{n\to\infty} \|\mu TUx_n - x_n\| = 0$ and so for some further subsequence $\mathbb{M}'$ we have that TU is an isomorphism on $[x_n]_{n\in\mathbb{M}'}$ which contradicts the strict singularity of T. Thus $(x_n)_{n=1}^\infty$ has no unconditional basic subsequence. In particular, it has no subsequence equivalent to the canonical basis of ℓ_1. By Rosenthal's theorem (Theorem 2.3) every subsequence of $(x_n)_{n=1}^\infty$ has a further weakly Cauchy subsequence.

Now if $x^* \in X^*$ consider the operator $R : X \to \ell_1$ given by $R(x) = (\langle P_n x, x^*\rangle)_{n=1}^\infty$; this is well-defined and bounded by (4.7). As ℓ_1 has the Schur property ([1, p. 37]), R maps weakly Cauchy sequences to norm convergent sequences, and so $(Rx_n)_{n=1}^\infty$ is relatively norm compact in ℓ_1. In particular we must have

$$\lim_{n\to\infty} \langle P_n x_n, x^*\rangle = 0.$$

Since this is true for all $x^* \in X^*$, the sequence $(P_n x_n)_{n=1}^\infty$ is weakly null. Thus $TP_n x_n$ is also weakly null and this implies by (4.8) that $(x_n)_{n=1}^\infty$ is also weakly null. If X embeds in a space with an unconditional basis this implies that $(x_n)_{n=1}^\infty$ has a subsequence which is an unconditional basic sequence (Proposition 2.2) and thus we have a contradiction.

Next we prove (ii). Notice that if $m > n$,

$$\begin{aligned}\lambda_m\lambda_n R(\lambda_m, A)R(\lambda_n, A) &= \frac{\lambda_m\lambda_n}{\lambda_n - \lambda_m}(R(\lambda_m, A) - R(\lambda_n, A)) \\ &= \lambda_m R(\lambda_m, A) + \frac{\lambda_m}{\lambda_n - \lambda_m}\left(\lambda_m R(\lambda_m, A) - \lambda_n R(\lambda_n, A)\right).\end{aligned}$$

This implies an estimate

$$\|P_m P_n\| \le C_1 4^{-\max(m,n)} \le C_1 2^{-(m+n)} \qquad |m-n| > 1.$$

Thus

$$\sum_{|m-n|>1} \|P_m P_n\| < \infty.$$

Let $Q_0 = P_0 + P_1$ and then $Q_n = P_{3n-1} + P_{3n} + P_{3n+1}$. Of course by (4.7) we have that for every $x \in X$, $\sum_{n=0}^\infty Q_n x = x$ unconditionally. We also have estimates

$$\sum_{n=1}^\infty \|Q_n P_{3n} - P_{3n}\| < \infty$$

and

$$\sum_{m \neq n} \|Q_n P_{3m}\| < \infty.$$

Let us now construct an operator V by defining

$$Vx = \mu \sum_{n=1}^{\infty} P_{3n} T Q_n x.$$

Since

$$Vx = \mu \mathbb{E}\Big(\sum_{j=1}^{\infty} \epsilon_j P_{3j}\Big) T \Big(\sum_{k=1}^{\infty} \epsilon_k Q_k\Big) x$$

where (ϵ_j) is a sequence of independent Rademachers, V is bounded. Furthermore $V \in \mathcal{I}$ since $\mathcal{I}$ is maximal; thus V is also strictly singular.

We clearly have

$$\sum_{n=1}^{\infty} \|VP_{3n} - \mu P_{3n} T P_{3n}\| < \infty$$

and so by (4.8),

$$\sum_{n=1}^{\infty} \|P_{3n} x_{3n} - VP_{3n} x_{3n}\| < \infty$$

which implies that

$$\sum_{n=1}^{\infty} \|y_{3n} - Vy_{3n}\| < \infty.$$

Now since V is strictly singular, we have a contradiction by standard perturbation arguments. This completes the proof of (ii). $\square$

Theorem 4.4

Let A be a sectorial operator on X with an H^∞-calculus and suppose that B is a closed operator on X. Suppose either that

(i) *X embeds into a Banach space with an unconditional basis and B is strictly singular perturbation of A, or*
(ii) *B is an $\mathcal{I}-$perturbation of A, where $\mathcal{I}$ is a maximal operator ideal contained in the strictly singular operators.*

Then if $\omega_H(A) < \phi < \pi$ and B fails to be sectorial with $\omega(B) \leq \phi$ there exists a eigenvalue λ of B with $|\arg \lambda| \geq \phi$.

Proof. Let $B = (1+T)A$. Then

$$(1 - TAR(\lambda, A)) = (1+T)(1 - \lambda T(1+T)^{-1} R(\lambda, A)).$$

We can then apply Lemma 4.3 to T for large λ and to $T(1+T)^{-1}$ for small λ and the result follows as in Proposition 4.1 noting that in each case $(1 - TAR(\lambda, A))$ is a Fredholm operator. $\square$

Note that (ii) holds if B is a p-absolutely summing perturbation of A for some $p < \infty$.

5. Triangular operators

For any bounded operator $T : X \to Y$ we define $\Theta(T)$ with $0 \le \Theta(T) \le \infty$ to be the least constant C such that if $x_1, \dots, x_n \in X$ and $y_1^*, \dots, y_n^* \in Y^*$ then

$$\Big|\sum_{j=1}^{n}\sum_{k=1}^{j}\langle Tx_j, y_k^*\rangle\Big| \le C \sup_{|a_j|=1}\Big\|\sum_{j=1}^{n} a_j x_j\Big\| \sup_{|a_k|=1}\Big\|\sum_{k=1}^{n} a_k y_k^*\Big\|. \tag{5.1}$$

In (5.1) the summation is over the lower triangle $k \le j$; however this is equivalent to summing over the upper triangle, i.e. one can equivalently define $\Theta(T)$ as the least constant such that

$$\Big|\sum_{k=1}^{n}\sum_{j=1}^{k}\langle Tx_j, y_k^*\rangle\Big| \le C \sup_{|a_j|=1}\Big\|\sum_{j=1}^{n} a_j x_j\Big\| \sup_{|a_k|=1}\Big\|\sum_{k=1}^{n} a_k y_k^*\Big\|. \tag{5.2}$$

To see this simply consider the sequences $(x_1, \dots, x_n)$ and $(y_1^*, \dots, y_n^*)$ in reverse order.

Proposition 5.1

We have $\|T\| \le \Theta(T)$ with equality if T has rank one.

Proof. The inequality $\|T\| \le \Theta(T)$ follows by taking $n = 1$ in (5.1). If $\|T\| = 1$ and T has rank one then $Tx = \langle x, x^*\rangle y$ for some fixed $x^* \in X^*$ and $y \in Y$ with $\|x^*\| = \|y\| = 1$. Thus for $x_1, \dots, x_n \in X$ and $y_1^*, \dots, y_n^* \in Y^*$

$$\Big|\sum_{j=1}^{n}\sum_{k=1}^{j}\langle Tx_j, y_k^*\rangle\Big| \le \sum_{j=1}^{n}\sum_{k=1}^{n}|\langle Tx_j, y_k^*\rangle| = \Big(\sum_{j=1}^{n}|\langle x_j, x^*\rangle|\Big)\Big(\sum_{k=1}^{n}|\langle y, y_k^*\rangle|\Big)$$

so that $\Theta(T) \le 1$. $\square$

An operator T will be called *triangular* if $\Theta(T) < \infty$. We denote the set of triangular operators by $\mathcal{T}(X,Y)$ or $\mathcal{T}(X)$ when $Y = X$.

We now collect together some elementary properties of triangular operators. We will denote by Δ the lower triangular projection defined for $n \times n$ matrices, i.e.

$$\Delta\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \cdot & \cdot & \dots & \cdot \\ a_{n-1,1} & a_{n-1,2} & \dots & a_{n-1,n} \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} = \begin{pmatrix} a_{11} & 0 & \dots & 0 \\ a_{21} & a_{22} & \dots & 0 \\ \cdot & \cdot & \dots & \cdot \\ a_{n-1,1} & a_{n-1,2} & \dots & 0 \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}.$$

Proposition 5.2

(i) *For any pair of Banach spaces X, Y, $\mathcal{T}(X,Y)$ is a Banach space under the norm Θ.*

(ii) *If*

$$W \xrightarrow{U} X \xrightarrow{T} Y \xrightarrow{V} Z$$

are bounded operators with $T \in \mathcal{T}(X,Y)$ then $VTU \in \mathcal{T}(W,Z)$ with $\Theta(VTU) \le \|V\|\Theta(T)\|U\|$.

(iii) *$T : X \to Y$ is triangular if and only if $T^* : Y^* \to X^*$ is triangular and $\Theta(T) = \Theta(T^*)$.*

(iv) *If $T : X \to Y$ is a bounded operator then*

$$\Theta(T) = \sup \|\Delta(VTU)\|_{\ell_\infty^n \to \ell_1^n} = \sup \Theta(VTU)$$

where the supremum is taken over all n and all operators $U : \ell_\infty^n \to X,\ V : Y \to \ell_1^n$ with $\|U\|, \|V\| \le 1$.

Remark. (i) and (ii) (with Proposition 5.1) show that $\mathcal{T}(X,Y)$ is an operator ideal; it is clearly also maximal. More generally $\mathcal{T}$ defines an operator ideal in the sense of Pietsch [8]. In (iv) we could, of course, replace Δ with the upper-triangular projection.

Proof. The proofs are quite routine. We omit (i) and (ii). For (iii), we use the Principle of Local Reflexivity [34, p. 76]. For $\epsilon > 0$, if $y_1^*, \dots, y_n^* \in X^*$ and $x_1^{**}, \dots, x_n^{**} \in X^{**}$ we may find $x_1, \dots, x_n \in X$ so that

$$\langle x_j, T^* y_k^* \rangle = \langle T^* y_k^*, x_j^{**} \rangle \quad 1 \le j, k \le n$$

and

$$\sup_{|a_j|=1} \Big\| \sum_{j=1}^n a_j x_j \Big\| \le (1+\epsilon) \sup_{|a_j|=1} \Big\| \sum_{j=1}^n a_j x_j^{**} \Big\|.$$

We now turn to (iv). If $x_1, \dots, x_n \in X$ then we have a natural induced operator $U : \ell_\infty^n \to X$ defined by $U\xi = \sum_{j=1}^n \xi_k x_k$ where $\xi = (\xi_1, \dots, \xi_n)$. Similarly if $y_1^*, \dots, y_n^* \in Y^*$ then we may induce an operator $V : X \to \ell_1^n$ by $Vx = (\langle x, x_k^* \rangle)_{k=1}^n$. Then

$$\|U\| = \sup_{|a_j| \le 1} \Big\| \sum_{j=1}^n a_j x_j \Big\|, \qquad \|V\| = \sup_{|a_k| \le 1} \Big\| \sum_{k=1}^n a_k y_k^* \Big\|.$$

Then $VTU(\xi) = (\sum_{j=1}^n \xi_j \langle Tx_j, y_k^* \rangle)_{k=1}^n$. Thus

$$\|VTU\| \le \sup_{|a_j|=1} \sup_{|b_k|=1} \Big| \sum_{k=1}^n \sum_{j=1}^k a_j b_k \langle Tx_j, y_k^* \rangle \Big| \le \Theta(T) \|U\| \|V\|,$$

by (5.2). It is also clear that by appropriate choices of V, U we obtain (iv). $\square$

Proposition 5.3

If $T : X \to Y$ is absolutely summing then T is triangular and $\Theta(T) \le \pi_1(T)$.

Proof. First note that if T is rank-one then $\Theta(T) = \|T\|$ by Proposition 5.1 and hence $\Theta(T) \le \nu(T)$ for all T. If $U : \ell_\infty^n \to X$ and $V : Y \to \ell_1^n$ are bounded we have

$$\Theta(VTU) \le \nu(VTU) = \pi_1(VTU) \le \|V\| \|U\| \pi_1(T).$$

Here we exploit the well-known fact that an absolutely summing operator S on ℓ_∞^n is nuclear with $\nu(S) = \pi_1(S)$ by Proposition 2.5. Use (iv) of Proposition 5.2. $\square$

This proposition shows that triangular operators need not be compact. For example the quotient map of ℓ_1 onto ℓ_2 (see [22, p. 108]) is absolutely summing (Theorem 2.6). Similarly an embedding of ℓ_2 into $\mathcal{C}[0,1]$ has an absolutely summing adjoint and therefore is triangular.

We first describe triangular operators on a Hilbert space. We will need the following well-known result concerning the boundedness of the infinite version of the lower triangular projection Δ.

Proposition 5.4

Let $T : \ell_2 \to \ell_2$ be a bounded operator such that

$$\sum_{n=1}^{\infty} \frac{s_n(T)}{n} < \infty.$$

Then the operator $S = \Delta(T)$ (the lower triangular part of T) represented by the matrix $s_{jk} = t_{jk}$ if $k \leq j$ and $s_{jk} = 0$ otherwise is bounded and for a suitable constant C independent of T,

$$\|\Delta(T)\| \leq C \sum_{n=1}^{\infty} \frac{s_n(T)}{n}.$$

Proof. This proposition is essentially known. The operators $T : \ell_2 \to \ell_2$ for which

$$\sum_{n=1}^{\infty} \frac{s_n(T)}{n} < \infty$$

is the Matsaev ideal [12]. The proposition asserts the boundedness of the lower triangular projection (and hence also the upper triangular projection) from this ideal into $\mathcal{L}(\ell_2)$. This is equivalent to the boundedness of the same map from the trace-class $\mathcal{S}_1$ into the dual of the Matsaev ideal, a result which goes back to [20] (also see [25]). See also [12, 13].

Let us show that how the proposition follows from e.g. [25, Theorem 2]. Given T we may choose $v \in H$ with $\|v\| = 1$ so that

$$\|\Delta(T)\| \leq 2|(\Delta(T)v, v)|.$$

Let R be the rank-one projection $Rx = (x, v)v$. Let $L = 2i(R - (\Delta(R))^*)$ (while we are working on a Hilbert space, we denote the Hilbert space adjoint of S by S^*). Then L is compact, lower-triangular and has diagonal zero. Thus L is quasi-nilpotent. The imaginary part of L i.e. $K = (L - L^*)/(2i)$ is $2R - (\Delta(R) + \Delta(R)^*) = R - D$ where D is the diagonal of R. Now $\nu(D) \leq 1$ and so $\nu(K) \leq 2$. Thus, applying [25, Theorem 2] we have an estimate $s_n(L) \le C/n$ where C is an absolute constant. Hence

$$\begin{aligned}\|\Delta(T)\| &\le 2|\mathrm{tr}\ \Delta(T)R| \\ &= 2|\mathrm{tr}\ T(\Delta(R))^*| \\ &\le 2|\mathrm{tr}\ TR| + |\mathrm{tr}\ TL| \\ &\le 2\|T\| + \sum_{n=1}^{\infty} s_n(T)s_n(L) \\ &\le 2s_1(T) + C\sum_{n=1}^{\infty}\frac{s_n(T)}{n}.\end{aligned}$$

□

This leads to the following result.

Theorem 5.5

Suppose H is a Hilbert space. Then there is a constant C so that if $T : H \to H$ is a bounded operator then

$$C^{-1}\sum_{n=1}^{\infty}\frac{s_n(T)}{n} \le \Theta(T) \le C\sum_{n=1}^{\infty}\frac{s_n(T)}{n}.$$

In particular T is triangular if and only if

$$\sum_{n=1}^{\infty}\frac{s_n(T)}{n} < \infty.$$

Proof. Suppose $N \in \mathbb{N}$ and $U : \ell_\infty^N \to H$ and $V : H \to \ell_1^N$ are operators with $\|U\|, \|V\| \le 1$.

We now use the Theorem 2.8. The operator $U : \ell_\infty^N \to H$ can be factored in the form

$$\ell_\infty^N \xrightarrow{D_1} \ell_2^N \xrightarrow{U_1} H$$

where D_1 is a diagonal operator with $\|D_1\|_{\ell_\infty^N \to \ell_2^N} \le 1$ and $\|U_1\| \le K_G$, where K_G is the (complex) Grothendieck constant. Dually $V : H \to \ell_1^N$ can be factored in the form

$$H \xrightarrow{V_1} \ell_2^N \xrightarrow{D_2} \ell_1^N$$

where D_2 is diagonal with $\|D_2\|_{\ell_2^N \to \ell_1^N} \le 1$ and $\|V_1\| \le K_G$. Now,

$$\Delta(VTU) = D_2\Delta(V_1TU_1)D_1$$

so that, using Proposition 5.4, for a suitable constant C_0, we have

$$\begin{aligned}\|\Delta(VTU)\|_{\ell_\infty^N \to \ell_1^N} &\le \|\Delta(V_1TU_1)\|_{\ell_2^N \to \ell_2^N} \\ &\le C_0\sum_{n=1}^{\infty}\frac{s_n(V_1TU_1)}{n} \\ &\le C_0K_G^2\sum_{n=1}^{\infty}\frac{s_n(T)}{n}.\end{aligned}$$

Thus we obtain an inequality (letting $C = C_0 K_G^2$).

$$\Theta(T) \leq C \sum_{n=1}^{\infty} \frac{s_n(T)}{n}.$$

For the converse direction, suppose $\Theta(T) < \infty$. If $(e_j)_{j=1}^n, (f_j)_{j=1}^n$ are orthonormal sets and $\sum_{j=1}^n |\xi_j|^2 = \sum_{j=1}^n |\eta_j|^2 = 1$ then we have

$$\Big| \sum_{k=1}^{n} \sum_{j=1}^{k} \xi_j \overline{\eta}_k (Te_j, f_k) \Big| \leq \Theta(T).$$

Now suppose $U : \ell_2^N \to H$ is an isometric embedding and $V : H \to \ell_2^N$ is such that $V^* : \ell_2^N \to H$ is an isometric embedding. Then $U(\xi) = \sum_{j=1}^N \xi_j e_j$ for some orthonormal set $e_1, \ldots, e_N$ and similarly $V^*\eta = \sum_{j=1}^N \eta_j f_j$ for some orthonormal set $f_1, \ldots, f_N$. Then VTU has the matrix $a_{jk} = (Te_k, f_j)$. Thus

$$(\Delta(VTU)\xi, \eta) = \sum_{j=1}^{n} \sum_{k=1}^{j} a_{jk} \xi_k \overline{\eta}_j.$$

It thus follows that $\|\Delta(VTU)\| \leq \Theta(T)$. However by a simply convexity argument this implies the same result for all bounded operators $U : \ell_2^N \to H$ and $V : H \to \ell_2^N$.

We now argue that for each n we may find a rank-one orthogonal projection $R : \ell_2 \to \ell_2$ such that

$$\sum_{j=1}^{k} s_j(\Delta(R)) \geq c \log k \qquad 1 \leq k \leq n$$

where $c > 0$ is an absolute constant. We work on ℓ_2^n. Suppose n is a power of two and let R be represented by the $n \times n$ matrix

$$R = \frac{1}{n} \begin{pmatrix} 1 & \ldots & 1 \\ 1 & \ldots & 1 \\ \cdot & \cdot & \cdot \\ 1 & \ldots & 1 \end{pmatrix}.$$

Recall that the Hilbert matrix

$$H = \begin{pmatrix} 0 & -\frac{1}{2} & -\frac{1}{3} & \ldots & -\frac{1}{n} \\ \frac{1}{2} & 0 & -\frac{1}{2} & \ldots & -\frac{1}{n-1} \\ \cdot & \cdot & \cdot\cdot & \cdot & \\ \frac{1}{n} & \frac{1}{n-1} & \frac{1}{n-2} & \ldots & 0 \end{pmatrix}$$

defines an operator with norm at most π (see e.g. [14]). Then

$$\operatorname{tr} \Delta(R)H \geq c_1 \log n$$

where $c_1 > 0$ is an absolute constant. Hence

$$\sum_{k=1}^{n} s_k(\Delta(R)) \geq c_2 \log n$$

where $c_2 > 0$. Now if $m \le n$ is a power of 2 then compressing $\Delta(R)$ to the subspace spanned by

$$f_k = \sqrt{\frac{m}{n}} \sum_{j=(k-1)n/m+1}^{kn/m} e_j$$

for $1 \le k \le m$ gives us an operator R' with $m \times m$-matrix (for some choice of b)

$$R' = \frac{1}{m} \begin{pmatrix} b & 0 & 0 & \dots & 0 \\ 1 & b & 0 & \dots & 0 \\ \cdot & \cdot & \cdot & \dots & \cdot \\ 1 & 1 & 1 & \dots & b \end{pmatrix}.$$

Thus we have

$$\sum_{k=1}^m s_k(\Delta(R)) \ge \sum_{k=1}^m s_k(R') \ge c_2 \log m$$

and so we get an estimate

$$\sum_{k=1}^m s_k(\Delta(R)) \ge c \log m$$

for every $1 \le m \le n$.

Now

$$\|\Delta(VTU)\| \ge \text{tr } \Delta(VTU)R = \text{tr } VTU(\Delta(R))^*.$$

Maximizing over V, U we obtain

$$\begin{aligned} \Theta(T) &\ge \sum_{k=1}^n s_k(\Delta(R)) s_k(T) \\ &\ge \sum_{k=1}^n (s_k(T) - s_{k+1}(T)) \Big(\sum_{j=1}^k s_j(\Delta(R)) \Big) + s_{n+1}(T) \Big(\sum_{j=1}^n s_j(\Delta(R)) \Big) \\ &\ge c \sum_{k=1}^n (s_k(T) - s_{k+1}(T)) \log k + s_{n+1}(T) \Big(\sum_{j=1}^n s_j(\Delta(R)) \Big) \\ &\ge c \sum_{k=2}^n s_k(T)(\log k - \log(k-1)) \\ &\ge c' \sum_{k=2}^n \frac{s_k(T)}{k}. \end{aligned}$$

This establishes the theorem. □

For operators between arbitrary Banach spaces we cannot give quite such a clean description of triangular operators, but the following theorem suggests that the class is still rather large.

Theorem 5.6

Suppose X, Y are arbitrary Banach spaces. Then there is a constant C so that if $T : X \to Y$ is a bounded operator then

$$\Theta(T) \le C \sum_{n=1}^\infty (1 + \log n) \frac{a_n(T)}{n}.$$

Proof. Suppose $U : \ell_\infty^N \to X$ and $V : Y \to \ell_1^N$ are bounded operators with $\|U\|, \|V\| \leq 1$.

We pick a sequence of finite-rank operators F_n so that rank $F_n < 2^n$ and $\|T - F_n\| \leq 2a_{2^n}(T)$. Let $E_1 = F_1$ and then $E_n = F_n - F_{n-1}$ for $n \geq 2$. Thus rank $E_n < 2^{n+1}$.

We now use Theorem 2.8 again. We will factorize the operators $S_n = VE_nU : \ell_\infty^N \to \ell_1^N$. For each fixed n there is a positive diagonal operator (with nonzero diagonal entries) $D_{n1} : \ell_\infty^N \to \ell_2^N$ so that $\|D_{n1}\| \leq 1$ and S_n can be factored in the form

$$\ell_\infty^N \xrightarrow{D_{n1}} \ell_2^N \xrightarrow{R_n} \ell_1^N$$

where $\|R_n\|_{\ell_2^N \to \ell_1^N} \leq 2K_G \|S_n\|_{\ell_\infty^N \to \ell_1^N}$. (Here the factor 2 is introduced simply to take care of the fact that we require each diagonal entry of D_{n1} to be nonzero). Now, using a dual form of Theorem 2.8, R_n can similarly be factorized in the form

$$\ell_2^N \xrightarrow{Q_n} \ell_2^N \xrightarrow{D_{n2}} \ell_1^N$$

where D_{n2} is a positive diagonal operator (with nonzero entries) with $\|D_{n2}\|_{\ell_2^N \to \ell_1^N} \leq 1$ and $\|Q_n\| \leq 2K_G\|R_n\| \leq 4K_G^2\|S_n\|$.

Since D_{n1} and D_{n2} are representable by invertible matrices, we have

$$\text{rank } (Q_n) = \text{rank } (S_n) \leq \text{rank } (E_n) < 2^{n+1}.$$

Hence by Proposition 5.4 we have an estimate

$$\|\Delta(Q_n)\|_{\ell_2^N \to \ell_2^N} \leq Cn\|S_n\|$$

for a suitable constant C. Hence

$$\|\Delta(D_{n2}Q_nD_{n1})\|_{\ell_\infty^N \to \ell_1^N} \leq C_1 n\|E_n\|.$$

Thus

$$\|\Delta(VTU)\|_{\ell_\infty^N \to \ell_1^N} \leq C_2 \sum_{n=1}^{\infty} n\|E_n\|.$$

Now

$$\|E_1\| \leq \|T\| + 2a_2(T) \leq 2(a_1(T) + a_2(T))$$

and then

$$\|E_n\| \leq 2(a_{2^{n-1}}(T) + a_{2^n}(T)) \qquad n = 2, 3, \ldots.$$

Thus

$$\sum_{n=1}^{\infty} n\|E_n\| \leq \sum_{n=1}^{\infty} (4n+2)a_{2^n}(T)$$

which yields the result. □

At the other extreme we may ask whether there is a Banach space X so that $\mathcal{L}(X) = \mathcal{T}(X)$ (or, equivalently the identity operator is triangular).

Proposition 5.7

Let X be a GT-space such that X^* has cotype 2. Then $\mathcal{L}(X) = \mathcal{T}(X)$.

Proof. We show $\Theta(1_X) < \infty$. Let $U : \ell_\infty^n \to X$ and $V : X \to \ell_1^n$ be operators of norm one. Since X^* has cotype 2 we can factorize V in the form

$$X \xrightarrow{V'} \ell_2^n \xrightarrow{D} \ell_1^n$$

where $\|D\| \leq 1$ and $\|V'\| \leq C_1$ for some constant C_1 independent of n, V. But then $\pi_1(V) \leq C_2$ independent of n, V which implies that $\pi_1(VU) \leq C_2$ and hence $\Theta(VU) \leq C$. □

There is an example satisfying these hypotheses; this space is known as Pisier's space P (and it is even of cotype 2). It was constructed by Pisier as a counterexample to a number of conjectures in Banach space theory [30, 31].

On the other hand for most nice spaces this cannot happen.

Proposition 5.8

Let X be a Banach space so that X has an unconditional basis. Then the identity $I = I_X$ is not triangular.

Proof. We first observe that $\Theta(I_{\ell_2^n}) \to \infty$ by Theorem 5.5. Next we have $\Theta(I_{\ell_\infty^n}) \to \infty$ because of Proposition 5.2 (iv). Similarly $\Theta(I_{\ell_1^n}) \to \infty$. Now X must contain uniformly complemented ℓ_1^n's, ℓ_∞^n's or ℓ_2^n's by an old result of Tzafriri [33]. □

Proposition 5.9

Let X be a Banach space of non-trivial type. Then every triangular operator on X is strictly singular.

Proof. Here we use the result of Figiel and Tomczak-Jaegermann ([11], [28, p. 122]) that every infinite-dimensional subspace of X contains uniformly complemented (in X) ℓ_2^n's. □

Proposition 5.9 implies that if X has non-trivial type then $\mathcal{T}(X)$ is a maximal operator ideal contained in the strictly singular operators and thus Theorem 4.4 can be applied to triangular perturbations.

It will be useful later to introduce a smaller class than the triangular operators. We shall say that $T : X \to Y$ is *strongly triangular* if for every $\epsilon > 0$ there exists $S : X \to Y$ with S absolutely summing that $\Theta(T - S) < \epsilon$. We only know of applications of this idea when S can be chosen to be finite rank.

Proposition 5.10

(i) *Every triangular operator on a Hilbert space is strongly triangular.*

(ii) *If X, Y are arbitrary Banach spaces and $T : X \to Y$ satisfies*

$$\sum_{n=1}^{\infty} (1 + \log n) \frac{a_n(T)}{n} < \infty$$

then T is strongly triangular.

Proof. As the two proofs are similar, we prove only (ii). For each $n \in \mathbb{N}$ pick a finite rank operator F_n with rank $F_n < n$ so that $\|T - F_n\| \le 2a_n(T)$. Then

$$a_k(T - F_n) \le \begin{cases} 2a_n(T) & 1 \le k \le 2n \\ a_{k-n}(T) & 2n+1 \le k < \infty. \end{cases}$$

By Theorem 5.6,

$$\Theta(T - F_n) \le C\Big(2a_n(T)\Big(\sum_{k=1}^{2n} \frac{(1+\log k)}{k}\Big) + \sum_{k=n+1}^{\infty} a_k(T)\frac{1+\log(n+k)}{n+k}\Big)$$

and it is easy to see that both terms in the bracket converge to zero as $n \to \infty$. □

6. Applications to functions

Let X be a Banach space and suppose $f : [a,b] \to X$ is a continuous function, where $0 < a, b < \infty$. The Bochner L_1-norm of f is defined by

$$\|f\|_{L_1(a,b)} = \int_a^b \|f(t)\| dt.$$

The Pettis norm of f is defined by

$$\|f\|_{\mathcal{P}(a,b)} = \sup_{\|x^*\| \le 1} \int_a^b |\langle f(t), x^* \rangle| dt.$$

Let us note the following elementary proposition.

Proposition 6.1

Let $T : X \to Y$ be an absolutely summing operator and suppose $f : (a,b) \to X$ is a bounded continuous function. Then

$$\|Tf\|_{L_1(a,b)} \le \pi_1(T)\|f\|_{\mathcal{P}(a,b)}.$$

Proof. This is well-known and almost immediate from the definition. For $n \in \mathbb{N}$ let $t_k = a + k(b-a)/n$ for $0 \le k \le n$. Then

$$\sum_{k=1}^{n} \| \int_{t_{k-1}}^{t_k} Tf(s)ds\| \le \pi_1(T)\|f\|_{\mathcal{P}(a,b)}.$$

Taking limits as $n \to \infty$ gives the conclusion. □

The following elementary estimate will be useful.

Proposition 6.2

Suppose $f : [a,b] \to X$ and $g : [a,b] \to X^*$ are continuous functions. Then

$$\Big| \int_a^b \int_a^t \langle f(t), g(s) \rangle ds\, dt \Big| \le \|f\|_{L_1(a,b)} \|g\|_{\mathcal{P}(a,b)}.$$

and

$$\Big|\int_a^b\int_a^t \langle f(s),g(t)\rangle ds\,dt\Big| \le \|f\|_{L_1(a,b)}\|g\|_{\mathcal{P}(a,b)}.$$

Proof. Both integrals are dominated by

$$\int_a^b\int_a^b |\langle f(s),g(t)\rangle| ds\,dt \le \int_a^b \|f(s)\|\|g\|_{\mathcal{P}(a,b)}ds.$$

□

If, in Proposition 6.2 we only have an estimate on the Pettis norms on f,g then we can only get a similar result if we compose f with a triangular operator.

Proposition 6.3

Suppose $f:[a,b]\to X$ and $g:[a,b]\to X^*$ are continuous functions and $T:X\to X$ is a triangular operator. Then

$$\Big|\int_a^b\int_a^t \langle Tf(t),g(s)\rangle ds\,dt\Big| \le \Theta(T)\|f\|_{\mathcal{P}(a,b)}\|g\|_{\mathcal{P}(a,b)}$$

and

$$\Big|\int_a^b\int_a^t \langle Tf(s),g(t)\rangle ds\,dt\Big| \le \Theta(T)\|f\|_{\mathcal{P}(a,b)}\|g\|_{\mathcal{P}(a,b)}.$$

Proof. This follows from the definition by approximation. Let N be an integer and $I_j=[a+(j-1)(b-a)/N, a+j(b-a)/N]$. Let

$$x_j=\int_{I_j} f(t)dt,\ x_j^*=\int_{I_j} g(t)dt \qquad 1\le j\le N.$$

Then

$$\sum_{j=1}^N\sum_{k=1}^j \langle Tx_j,x_k^*\rangle \le \Theta(T)\|f\|_{\mathcal{P}(a,b)}\|g\|_{\mathcal{P}(a,b)}.$$

Letting $N\to\infty$ gives the conclusion. □

Now suppose $F:[a,b]\to\mathcal{L}(X)$ is a C^1-function. We will define two seminorms to measure the variation of F. Let

$$\|F\|_{BV}=\sup_{\|x\|\le 1}\|F'(t)x\|_{L_1(a,b)}=\sup_{\|x\|\le 1}\int_a^b\|F'(t)x\|dt$$

and

$$\|F\|_{bv}=\sup_{\|x\|\le 1}\|F'(t)x\|_{\mathcal{P}(a,b)}=\sup_{\substack{\|x\|\le 1\\ \|x^*\|\le 1}}\int_a^b|\langle F'(t)x,x^*\rangle|dt.$$

Notice that $\|F\|_{bv}\le\|F\|_{BV}$ and also that $\|F\|_{bv}=\|F^*\|_{bv}$ where $F^*:[a,b]\to X^*$ is define by $F^*(t)=(F(t))^*$ (this follows easily from Goldstine's theorem.)

We denote by $\|F\|_\infty$ the usual sup norm i.e. $\|F\|_\infty=\max_{a\le t\le b}\|F(t)\|$.

Proposition 6.4

Suppose $F, G : [a,b] \to \mathcal{L}(X)$ are C^1- functions and $T : X \to X$ is a bounded operator. Then

$$\|FG\|_{BV} \le (\|F\|_\infty \|G\|_{BV} + \|F\|_{BV}\|G\|_\infty). \tag{6.1}$$

$$\|FG\|_{bv} \le (\|F\|_\infty \|G\|_{bv} + \|F\|_{bv}\|G\|_\infty) + 2\|F\|_{bv}\|G\|_{BV}. \tag{6.2}$$

$$\|FTG\|_{bv} \le \|T\|\|F\|_\infty \|G\|_{bv} + \|F\|_{bv}(\|TG\|_\infty + 2\Theta(T)\|G\|_{bv}). \tag{6.3}$$

Proof. (6.1) follows trivially from $(FG)'(t) = F'(t)G(t) + F(t)G'(t)$. For (6.2) we note that

$$(FG)'(t) = F'(t)(G(t) - G(a)) + (F(t) - F(a))G'(t) + F'(t)G(a) + F(a)G'(t).$$

Now

$$\int_a^b \langle (F(t) - F(a))G'(t)x, x^*\rangle dt = \int_a^b \int_a^t \langle G'(t)x, (F'(s))^* x^*\rangle ds\, dt$$

so that by Proposition 6.2 we have

$$\begin{aligned}\left|\int_a^b \int_a^t \langle G'(t)x, (F'(s))^* x^*\rangle ds\, dt\right| &\le \sup_{\|x\|\le 1} \|G'(t)x\|_{L_1(a,b)} \sup_{\|x^*\|\le 1} \|(F'(t))^* x^*\|_{\mathcal{P}(a,b)} \\ &= \|G\|_{BV}\|F\|_{bv}.\end{aligned}$$

Similarly

$$\int_a^b \langle F'(t)(G(t) - G(a))x, x^*\rangle| dt = \int_a^b \int_a^t \langle F'(t)x, (G'(s))^* x^*\rangle ds\, dt$$

and (6.2) now follows simply.

For (6.3) we have

$$(FTG)'(t) = F'(t)T(G(t) - G(a)) + (F(t) - F(a))TG'(t) + F'(t)TG(a) + F(a)TG'(t)$$

and the proof is similar to the above but using Proposition 6.3. □

The following proposition is quite elementary but will be used several times later.

Proposition 6.5

Suppose $F : [a,b] \to \mathcal{L}(X)$ is a C^1- function. Then

$$\|F^n\|_{BV} \le n\|F\|_{BV}\|F\|_\infty^{n-1} \qquad n = 1, 2, \ldots$$

and if $\|F\|_\infty < 1$,

$$\|(1 - F)^{-1}\|_{BV} \le \|F\|_{BV}(1 - \|F\|_\infty)^{-2}.$$

Proof. These follow from the composition law. We have

$$(F^n)' = F'F^{n-1} + FF'F^{n-2} + \cdots + F^{n-1}F'$$

and

$$((1-F)^{-1})' = (1-F)^{-1}F'(1-F)^{-1}$$

whence the results follow. □

We need a version of Proposition 6.5 for the bv-norm. This requires more assumptions, however.

Theorem 6.6

Suppose $F : [a,b] \to \mathcal{L}(X)$ is a C^1-function and $T : X \to X$ is a triangular operator with $\Theta(T)(\|F\|_\infty + 2\|F\|_{bv}) = r < 1$. Then

$$\|F(1-TF)^{-1}\|_{bv} \leq (1-r)^{-2}\|F\|_{bv}.$$

Proof. Let $r = \Theta(T)(\|F\|_\infty + 2\|F\|_{bv}) < 1$. We let $F_n = F(TF)^n$ for $n \geq 0$. Thus $F_n = FTF_{n-1}$. Note that $\|TF\|_\infty \leq r$ so that $\|TF_{n-1}\|_\infty \leq r^n$. We also note that

$$F_n' = F'TF_{n-1} + FTF_{n-1}' \qquad n = 1, 2, \ldots$$

so that

$$\|F_n'\|_\infty \leq r^n\|F'\|_\infty + r\|F_{n-1}'\|_\infty \qquad n = 1, 2, \ldots.$$

Thus

$$r^{-n}\|F_n'\|_\infty \leq r^{1-n}\|F_{n-1}'\|_\infty + \|F'\|_\infty \qquad n = 1, 2, \ldots.$$

This implies that

$$\|F_n'\|_\infty \leq (n+1)r^n\|F'\|_\infty$$

so that $\sum_{n=0}^\infty F_n'$ converges uniformly to $d/dt(F(1-TF)^{-1})$.

Using Proposition 6.4 (6.3) gives

$$\|F_n\|_{bv} \leq (\|T\|\|F\|_\infty + 2\Theta(T)\|F\|_{bv})\|F_{n-1}\|_{bv} + \|TF_{n-1}\|_\infty\|F\|_{bv} \qquad n = 1, 2, \ldots.$$

Then since $\|T\| \leq \Theta(T)$,

$$\|F_n\|_{bv}r^{-n} \leq \|F_{n-1}\|_{bv}r^{1-n} + \|F\|_{bv} \qquad n = 1, 2, \ldots.$$

This implies that

$$\|F_n\|_{bv} \leq (n+1)r^n\|F\|_{bv}$$

and so

$$\|\sum_{n=0}^\infty F_n\|_{bv} \leq (1-r)^{-2}\|F\|_{bv}.$$

This implies that

$$\|F(1-TF)^{-1}\|_{bv} \leq (1-r)^{-2}\|F\|_{bv}.$$

□

For ease of exposition we have restricted ourselves to C^1-functions defined on a closed bounded interval. However it is natural to extend our definitions to open, possibly unbounded, intervals. If I is an open interval and $F : I \to \mathcal{L}(X)$ is a bounded C^1-function we define

$$\|F\|_{BV(I)} = \sup_{[a,b]\subset I} \|F\|_{BV[a,b]}$$

and

$$\|F\|_{bv(I)} = \sup_{[a,b]\subset I} \|F\|_{bv[a,b]}$$

and we say that $F \in BV(I)$ (respectively $F \in bv(I)$) if $\|F\|_{BV(I)} < \infty$ (respectively $\|F\|_{bv(I)} < \infty$). Naturally our main results extend without difficulty, and we will use them in the extended form.

7. Perturbing the H^∞-calculus

The following two propositions are well-known criteria for the existence of an H^∞-calculus. They go back to [4, 5]; for a recent simple proof see [21].

Proposition 7.1

Let A be a sectorial operator on a Banach space X. Suppose A has an H^∞-calculus and that $\phi > \omega_H(A)$. Then there is a constant C so that

$$\int_0^\infty |\langle AR(e^{i\theta}, tA)^2 x, x^*\rangle| dt \le C\|x\|\|x^*\| \qquad x \in X,\ x^* \in X^*,\ \phi \le |\theta| \le \pi. \tag{7.1}$$

Proposition 7.2

Let A be a sectorial operator on a Banach space X. Suppose $0 < \omega(A) < \phi < \pi$ and for some constant C we have an estimate

$$\int_0^\infty |\langle AR(e^{i\theta}, tA)^2 x, x^*\rangle| dt \le C\|x\|\|x^*\| \qquad x \in X,\ x^* \in X^*,\ \phi \le |\theta| \le \pi. \tag{7.2}$$

Then A has an H^∞-calculus and $\omega_H(A) \le \phi$.

Remark. The criterion (7.2) can be rewritten as

$$\int_0^\infty |\langle AR(te^{i\theta}, A)^2 x, x^*\rangle| dt \le C\|x\|\|x^*\| \qquad x \in X,\ x^* \in X^*,\ \phi \le |\theta| \le \pi \tag{7.3}$$

or as

$$\int_0^\infty |\langle \lambda AR(\lambda, tA)^2 x, x^*\rangle| dt \le C\|x\|\|x^*\| \qquad x \in X,\ x^* \in X^*,\ |\arg\lambda| \ge \phi. \tag{7.4}$$

Now if A is a sectorial operator let us define for $\omega(A) < |\theta| \le \pi$ the C^1-function $F_\theta = F_{\theta,A}$ by

$$F_\theta(t) = AR(te^{i\theta}, A).$$

Note that

$$1 + F_\theta(t) = te^{i\theta}R(te^{i\theta}, A).$$

Then

$$F_\theta'(t) = -e^{i\theta}AR(te^{i\theta}, A)^2$$

and we deduce that

Proposition 7.3

In order that A has an H^∞-calculus with $\omega_H(A) < \psi$ it is is necessary and sufficient that for some $\omega_H(A) < \phi < \psi$ we have a uniform bound:

$$\sup_{\phi \le |\theta| \le \pi} \|F_\theta\|_{bv(0,\infty)} < \infty.$$

We now consider perturbations.

Proposition 7.4

Suppose A is a sectorial operator with an H^∞-calculus and suppose B is a perturbation of A which is sectorial. Let $B = (I+T)A$ and suppose $\phi > \max(\omega_H(A), \omega(B))$. If

$$\sup_{|\theta| \ge \phi} \|F_\theta(I - TF_\theta)^{-1}\|_{bv(0,\infty)} < \infty \qquad \phi \le |\theta| \le \pi \tag{7.5}$$

then B has an H^∞-calculus and $\omega_H(B) \le \phi$.

More generally, let $T_0 = T(1+T)^{-1}$. Suppose there exist $0 < a < b < \infty$ such that:

$$\sup_{|\theta| \ge \phi} \|F_\theta(I - TF_\theta)^{-1}\|_{bv(b,\infty)} < \infty \qquad \phi \le |\theta| \le \pi \tag{7.6}$$

and

$$\sup_{|\theta| \ge \phi} \|(I + F_\theta)(1 - T_0(1 + F_\theta))^{-1}\|_{bv(0,a)} < \infty \qquad \phi \le |\theta| \le \pi. \tag{7.7}$$

Then B has an H^∞-calculus and $\omega_H(B) \le \phi$.

Proof. Let $G_\theta(t) = BR(te^{i\theta}, B)$. It suffices to show that $\sup_{\phi \le |\theta|} \|G_\theta\|_{bv(0,\infty)} < \infty$.

$$R(\lambda, B) = R(\lambda, A)(1 - TAR(\lambda, A))^{-1}$$

so that

$$(1+T)^{-1} G_\theta(t) = F_\theta(t)(1 - TF_\theta(t))^{-1}.$$

Then (7.5) gives the conclusion.

For (7.6) and (7.7) we observe that since $\phi > \omega(B)$ it is immediate that

$$\sup_{\phi \le |\theta|} \|G_\theta\|_{bv[a,b]} < \infty.$$

Thus we can treat $(0, a)$ and (b, ∞) separately. In this case we observe that

$$(1 + G_\theta(t))(1 + T) = (1 + F_\theta(t))(1 - T_0(1 + F_\theta(t)))^{-1}$$

and the proposition follows. □

We first give an improved form of Arendt and Batty [2], who proved the same result for nuclear perturbations when A is invertible.

Theorem 7.5

Suppose A is a sectorial operator with an H^∞-calculus and suppose B is an absolutely summing perturbation of A which is sectorial. Then B has an H^∞-calculus and $\omega_H(B) \le \max(\omega(B), \omega_H(A))$.

Proof. We use Proposition 7.3. Suppose $\max(\omega(B), \omega_H(A)) < \phi < \pi$. Let $B = (1+T)A$ where T is absolutely summing.

Since T is absolutely summing, we can apply Proposition 6.1: we thus have a uniform bound

$$\sup_{|\theta|\ge\phi} \|TF_\theta\|_{BV(0,\infty)} \le \pi_1(T) \sup_{|\theta|\ge\phi} \|F_\theta\|_{bv(0,\infty)} < \infty.$$

Let $F_\theta(t) = AR(te^{i\theta}, A)$ for $\phi \le |\theta| \le \pi$. Then

$$BR(te^{i\theta}, B) = (1+T)F_\theta(1-TF_\theta)^{-1} \qquad \phi \le |\theta| \le \pi.$$

Now combining Proposition 4.2 and Lemma 4.3 yields the existence of $0 < a < b < \infty$ and $n \in \mathbb{N}$ so that

$$\|(TF_\theta(t))^n\| \le \frac{1}{2} \qquad b \le t < \infty,\ \phi \le |\theta| \le \pi,$$

and

$$\|\,(T_0(1+F_\theta(t)))^n\,\| \le \frac{1}{2} \qquad 0 < t \le a,\ \phi \le |\theta| \le \pi.$$

We now verify the criteria of Proposition 7.4. For (7.6) we note by Proposition 6.5 (and Proposition 6.4) that we have a uniform bound

$$\sup_{|\theta|\ge\phi} \|(1-(TF_\theta)^n)^{-1}\|_{BV(b,\infty)} < \infty.$$

Now

$$(1-TF_\theta)^{-1} = \left(1 + \sum_{k=1}^{n-1}(TF_\theta)^k\right)(1-(TF_\theta)^n)^{-1}$$

and so we also get a uniform bound

$$\sup_{|\theta|\ge\phi} \|(1-(TF_\theta))^{-1}\|_{BV(b,\infty)} < \infty.$$

Again applying Proposition 6.4 in particular (6.2) gives (7.6).

The treatment of (7.7) is similar. □

Theorem 7.6

Let A be a sectorial operator with an H^∞-calculus. Suppose $\omega_H(A) < \phi < \pi$. Then there exists a $\delta = \delta(A) > 0$ such that if $T : X \to X$ is a triangular operator with $\Theta(T) < \delta$ then $1+T$ is invertible and $B = (1+T)A$ is sectorial and admits an H^∞-calculus with $\omega_H(B) \le \phi$.

Proof. Since $\|T\| \le \Theta(T)$, it is immediate that if δ is small enough then B is sectorial with $\omega(B) < \phi$. The theorem is now a direct application of Theorem 6.6 and Proposition 7.4. □

We can now put these results together to obtain a cleaner result for strongly triangular perturbations. Recall that every triangular operator on a Hilbert space is strongly triangular, and if X is an arbitrary Banach space then an operator $T : X \to X$ for which

$$\sum_{n=1}^{\infty} \frac{a_n(T)\log n}{n} < \infty$$

is strongly triangular (see Proposition 5.10).

Theorem 7.7

Let A be a sectorial operator with an H^∞-calculus, and let B be a strongly triangular perturbation of A which is sectorial. Then B admits an H^∞-calculus and $\omega_H(B) \le \max(\omega_H(A), \omega(B))$.

Proof. Suppose $\max(\omega_H(A), \omega(B)) < \phi < \pi$. Choose $\delta = \delta(A) > 0$ as given by Theorem 7.6 and then choose an absolutely summing operator V so that $\Theta(T-V) < \delta$. Let $B_0 = (1 + T - V)A$ so that B_0 is a sectorial operator with $\omega_H(B_0) \le \phi$. Now $B = B_0 + VA = (1 + V(1 + T - V)^{-1})B_0$ so we can apply Theorem 7.5 to deduce that B has an H^∞-calculus and $\omega_H(B) \le \max(\omega_H(B_0), \omega(B)) \le \max(\phi, \omega(B))$. ☐

Remark. By Theorem 4.4, if X has non-trivial type, $\omega_H(B) > \omega_H(A)$ if and only if there is an eigenvalue λ of B with $|\arg \lambda| > \omega_H(A)$.

We now give a special result for GT-spaces. Special results for operators with an H^∞-calculus in GT-spaces were first obtained in [19]. We remind the reader that the Banach spaces L_1 and ℓ_1 are GT-spaces.

Theorem 7.8

Let X be a GT-space and suppose A is a sectorial operator on X with an H^∞-calculus.

(i) *Suppose B is a compact perturbation of A which is sectorial; then B admits an H^∞-calculus with $\omega_H(B) \le \max(\omega_H(A), \omega(B))$.*

(ii) *Given $\omega_H(A) < \phi < \pi$ there exists $\delta > 0$ so that if $T : X \to X$ is a bounded operator with $\|T\| < \delta$ then $B = (1+T)A$ has an H^∞-calculus with $\omega_H(B) \le \phi$.*

Proof. If $\phi > \omega_H(A)$ we observe that for a suitable constant C we have

$$\int_0^\infty \|AR(te^{i\theta}, A)^2 x\| dt \le C\|x\| \qquad |\theta| \ge \phi.$$

This follows from [19, Proposition 7.1]; note that this proposition has a redundant assumption that X has cotype 2, as was pointed out in [18]. To derive the statement from in [19, Proposition 7.1] take $s = \frac{1}{2}$ and note that

$$\int_0^\infty \|AR(te^{i\theta}, A)^2 x\| dt \le \sup_{0<t<\infty} \|t^{1/2} A^{1/2} R(te^{i\theta}, A)\| \int_0^\infty \|A^{1/2} R(te^{i\theta}, A)x\| \frac{dt}{t^{1/2}}.$$

Thus in this if $F_\theta(t) = AR(te^{i\theta}, A)$ then $F_\theta \in BV(0,\infty)$ for $|\theta| \ge \phi$ and

$$\sup_{|\theta|\ge\phi} \|F_\theta\|_{BV(0,\infty)} < \infty.$$

Now in case (i) we observe that

$$\lim_{t\to\infty} \sup_{|\theta|\ge\phi} \|F_\theta(t)T\| = 0$$

and so

$$\lim_{t\to\infty} \sup_{|\theta|\ge\phi} \|(TF_\theta(t))^2\| = 0.$$

Similarly

$$\lim_{t\to 0} \sup_{|\theta|\ge\phi} \| \left(T_0(1+F_\theta(t))\right)^2 \| = 0.$$

Thus the method of Theorem 7.5 applies to give the conclusion (using Proposition 6.5).

In case (ii) the proof follows directly from Proposition 6.5. □

8. Examples

We now present an example to show that in a Hilbert space our result is essentially best possible.

First suppose X is any Banach space with a 1-unconditional basis $(e_n)_{n=1}^\infty$ and let $(e_n^*)_{n=1}^\infty$ be the biorthogonal functionals. Then any operator $T : X \to X$ is represented by a matrix $(t_{jk})_{j,k=1}^\infty$ where $t_{jk} = e_j^*(Te_k)$. We will therefore talk in terms of matrix transformations.

Let $(\mu_j)_{j=1}^\infty$ be a sequence of distinct complex numbers contained in a sector Σ_ϕ where $0 < \phi < \pi$. Let A be the sectorial operator defined by

$$A\Big(\sum_{j=1}^\infty \xi_j e_j\Big) = \sum_{j=1}^\infty \mu_j \xi_j e_j$$

with domain consisting of all $\sum_{j=1}^\infty \xi_j e_j$ so that the right-hand side converges. Then A is sectorial and has an H^∞-calculus with $\omega_H(A) \le \phi$.

Proposition 8.1

Under the above assumptions, let $T : X \to X$ be a bounded operator with matrix $(t_{jk})_{j,k}$. Suppose for some $\delta > 0$, $\phi < \psi < \pi$ and constant C the perturbations $B(\zeta) = (1+\zeta T)A$ are sectorial for $|\zeta| \le \delta$ and have an $H^\infty(\Sigma_\psi)$-calculus with uniform bounds i.e

$$\|f(B(\zeta))\| \le C\|f\|_{H^\infty(\Sigma_\psi)} \qquad f \in H^\infty(\Sigma_\psi),\ |\zeta| \le \delta.$$

Then for each $f \in H^\infty(\Sigma_\psi)$ into $\mathcal{L}(X)$ the matrix $V(f) = (v_{jk})_{j,k}$ defines a bounded operator with $\|V(f)\| \le C\|f\|_{H^\infty(\Sigma_\psi)}$ where $v_{jj} = 0$ for all j

$$v_{jk} = t_{jk}\frac{\mu_k(f(\mu_j) - f(\mu_k))}{\mu_j - \mu_k} \qquad j \ne k.$$

Proof. It suffices to prove the estimate for $V(f)$ where $f \in H_0^\infty(\Sigma_\psi)$. Then for $\phi < \nu < \psi$ we can write

$$f(B(\zeta)) = \frac{1}{2\pi i}\int_{\Gamma_\nu} f(\lambda)R(\lambda, B(\zeta))d\lambda$$

and we have a uniform estimate (for a suitable choice of C_1),

$$\|f(B(\zeta))\| \le C_1 \|f\|_{H^\infty(\Sigma_\psi)} \qquad |\zeta| \le \delta.$$

The map $F(\zeta) = f(B(\zeta))$ is analytic on $|\zeta| < \delta$ and so

$$\|F'(0)\| \le C_1 \delta^{-1} \|f\|_{H^\infty(\Sigma_\psi)}.$$

Now

$$F'(0) = \frac{1}{2\pi i} \int_{\Gamma_\nu} f(\lambda) R(\lambda, A) T A R(\lambda, A) d\lambda$$

and so

$$\begin{aligned} \langle F'(0) e_k, e_j^* \rangle &= \frac{1}{2\pi i} \int_{\Gamma_\nu} f(\lambda) \langle T A R(\lambda, A) e_k, R(\lambda, A)^* e_j^* \rangle d\lambda \\ &= \frac{1}{2\pi i} \int_{\Gamma_\nu} \mu_k t_{jk} \frac{f(\lambda)}{(\lambda - \mu_j)(\lambda - \mu_k)} d\lambda. \end{aligned}$$

If $j \ne k$ we rewrite

$$\frac{f(\lambda)}{(\lambda - \mu_j)(\lambda - \mu_k)} = \frac{1}{\mu_j - \mu_k} \left(\frac{f(\lambda)}{\lambda - \mu_j} - \frac{f(\lambda)}{\lambda - \mu_k} \right)$$

and integrating gives

$$\langle F'(0) e_k, e_j^* \rangle = t_{jk} \frac{\mu_k (f(\mu_j) - f(\mu_k))}{\mu_j - \mu_k}.$$

If $j = k$ we obtain

$$\langle F'(0) e_j, e_j^* \rangle = t_{jj} \mu_j f'(\mu_j).$$

If we consider the matrix $V(f)$ defined by $F'(0)$ with the diagonal replaced by zero then $\|V(f)\| \le 2\|F'(0)\|$ and so we are done. $\square$

Proposition 8.2

Under the hypotheses of the preceding proposition, suppose that $\mu_j = 2^j$. Then $\Delta(T)$ i.e. the operator whose matrix corresponds to the lower triangular part of T is bounded on X.

Proof. Let us denote by $\delta(j,k)$ the Kronecker delta and let $\Delta(j,k) = 1$ if $j \ge k$ and 0 otherwise.

We first note for any matrix $V = (v_{jk})_{j,k}$ with zero diagonal, we have an estimate on each super-diagonal or sub-diagonal i.e

$$\|(\delta(n, j-k) v_{jk})_{j,k}\| \le \|V\| \qquad n = \pm 1, \pm 2, \ldots.$$

This implies that

$$\Big\| \Big(\frac{\mu_{\min(j,k)}}{\mu_j - \mu_k} \delta(n, j-k) v_{jk} \Big)_{j,k} \Big\| \le \frac{1}{2^{|n|} - 1} \|V\| \qquad n = \pm 1, \pm 2, \ldots.$$

and so

$$\Big\|\Big(\frac{\mu_{\min(j,k)}}{\mu_j-\mu_k}v_{jk}\Big)_{j,k}\Big\| \le 4\|V\|.$$

It thus follows that if $\epsilon_j = \pm 1$ we have

$$\Big\|\Big(\frac{\mu_{\min(j,k)}}{\mu_j-\mu_k}(\epsilon_j-\epsilon_k)t_{jk}\Big)_{j,k}\Big\| \le 8\|T\|.$$

Now for any sequence $\epsilon_j = \pm 1$ we can find $f = f_{(\epsilon_j)} \in H^\infty(\Sigma_\psi)$ with $\|f\|_{H^\infty(\Sigma_\psi)} \le K$ independent of (ϵ_j) so that

$$f(\mu_j) = \epsilon_j \qquad j = 1, 2, \ldots.$$

By Proposition 8.1,

$$\Big\|\Big(\frac{\mu_k}{\mu_j-\mu_k}(\epsilon_j-\epsilon_k)t_{jk}\Big)_{j,k}\Big\| \le CK$$

and so

$$\|((1-\Delta(j,k))(\epsilon_j-\epsilon_k)t_{jk})_{j,k}\| \le CK + 8\|T\|.$$

Therefore

$$\|((1-\Delta(j,k))(1-\epsilon_j\epsilon_k)t_{jk})_{j,k}\| \le CK + 8\|T\|.$$

Now treating the ϵ_j as Rademachers and averaging gives

$$\|((1-\Delta(j,k))t_{jk})_{j,k}\| \le CK + 8\|T\|,$$

and so

$$\|(\Delta(j,k)t_{jk})_{j,k}\| \le CK + 9\|T\|.$$

□

Theorem 8.3

Let $\mathcal{H}$ be a separable Hilbert space and let $(e_n)_{n=1}^\infty$ be an orthonormal basis. Let A be the sectorial operator defined by $Ae_n = 2^n e_n$ with domain all $x \in \mathcal{H}$ such that $\sum_{n=1}^\infty 2^{2n}|(x,e_n)|^2 < \infty$. Suppose $L : \mathcal{H} \to \mathcal{H}$ is any non-triangular compact operator. Then there exist bounded operators $U, V : \mathcal{H} \to \mathcal{H}$ such that for for every $m \in \mathbb{N}$, $B_m = (1 + 2^{-m}VLU)A$ fails to have an H^∞-calculus.

Proof. We may assume that $\|L\|$ is small enough so that $(1+L')A$ is always sectorial when $\|L'\| \le \|L\|$. Since L is non-triangular we have $\sum_{n=1}^\infty s_n(L)/n = \infty$. It follows that we can find a sequence $r_0 = 0 < r_1 < r_2 < \cdots$ of natural numbers and operators $V_n : H_n \to H_n$ where $H_n = [e_j]_{j=r_{n-1}+1}^{r_n}$ so that $\Theta(V_n) \to \infty$ and $V_1 \oplus V_2 \oplus \cdots = V$ satisfies $s_k(V) = s_k(L)$ for all k. Now using the argument of Theorem 5.5 we can find a unitary operator $U_n : H_n \to H_n$ so that $\|\Delta(U_nV_n)\| \to \infty$. Indeed we can find U_n so that

$$\|\Delta(U_nV_n)\| \ge c_1 \sum_{k=1}^{r_n-r_{n-1}} \frac{s_k(V_n)}{k} \ge c_2\Theta(V_n)$$

where $c_1, c_2 > 0$. Let $T = U_1V_1 \oplus U_2V_2 \oplus \cdots$. Note that $\|T\| \le \|V\| = \|L\|$.

We now partition $\mathbb{N}$ into countably many infinite subsets $\mathbb{M}_{l,m,n}$. Let

$$H_{l,m,n} = [\cup\{e_j :\ r_{k-1}+1 \le j \le r_k,\ k \in \mathbb{M}_{l,m,n}\}].$$

Let $P_{l,m,n}$ be the orthogonal projection of H onto $H_{l,m,n}$ and let $T_{l,m,n} = TP_{l,m,n} = P_{l,m,n}TP_{l,m,n}$ be the compression of T to $H_{l,m,n}$. Then we may argue that the operator $(1+\zeta T_{l,m,n})A$ cannot have a uniformly bounded $H^\infty(\Sigma_{\pi-2^{-m}})-$calculus for all $|\zeta| \le 2^{-n}$. Indeed if it did, we could conclude from Proposition 8.2 that $\Delta(T_{l,m,n})$ is a bounded operator, which is false from our construction. Thus we may find $\zeta_{l,m,n}$ with $|\zeta_{l,m,n}| \le 2^{-n}$ and such that for some $f_{m,n,l} \in H_0^\infty(\Sigma_{\pi-2^{-m}})$ with $\|f_{l,m,n}\|_{H^\infty(\Sigma_{\pi-2^{-m}})} \le 1$ but

$$\|f_{l,m,n}((1+\zeta_{l,m,n}T_{l,m,n})A)\| > 2^l.$$

Since A has an $H^\infty(\Sigma_{\pi-1/2})$-calculus we certainly have an estimate $\|f_{l,m,n}(A)\| \le C$ for some fixed constant C independent of l, m and n. Now this implies

$$\|(1-P_{l,m,n})f_{l,m,n}((1+\zeta_{l,m,n}T_{l,m,n})A)(1-P_{l,m,n})\| \le C$$

and hence

$$\|P_{l,m,n}f_{l,m,n}((1+\zeta_{l,m,n}T_{l,m,n})A)P_{l,m,n}\| \ge 2^l$$

if $2^l > C$.

Finally let $T' = z_1U_1V_1 \oplus z_2U_2V_2 \oplus \cdots$ where $z_j = 2^n\zeta_{l,m,n}$ if $j \in \mathbb{M}_{l,m,n}$. Suppose $B_k = (1+2^{-k}T')A$; then B_k is sectorial. Let us suppose that B_k has an $H^\infty(\Sigma_\phi)$-calculus for some $0<\phi<\pi$. For $f \in H_0^\infty(\Sigma_\phi)$ we have

$$P_{l,m,n}f(B_k)P_{l,m,n} = P_{l,m,n}f((1+2^{n-k}\zeta_{l,m,n}T_{l,m,n})A))P_{l,m,n}.$$

In particular if $2^l > C$ and $\phi < \pi - 2^{-m}$ we have

$$\|P_{l,m,k}f_{l,m,n}(B_k)P_{l,m,k}\| > 2^l.$$

This implies $\|f_{l,m,k}(B_k)\| > 2^l$ and gives a contradiction. $\square$

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