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Multilinear Calderón-Zygmund operators on Hardy spaces

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Abstract

It is shown that multilinear Calderón-Zygmund operators are bounded on products of Hardy spaces.

1. Introduction

The study of multilinear singular integral operators has recently received increasing attention. In analogy to the linear theory, the class of multilinear singular integrals with standard Calderón-Zygmund kernels provide the foundation and starting point of investigation of the theory. The class of multilinear Calderón-Zygmund operators was introduced and first investigated by Coifman and Meyer [3], [4], [5], and was later systematically studied by Grafakos and Torres [7].

In this article we take up the issue of boundedness of multilinear Calderón-Zygmund operators on products of Hardy spaces. As in the linear theory, a certain amount of extra smoothness is required for these operators to have such boundedness properties. We will assume that $K(y_0, y_1, \ldots, y_m)$ is a function defined *away from the diagonal* $y_0 = y_1 = \ldots = y_m$ in $(\mathbb{R}^n)^{m+1}$ which satisfies the following estimates

$$\big|\partial_{y_0}^{\alpha_0} \ldots \partial_{y_m}^{\alpha_m} K(y_0, y_1, \ldots, y_m)\big| \le \frac{A_\alpha}{\big(\sum\limits_{k,l=0}^{m} |y_k - y_l|\big)^{mn+|\alpha|}}, \qquad \text{for all } |\alpha| \le N, \tag{1}$$

where $\alpha = (\alpha_0, \ldots, \alpha_m)$ is an ordered set of n-tuples of nonnegative integers, $|\alpha| = |\alpha_0| + \ldots + |\alpha_m|$, where $|\alpha_j|$ is the order of each multiindex α_j, and N is a large integer

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to be determined later. We will call such functions K multilinear standard kernels. We assume throughout that T is a weakly continuous m-linear operator defined on products of test functions such that for some multilinear standard kernel K, the integral representation below is valid

$$T(f_1,\ldots,f_m)(x)=\int_{\mathbb{R}^n}\cdots\int_{\mathbb{R}^n}K(x,y_1,\ldots,y_m)\prod_{j=1}^m f_j(y_j)\,dy_1,\ldots dy_m, \tag{2}$$

whenever f_j are smooth functions with compact support and $x\notin\cap_{j=1}^m\operatorname{supp}f_j$. In the case $m=1$ conditions (1) are called standard estimates and operators given by (2) are called Calderón-Zygmund if they are bounded from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$. We will adopt the same terminology in the multilinear case and call T a multilinear Calderón-Zygmund operator if it is associated to a multilinear standard kernel as in (2) and has a bounded extension from a product of some L^{q_j} spaces into another L^q space with $1/q=1/q_1+\ldots+1/q_m$. If this is the case, it was shown in [7] that these operators map any other product of Lebesgue spaces $\prod_{j=1}^m L^{p_j}(\mathbb{R}^n)$ with $p_j>1$ into the corresponding L^p space.

When $m=1$, bounded extensions for Calderón-Zygmund operators on the Hardy spaces H^p were obtained by Fefferman and Stein [6]. Here $H^p=H^p(\mathbb{R}^n)$ denotes the real Hardy space in [6] defined for $0<p\le 1$. In this note we provide analogous bounded extensions for multilinear Calderón-Zygmund on products of Hardy spaces. The following theorem is our main result:

Theorem 1.1

Let $1<q_1,\ldots,q_m,q<\infty$ be fixed indices satisfying

$$\frac{1}{q_1}+\ldots+\frac{1}{q_m}=\frac{1}{q} \tag{3}$$

and let $0<p_1,\ldots,p_m,p\le 1$ be real numbers satisfying

$$\frac{1}{p_1}+\ldots+\frac{1}{p_m}=\frac{1}{p}. \tag{4}$$

Suppose that K satisfies (1) with $N=[n(1/p-1)]$. Let T be related to K as in (2) and assume that T admits an extension that maps $L^{q_1}(\mathbb{R}^n)\times\ldots\times L^{q_m}(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$ with norm B. Then T extends to a bounded operator from $H^{p_1}(\mathbb{R}^n)\times\ldots\times H^{p_m}(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$ which satisfies the norm estimate

$$\|T\|_{H^{p_1}\times\ldots\times H^{p_m}\to L^p}\le C\Big(B+\sum_{|\alpha|\le N+1}A_\alpha\Big),$$

for some constant $C=C(n,p_j,q_j)$.

2. The proof of Theorem 1.1

Proof. We prove the theorem using the atomic decomposition of H^p. See Coifman [1] and Latter [11]. Since finite sums of atoms are dense in H^p we will work with such sums and we will obtain estimates independent of the number of terms in each sum. The general case will follow by a simple density argument. Write each f_j, $1 \le j \le m$ as a finite sum of H^{p_j}-atoms $f_j = \sum_k \lambda_{j,k} a_{j,k}$, where $a_{j,k}$ are H^{p_j}-atoms. This means that the $a_{j,k}$'s are functions supported in cubes $Q_{j,k}$ and satisfy the properties

$$|a_{j,k}| \le |Q_{j,k}|^{-1/p_j} \tag{5}$$

$$\int_{Q_{j,k}} x^\gamma a_{j,k}(x)\,dx = 0, \tag{6}$$

for all $|\gamma| \le [n(1/p_j - 1)]$. By the theory of H^p spaces, see [12] page 112, we can take the atoms $a_{j,k}$ to have vanishing moments up to any large fixed specified integer. In this article we will assume that all the $a_{j,k}$'s satisfy (6) for all $|\gamma| \le [n(1/p-1)]$. For a cube Q, let Q^* denote the cube with the same center and $2\sqrt{n}$ its side length, i.e. $l(Q^*) = 2\sqrt{n}l(Q)$. Using multilinearity we write

$$T(f_1,\dots,f_m)(x) = \sum_{k_1}\dots\sum_{k_m} \lambda_{1,k_1}\dots\lambda_{m,k_m} T(a_{1,k_1},\dots,a_{m,k_m})(x). \tag{7}$$

We now fix $k_1,\dots,k_m$ and $x \in \mathbb{R}^n$ and we consider the following cases:

Case 1: $x \in Q^*_{1,k_1} \cap \dots \cap Q^*_{m,k_m}$.

Case 2: x lies in the complement of at least one of the cubes Q^*_{j,k_j}.

Let us begin with case 2. We fix $1 \le r \le m$ and without loss of generality (by permuting the indices) we assume that $x \in Q^*_{r+1,k_{r+1}} \cap \dots \cap Q^*_{m,k_m}$ and that $x \notin Q^*_{1,k_1} \cup \dots \cup Q^*_{r,k_r}$. Assume without loss of generality that the side length of the cube Q_{1,k_1} is the smallest among the side lengths of the cubes $Q_{1,k_1},\dots,Q_{r,k_r}$. Let c_{j,k_j} be the center of the cube Q_{j,k_j}. Since a_{1,k_1} has zero vanishing moments up to order $N = [n(1/p_1 - 1)]$, we can subtract the Taylor polynomial $P^N_{c_{1,k_1}}(x,\,\cdot\,,y_2,\dots,y_m)$ of the function $K(x,\,\cdot\,,y_2,\dots,y_m)$ at the point c_{1,k_1} to obtain

$$\begin{aligned}
&T(a_{1,k_1},\dots,a_{m,k_m})(x)\\
&= \int_{(\mathbb{R}^n)^{m-1}} \prod_{j=2}^m a_{j,k_j}(y_j) \int_{\mathbb{R}^n} a_{1,k_1}(y_1)\big[K(x,y_1,\dots,y_m) - P^N_{c_{1,k_1}}(x,y_1,y_2,\dots,y_m)\big]\,d\vec{y}\\
&= \int_{(\mathbb{R}^n)^{m-1}} \prod_{j=2}^m a_{j,k_j}(y_j) \int_{\mathbb{R}^n} a_{1,k_1}(y_1) \sum_{|\gamma|=N+1} (\partial^\gamma_{y_1}K)(x,\xi,y_2,\dots,y_m)\frac{(y_1 - c_{1,k_1})^\gamma}{\gamma!}\,d\vec{y},
\end{aligned}$$

for some ξ on the line segment joining y_1 to c_{1,k_1} by Taylor's theorem. We have

$$|x-\xi| \ge |x - c_{1,k_1}| - |\xi - c_{1,k_1}| \ge |x - c_{1,k_1}| - \frac{1}{2}\sqrt{n}\,l(Q_{1,k_1}) \ge \frac{1}{2}|x - c_{1,k_1}|\,,$$

since $x \notin Q^*_{1,k_1} = 2\sqrt{n}Q_{1,k_1}$. Similarly we obtain $|x-y_j| \geq \frac{1}{2}|x-c_{j,k_j}|$ for $j \in \{2,3,\dots,r\}$. Set

$$A = \sum_{|\gamma|\leq N+1} A_\gamma \tag{8}$$

and note that $A \geq \sum_{|\gamma|=N+1} A_\gamma$. The estimates for the kernel K and the size estimates for the atoms give the following pointwise bound for the expression above:

$$\underbrace{\int_{\mathbb{R}^n}\cdots\int_{\mathbb{R}^n}}_{m-r \text{ times}} \Big(\sum_{|\gamma|=N+1} A_\gamma\Big)\Big(\int_{Q_{1,k_1}} |a_{1,k_1}(y_1)|\,|y_1-c_{1,k_1}|^{N+1}dy_1\Big)$$
$$\frac{|Q_{2,k_2}|^{1-1/p_2}\dots|Q_{r,k_r}|^{1-1/p_r}|Q_{r+1,k_{r+1}}|^{-1/(p_{r+1})}\dots|Q_{m,k_m}|^{-1/p_m}dy_m\dots dy_{r+1}}{\big(\frac{1}{2}|x-c_{1,k_1}|+\dots+\frac{1}{2}|x-c_{r,k_r}|+|x-y_{r+1}|+\dots+|x-y_m|\big)^{nm+N+1}}.$$

Integrating the above over $y_m \in \mathbb{R}^n,\dots,y_{r+1}\in\mathbb{R}^n$ we obtain that the expression above is bounded by a constant multiple of

$$\frac{A|Q_{2,k_2}|^{1-1/p_2}\dots|Q_{r,k_r}|^{1-1/p_r}\displaystyle\int_{Q_{1,k_1}}|a_{1,k_1}(y_1)|\,|y_1-c_{1,k_1}|^{N+1}dy_1}{\big(\frac{1}{2}|x-c_{1,k_1}|+\dots+\frac{1}{2}|x-c_{r,k_r}|\big)^{nm+N+1-n(m-r)}|Q_{r+1,k_{r+1}}|^{1/(p_{r+1})}\dots|Q_{m,k_m}|^{1/p_m}}.$$

But the integral above is easily seen to be controlled by a constant multiple of $|Q_{1,k_1}|^{1-1/p_2+(N+1)/n}$. Since the cube Q_{1,k_1} was picked to have the smallest size among the $Q_{1,k_1},\dots,Q_{r,k_r}$, the expression above is bounded by a constant multiple of

$$\begin{aligned}
&A\prod_{j=1}^{r}\frac{|Q_{j,k_j}|^{1-1/p_j+(N+1)/nr}}{\big(|x-c_{j,k_j}|+l(Q_{j,k_j})\big)^{n+(N+1)/r}}\,|Q_{r+1,k_{r+1}}|^{-1/(p_{r+1})}\dots|Q_{m,k_m}|^{-1/p_m}\\
&\leq C\,A\prod_{j=1}^{m}\frac{|Q_{j,k_j}|^{1-1/p_j+(N+1)/nr}}{\big(|x-c_{j,k_j}|+l(Q_{j,k_j})\big)^{n+(N+1)/r}},
\end{aligned}$$

since $x \in Q^*_{r+1,k_{r+1}}\cap\dots\cap Q^*_{m,k_m}$.

Summing over all possible $1\leq r\leq m$ and all possible combinations of subsets of $\{1,\dots,m\}$ of size r we obtain the pointwise estimate

$$\big|T(a_{1,k_1},\dots,a_{m,k_m})(x)\big| \leq CA\prod_{j=1}^{m}\frac{|Q_{j,k_j}|^{1-1/p_j+(N+1)/nm}}{\big(|x-c_{j,k_j}|+l(Q_{j,k_j})\big)^{n+(N+1)/m}} \tag{9}$$

for all x which belong to the complement of at least one Q^*_{j,k_j} (case 2).

Now using (7) and (9) we obtain

$$\big|T(f_1,\dots,f_m)(x)\big| \leq G_1(x)+G_2(x),$$

where $G_1(x)$ and $G_2(x)$ correspond to cases 1 and 2 respectively and are given by

$$G_1(x) = \sum_{k_1} \dots \sum_{k_m} |\lambda_{1,k_1}| \dots |\lambda_{m,k_m}| |T(a_{1,k_1}, \dots, a_{m,k_m})(x)| \chi_{Q^*_{1,k_1} \cap \dots \cap Q^*_{m,k_m}}(x)$$

$$G_2(x) = C\,A \prod_{j=1}^{m} \Big(\sum_{k_j} |\lambda_{j,k_j}| \frac{|Q_{j,k_j}|^{1-1/p_j+(N+1)/nm}}{\big(|x-c_{j,k_j}| + l(Q_{j,k_j})\big)^{n+(N+1)/m}} \Big).$$

Applying Hölder's inequality with exponents $p_1, \dots, p_m$ and p we obtain the estimate

$$\begin{aligned} \|G_2\|_{L^p} \leq & CA \prod_{j=1}^{m} \Big\| \sum_{k_j} |\lambda_{j,k_j}| \frac{|Q_{j,k_j}|^{1-1/p_j+(N+1)/nm}}{\big(|x-c_{j,k_j}| + l(Q_{j,k_j})\big)^{n+(N+1)/m}} \Big\|_{L^{p_j}} \\ \leq & C' A \prod_{j=1}^{m} \Big(\sum_{k_j} |\lambda_{j,k_j}|^{p_j} \Big)^{1/p_j} \leq C' A \prod_{j=1}^{m} \|f_j\|_{H^{p_j}}, \end{aligned} \tag{10}$$

where we used the p_j-subadditivity of the L^{p_j} quasi-norm and the easy fact that the functions

$$\frac{|Q_{j,k_j}|^{1-1/p_j+(N+1)/nm}}{\big(|x-c_{j,k_j}| + l(Q_{j,k_j})\big)^{n+(N+1)/m}}$$

have L^{p_j} norms bounded by constants.

We now turn our attention to case 1. Here we will show that

$$\|G_1\|_{L^p} \leq C(A+B) \prod_{j=1}^{m} \|f_j\|_{H^{p_j}}. \tag{11}$$

To prove (11) we will need the following lemma whose proof we postpone until the next section.

Lemma 2.1

Let $0 < p \leq 1$. Then there is a constant $C(p)$ such that for all finite collections of cubes $\{Q_k\}_{k=1}^{K}$ in $\mathbb{R}^n$ and all nonnegative integrable functions g_k with supp $g_k \subset Q_k$ we have

$$\Big\| \sum_{k=1}^{K} g_k \Big\|_{L^p} \leq C(p) \Big\| \sum_{k=1}^{K} \Big(\frac{1}{|Q_k|} \int_{Q_k} g_k(x)\, dx \Big) \chi_{Q_k^*} \Big\|_{L^p}.$$

We momentarily assume Lemma 2.1 and we prove (11). Using the assumption that T maps $L^{q_1} \times \dots \times L^{q_m}$, it was proved in [7] that T maps all possible combinations of products

$$L^\infty_c \times \dots \times L^\infty_c \times L^2 \times L^\infty_c \times \dots \times L^\infty_c$$

into L^2 with norm at most a multiple of $A+B$. L^∞_c denotes here the space of all L^∞ functions with compact support.

Now fix atoms a_{1,k_1}, …, a_{m,k_m} supported in cubes Q_{1,k_1}, …, Q_{m,k_m} respectively. Assume that $Q^*_{1,k_1} \cap \ldots \cap Q^*_{m,k_m} \neq \emptyset$, otherwise there is nothing to prove. Since $Q^*_{1,k_1} \cap \ldots \cap Q^*_{m,k_m} \neq \emptyset$, we can pick a cube $R_{k_1,\ldots,k_m}$ such that

$$Q^*_{1,k_1} \cap \ldots \cap Q^*_{m,k_m} \subset R_{k_1,\ldots,k_m} \subset R^*_{k_1,\ldots,k_m} \subset Q^{**}_{1,k_1} \cap \ldots \cap Q^{**}_{m,k_m} \tag{12}$$

and $|R_{k_1,\ldots,k_m}| \geq c|Q_{1,k_1}|$.

Without loss of generality assume that Q_{1,k_1} has the smallest size among all these cubes. Since T maps $L^2 \times L^\infty \times \ldots \times L^\infty$ into L^2 we obtain that

$$\begin{aligned} &\int_{R_{k_1,\ldots,k_m}} |T(a_{1,k_1},\ldots,a_{m,k_m})(x)|\,dx \\ \leq &\Big(\int_{\mathbb{R}^n} |T(a_{1,k_1},\ldots,a_{m,k_m})(x)|^2\,dx\Big)^{1/2} |R_{k_1,\ldots,k_m}|^{1/2} \\ \leq &C(A+B)|Q^{**}_{1,k_1}|^{1/2}|Q_{1,k_1}|^{1/2-1/p_1}\prod_{j=2}^m |Q_{j,k_j}|^{-1/p_j}, \end{aligned} \tag{13}$$

since $\|a_{1,k_1}\|_{L^2} \leq |Q_{1,k_1}|^{1/2-1/p_1}$ and $\|a_{j,k_j}\|_{L^\infty} \leq |Q_{j,k_j}|^{-1/p_j}$. It follows from (13) that

$$\int_{R_{k_1,\ldots,k_m}} |T(a_{1,k_1},\ldots,a_{m,k_m})(x)|\,dx \leq C(A+B)|Q_{1,k_1}|\prod_{j=1}^m |Q_{j,k_j}|^{-1/p_j}$$

which combined with $|R_{k_1,\ldots,k_m}| \geq c|Q_{1,k_1}|$ gives

$$\frac{1}{|R_{k_1,\ldots,k_m}|}\int_{R_{k_1,\ldots,k_m}} |T(a_{1,k_1},\ldots,a_{m,k_m})(x)|\,dx \leq C(A+B)\prod_{j=1}^m |Q_{j,k_j}|^{-1/p_j}. \tag{14}$$

We now have the easy estimate

$$G_1(x) \leq \sum_{k_1}\ldots\sum_{k_m} |\lambda_{1,k_1}|\ldots|\lambda_{m,k_m}||T(a_{1,k_1},\ldots,a_{m,k_m})(x)|\chi_{R_{k_1,\ldots,k_m}}(x),$$

and using Lemma 2.1, estimate (14), and the last inclusion in (12) we obtain

$$\begin{aligned} \big\|G_1\big\|_{L^p} \leq &C(A+B)\Big\|\sum_{k_1}\ldots\sum_{k_m}|\lambda_{j,k_j}|\ldots|\lambda_{m,k_m}|\prod_{j=1}^m |Q_{j,k_j}|^{-1/p_j}\chi_{Q^{**}_{1,k_1}}\ldots\chi_{Q^{**}_{m,k_m}}\Big\|_{L^p} \\ \leq &C(A+B)\Big\|\prod_{j=1}^m\Big(\sum_{k_j}|\lambda_{j,k_j}||Q_{j,k_j}|^{-1/p_j}\chi_{Q^{**}_{j,k_j}}\Big)\Big\|_{L^p} \\ \leq &C(A+B)\prod_{j=1}^m\Big\|\Big(\sum_{k_j}|\lambda_{j,k_j}||Q_{j,k_j}|^{-1/p_j}\chi_{Q^{**}_{j,k_j}}\Big)\Big\|_{L^{p_j}} \\ \leq &C(A+B)\prod_{j=1}^m\big\|f_j\big\|_{H^{p_j}}. \end{aligned}$$

This proves (11) which combined with (10) completes the proof of the theorem. □

3. The proof of Lemma 2.1

It remains to prove Lemma 2.1. This lemma will be a consequence of the lemma below. Let $\mathcal{D}$ be the collection of all dyadic cubes on $\mathbb{R}^n$ and $\mathcal{D}_j$ be the set of all cubes in $\mathcal{D}$ with side length $l(Q) = 2^{-j}$.

Lemma 3.1

Suppose $0 < p \leq 1$. Then there is a constant $C(p)$ such that for all finite subsets $\mathcal{J}$ of $\mathcal{D}$ and all collections $\{f_Q : Q \in \mathcal{J}\}$ of non-negative integrable functions on $\mathbb{R}^n$ with supp $f_Q \subset Q$ *we have*

$$\Big\| \sum_{Q \in \mathcal{J}} f_Q \Big\|_{L^p} \leq C(p) \Big\| \sum_{Q \in \mathcal{J}} a_Q \chi_Q \Big\|_{L^p},$$

where

$$a_Q = |Q|^{-1} \int_Q f_Q(x)\, dx.$$

Proof. Let us set $\mathcal{J}_m = \mathcal{J} \cap \mathcal{D}_m$ for all $m \in \mathbb{Z}$. Given $Q \in \mathcal{J}$, we define $s(Q)$ to be the unique m such that $Q \in \mathcal{J}_m$. We also set

$$F = \sum_{Q \in \mathcal{J}} f_Q, \qquad G = \sum_{Q \in \mathcal{J}} a_Q \chi_Q, \qquad G_m = \sum_{k=-\infty}^{m} \sum_{Q \in \mathcal{J}_k} a_Q \chi_Q.$$

We now observe that if $Q \in \mathcal{J}$ and $m \leq s(Q)$, then G_m is constant on Q. Therefore for $j \in \mathbb{Z}$ the sets below are well-defined

$$\begin{aligned} \mathcal{R}_j &= \big\{ Q \in \mathcal{J} : \ G_{s(Q)} \leq 2^j \quad \text{on } \ Q \big\}, \\ \mathcal{R}'_j &= \big\{ Q \in \mathcal{J} : \ G_{s(Q)} > 2^j \quad \text{on } \ Q \ \text{ and } \ G_{s(Q)-1} \leq 2^j \quad \text{on } \ Q \big\}. \end{aligned}$$

For $Q \in \mathcal{R}'_j$ and any $t \in Q$, we let

$$\lambda_Q = \frac{2^j - G_{s(Q)-1}(t)}{G_{s(Q)}(t) - G_{s(Q)-1}(t)}.$$

Note that λ_Q is a constant since both functions $G_{s(Q)}$ and $G_{s(Q)-1}$ are constant on Q.

We claim that for all $x \in \mathbb{R}^n$ we have the identity

$$\sum_{Q \in \mathcal{R}_j} a_Q \chi_Q(x) + \sum_{Q \in \mathcal{R}'_j} \lambda_Q a_Q \chi_Q(x) = \min\big(2^j, G(x)\big). \tag{15}$$

To prove (15) observe that if $G(x) \leq 2^j$, then $\mathcal{R}'_j = \emptyset$ and the conclusion easily follows. Otherwise, there is a smallest $m = m(x)$ such that

$$2^j < \sum_{k=-\infty}^{m} \sum_{Q \in \mathcal{J}_k} a_Q \chi_Q(x).$$

Then all the cubes that contain x from the collection $\cup_{k\le m-1}\mathcal{J}_k$ belong to $\mathcal{R}_j$ and the cube that contains x from $\mathcal{J}_m$ belongs to $\mathcal{R}'_j$. It follows that

$$\begin{aligned}\sum_{Q\in\mathcal{R}_j} a_Q\chi_Q(x) + \sum_{Q\in\mathcal{R}'_j} \lambda_Q a_Q\chi_Q(x)&\\ &= G_{m-1}(x) + \sum_{Q\in\mathcal{R}'_j}\big(2^j - G_{m-1}(x)\big)\chi_Q(x) = 2^j,\end{aligned}$$

since the last sum has only one term. This proves (15). Next we set

$$F_j = \sum_{Q\in\mathcal{R}_j} f_Q + \sum_{Q\in\mathcal{R}'_j}\lambda_Q f_Q.$$

Then using (15) we obtain

$$\begin{aligned}\int_{\mathbb{R}^n} F_j(x)\,dx &= \int_{\mathbb{R}^n}\bigg(\sum_{Q\in\mathcal{R}_j} a_Q\chi_Q(x) + \sum_{Q\in\mathcal{R}'_j}\lambda_Q a_Q\chi_Q(x)\bigg)dx\\ &= \int_{\mathbb{R}^n}\min\big(2^j, G(x)\big)\,dx. \qquad (16)\end{aligned}$$

It is easy to see that the function $F_j - F_{j-1}$ is supported in the set $\{G > 2^{j-1}\}$. The function $\min\big(2^j, G\big) - \min\big(2^{j-1}, G\big)$ is also supported in the set $\{G > 2^{j-1}\}$ and is bounded by 2^j. Using these facts, Hölder's inequality, and (16) we obtain

$$\begin{aligned}\int_{\mathbb{R}^n}\big(F_j(x) - F_{j-1}(x)\big)^p\,dx &\le \big|\{G>2^{j-1}\}\big|^{1-p}\bigg(\int_{\mathbb{R}^n} F_j(x) - F_{j-1}(x)\,dx\bigg)^p\\ \le \big|\{G>2^{j-1}\}\big|^{1-p}&\bigg(\int_{\mathbb{R}^n}\min\big(2^j, G(x)\big) - \min\big(2^{j-1}, G(x)\big)\,dx\bigg)^p\\ \le \big|\{G>2^{j-1}\}\big|^{1-p}&\big(2^j|\{G>2^{j-1}\}|\big)^p = 2^{jp}\big|\{G>2^{j-1}\}\big|.\end{aligned}$$

Summing the above over all $j\in\mathbb{Z}$ and using the fact that

$$F(x) = \sum_{j\in\mathbb{Z}}\big(F_j(x) - F_{j-1}(x)\big),$$

and that $p\le 1$, we obtain the required estimate

$$\int_{\mathbb{R}^n}\big(F(x)\big)^p\,dx \le \sum_{j\in\mathbb{Z}} 2^{jp}\big|\{G>2^{j-1}\}\big| \le C(p)^p\int_{\mathbb{R}^n}(G(x))^p\,dx,$$

where the last inequality follows by summation by parts. □

Having established Lemma 3.1, we now proceed to the proof of Lemma 2.1.

Proof. Given the cubes $\{Q_k\}_{k=1}^K$ we can find a finite collection of dyadic cubes $\{Q_{kj}\}_{j=1}^{m_k}$ with

$$l(Q_k) \le l(Q_{kj}) \le 2l(Q_k)$$

and

$$Q_k \subset \bigcup_{j=1}^{m_k} Q_{kj} \subset Q_k^*, \tag{17}$$

where $m_k \le 2^n$. We apply Lemma 3.1 to the functions $\{g_k \chi_{Q_{kj}}\}_{1\le k\le K}^{1\le j\le m_k}$. (We collapse terms when the same dyadic cube is used twice). We obtain

$$\Big\|\sum_{k=1}^K g_k\Big\|_{L^p} \le \Big\|\sum_{k=1}^K\sum_{j=1}^{m_k} g_k \chi_{Q_{kj}}\Big\|_{L^p} \le C(p)\Big\|\sum_{k=1}^K\sum_{j=1}^{m_k} b_{kj}\chi_{Q_{kj}}\Big\|_{L^p}, \tag{18}$$

where $b_{kj} = |Q_{kj}|^{-1}\int_{Q_{kj}} g_k(x)\,dx \le |Q_k|^{-1}\int_{Q_k} g_k(x)\,dx$. Inserting this estimate in (18) gives

$$\Big\|\sum_{k=1}^K g_k\Big\|_{L^p} \le C(p)\Big\|\sum_{k=1}^K \Big(|Q_k|^{-1}\int_{Q_k} g_k(x)\,dx\Big)\sum_{j=1}^{m_k}\chi_{Q_{kj}}\Big\|_{L^p},$$

and the required conclusion follows from the last inclusion in (17). $\square$

4. Related results and comments

We note that Theorem 1.1 can be extended to the case when some p_j's are bigger than 1 and the remaining p_j's are less than or equal to 1. We have the following:

Theorem 4.1

Let $1 < q_1, \ldots, q_m, q < \infty$ be fixed indices satisfying (3) and let $0 < p_1, \ldots, p_m, p < \infty$ be any real numbers satisfying (4). Suppose that K satisfies (1) for all $|\alpha| \le N$ where N is sufficiently large. Let T be related to K as in (2) and assume that T admits an extension that maps $L^{q_1}(\mathbb{R}^n)\times\ldots\times L^{q_m}(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$ with norm B. Then T extends to a bounded operator from $H^{p_1}(\mathbb{R}^n)\times\ldots\times H^{p_m}(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$ (we set $H^{p_j} = L^{p_j}$ when $p_j > 1$), which satisfies the norm estimate $\|T\|_{H^{p_1}\times\ldots\times H^{p_m}\to L^p} \le C(A+B)$ for some constant $C = C(n, p_j, q_j)$. (A is as in (8).)

Proof. We discuss the multilinear interpolation needed to prove this theorem for all indices $0 < p_j < \infty$. Theorem 4.1 is valid when all the p_j's satisfy $1 < p_j < \infty$ as proved in [7]. In Theorem 1.1 we considered the case when all $0 < p_j \le 1$.

We now fix indices $0 < p_j < \infty$ so that some of them are bigger than 1 and some of them are less than or equal to 1. We pick $\varepsilon > 0$ and $\lambda > 0$ so that

$$0 < \varepsilon < \min\Big(\frac{1}{m}, \frac{1}{p_1}, \ldots, \frac{1}{p_m}\Big), \qquad \lambda > \Big(\min\big(\frac{1}{p_1}, \ldots, \frac{1}{p_m}\big) - \varepsilon\Big)^{-1}.$$

Using that T is bounded from $L^{q_1} \times \ldots \times L^{q_m}$ into L^q with norm at most B, it follows from [7] that

$$T: \; L^{1/\varepsilon} \times \ldots \times L^{1/\varepsilon} \to L^{1/m\varepsilon} \tag{19}$$

with norm at most a constant multiple of $A + B$. Now define s_j by setting

$$\frac{1}{s_j} = \lambda\Big(\frac{1}{p_j} - \varepsilon\Big) + \frac{1}{p_j}. \tag{20}$$

Then it is easy to see that $0 < s_j < 1$ for all $1 \le j \le m$ and by Theorem 1.1 we have

$$T: \; H^{s_1} \times \ldots \times H^{s_m} \to L^s, \tag{21}$$

with norm at most a constant multiple of $A + B$, where $1/s = 1/s_1 + \ldots + 1/s_m$. Here we need (1) with $N = [n(1/s - 1)]$. Identity (20) gives

$$\frac{1}{p_j} = \frac{\theta}{s_j} + \frac{1-\theta}{1/\varepsilon}$$

where $\theta = (\lambda + 1)^{-1}$. Interpolating between (19) and (21) we obtain that

$$T: \; [L^{1/\varepsilon}, H^{s_1}]_\theta \times \ldots \times [L^{1/\varepsilon}, H^{s_m}]_\theta \to [L^{1/m\varepsilon}, L^s]_\theta = L^p,$$

where $1/p = 1/p_1 + \ldots + 1/p_m$. But $[L^{1/\varepsilon}, H^{s_j}]_\theta = L^{p_j}$ if $p_j > 1$ or H^{p_j} if $p_j \le 1$ and the required conclusion follows (see e.g. [10]). $\square$

We note that mapping into the Hardy space H^p instead of L^p is hopeless even in the translation invariant case unless some further cancellation is imposed. We refer to [8] and [2] for results of this sort. Both references deal with bilinear operators but the techniques can be adapted to give similar results for m-linear operators as well.

We now discuss some analogous results for the maximal singular integral operator defined by

$$T_*(f_1, \ldots, f_m)(x) = \sup_{\delta > 0} \big|T_\delta(f_1, \ldots, f_m)(x)\big|,$$

where T_δ are the smooth truncations of T given by

$$T_\delta(f_1, \ldots, f_m)(x) = \int_{\mathbb{R}^n} K_\delta(x, y_1, \ldots, y_m) f_1(y_1) \ldots f_m(y_m)\, dy_1 \ldots dy_m.$$

Here $K_\delta(x, y_1, \ldots, y_m) = \eta\big(\sqrt{|x - y_1|^2 + \ldots + |x - y_m|^2}/\delta\big) K(x, y_1, \ldots, y_m)$ and η is a smooth function on $\mathbb{R}^n$ which vanishes in a neighborhood of the origin and is equal to 1 outside a larger neighborhood of the origin.

It is proved in [9] that the sublinear operator T_* satisfies similar boundedness estimates as T. We have the following result regarding T_*.

Theorem 4.2

Under the same hypotheses as Theorem 4.1, T_* maps the product $H^{p_1}(\mathbb{R}^n) \times \ldots \times H^{p_m}(\mathbb{R}^n)$ boundedly into $L^p(\mathbb{R}^n)$, and satisfies the norm estimate

$$\|T_*\|_{H^{p_1} \times \ldots \times H^{p_m} \to L^p} \le C(A + B)$$

for some constant $C = C(n, p_j, q_j)$. As usually, we set $H^{p_j} = L^{p_j}$ when $p_j > 1$.

Proof. The proof is similar to that for T. First we consider the case where all the p_j's are less than or equal to one. It follows from [9] that T_* is bounded on the same range as T with bound at most a multiple of $A+B$. Thus the estimates in case 1 follow as before. Next observe that the kernels K_δ satisfy (1) uniformly in $\delta > 0$. Hence the estimates in case 2 for K equally apply to K_δ uniformly in $\delta > 0$ and the same conclusion follows.

The remainder of the argument is then similar. One treats the multilinear maps

$$T_{\delta_1,\ldots,\delta_N}(f_1,\ldots,f_m)(x) = \big\{T_{\delta_k}(f_1,\ldots,f_m)(x)\big\}_{k=1}^{N},$$

as maps $T_{\delta_1,\ldots,\delta_N} : H^{s_1} \times \ldots \times H^{s_m} \to L^s(\ell^N_\infty)$, for any finite set $\delta_1,\ldots,\delta_N > 0$ and uses complex interpolation as before. $\square$

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