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Rearrangement-Invariant Functionals with Applications to Traces on Symmetrically Normed Ideals

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Abstract. We present a construction of singular rearrangement invariant functionals on Marcinkiewicz function/operator spaces. The functionals constructed differ from all previous examples in the literature in that they fail to be symmetric. In other words, the functional ϕ fails the condition that if $x \prec\prec y$ (Hardy-Littlewood-Polya submajorization) and $0 \leq x, y$, then $0 \leq \phi(x) \leq \phi(y)$. We apply our results to singular traces on symmetric operator spaces (in particular on symmetrically-normed ideals of compact operators), answering questions raised by Guido and Isola.

1 Introduction

Let us describe our results in an important special case, although in the remainder of the paper, we will treat a much more general situation.

In non-commutative geometry singular traces play a very important role (see [4, 5]). In the simplest setting one considers the operator ideal $\mathcal{L}^{(1,\infty)}$ of all compact operators T on a separable Hilbert space such that

$$\|T\|_{(1,\infty)} = \sup_n \frac{1}{1+\log n} \sum_{k=0}^{n} \mu_k(T) < \infty,$$

where $(\mu_k(T))_{k=1}^{\infty}$ is the sequence of singular values of T. This is the dual of the Macaev ideal [18]. Then a Dixmier trace [6] is a positive linear functional ϕ on $\mathcal{L}^{(1,\infty)}$ of the form

$$\phi(T) = \omega - \lim_{n\to\infty} \frac{1}{1+\log n} \sum_{k=0}^{n} \mu_k(T),$$

where ω is a dilation invariant state on ℓ_∞ (see [4] for more explanation).

All Dixmier traces have the additional property that if S, T are positive and $\sum_{k=0}^{n} \mu_k(S) \leq \sum_{k=0}^{n} \mu_k(T)$ for $n = 0, 1, 2, \dots$, then $\phi(S) \leq \phi(T)$. We call such functionals *symmetric*. Recently Guido and Isola [16] raised a question which in this context asks whether there could exist a positive unitarily invariant functional on $\mathcal{L}^{(1,\infty)}$ which is not symmetric in this sense.

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In this paper we answer this question by constructing an example where ϕ is non-zero but vanishes on any operator whose singular values satisfy an estimate $|\mu_n(T)| \leq C/n$. This is achieved in two steps. First we show the existence of a positive linear functional on the corresponding Marcinkiewicz sequence space which is rearrangement invariant but vanishes on the sequence $(1/n)_{n=1}^{\infty}$. Secondly, we prove a general result (also answering a question of Guido and Isola [16, 17]) that any rearrangement functional induces a linear functional on the corresponding space of operators.

2 Preliminaries

A Banach space $(X, \|\cdot\|_X)$ of real valued Lebesgue measurable functions (with identification λ a.e.) on the interval $J = [0, \infty)$ or else on $J = \mathbb{N}$ will be called *rearrangement invariant* if

(i) X is an ideal lattice, that is, if $y \in X$, and x is any measurable function on J with $0 \leq |x| \leq |y|$, then $x \in X$ and $\|x\|_X \leq \|y\|_X$;

(ii) X is rearrangement invariant in the sense that if $y \in X$ and if x is any measurable function on J with $x^* = y^*$, then $x \in X$ and $\|x\|_X = \|y\|_X$.

Here, λ denotes Lebesgue measure and x^* denotes the non-increasing, right-continuous rearrangement of $|x|$ given by

$$x^*(t) = \inf\{s \geq 0 \mid \lambda(\{|x| > s\}) \leq t\}, \quad t > 0.$$

In the case $J = \mathbb{N}$, it is convenient to identify x^* with the rearrangement of the sequence $|x| = \{x_n\}_{n=1}^{\infty}$ in the descending order. For basic properties of rearrangement invariant (RI) spaces we refer to the monographs [20, 23, 24]. We note that for any RI space $X = X(J)$, the following continuous embeddings hold

$$L_1 \cap L_\infty(J) \subseteq X \subseteq L_1 + L_\infty(J).$$

The RI space X is said to be fully symmetric Banach space if it has the additional property that if $y \in X$ and $L_1 + L_\infty(J) \ni x \prec\prec y$, then $x \in X$ and $\|x\|_X \leq \|y\|_X$. Here, $x \prec\prec y$ denotes submajorization in the sense of Hardy, Littlewood, and Polya:

$$\int_0^t x^*(s)ds \leq \int_0^t y^*(s)ds, \qquad \forall\, t > 0.$$

A classical example of non-separable fully symmetric function and sequence spaces $X(J)$ is given by Marcinkiewicz spaces.

2.1 Marcinkiewicz Function and Sequence Spaces

Let Ω denote the set of concave functions $\psi\colon [0, \infty) \to [0, \infty)$ such that

$$\lim_{t \to 0^+} \psi(t) = 0 \quad \text{and} \quad \lim_{t \to \infty} \psi(t) = \infty.$$

Important functions belonging to Ω include t, $\log(1+t)$, t^α and $(\log(1+t))^\alpha$ for $0<\alpha<1$, and $\log(1+\log(1+t))$. Let $\psi \in \Omega$. Define the weighted mean function

$$a(x,t) = \frac{1}{\psi(t)} \int_0^t x^*(s)\, ds, \quad t>0,$$

and denote by $M(\psi)$ the Marcinkiewicz space of measurable functions x on $[0,\infty)$ such that

$$\|x\|_{M(\psi)} := \sup_{t>0} a(x,t) = \|a(x,\cdot)\|_\infty < \infty.$$

The definition of the Marcinkiewicz sequence space $(m(\psi), \|x\|_{m(\psi)})$ is similar.

The norm closure of $M(\psi) \cap L^1([0,\infty))$ (respectively, $\ell_1 = \ell_1(\mathbb{N})$) in $M(\psi)$ (respectively, $m(\psi)$) is denoted by $M_1(\psi)$ (respectively, $m_1(\psi)$). For every $\psi \in \Omega$, we have $M_1(\psi) \neq M(\psi)$. The Banach spaces $(M(\psi), \|\cdot\|_{M(\psi)})$, $(m(\psi), \|\cdot\|_{m(\psi)})$, $(M_1(\psi), \|\cdot\|_{M(\psi)})$, $(m_1(\psi), \|\cdot\|_{m(\psi)})$ are examples of fully symmetric spaces [20,23]. The latter two spaces are separable and the former two spaces are non-separable.

Suppose now that $\psi \in \Omega$ satisfies the condition

$$\liminf_{t\to\infty} \frac{\psi(2t)}{\psi(t)} = 1. \tag{2.1}$$

We denote by $N(\psi)$ the norm closure in $M(\psi)$ of the (order) ideal

$$N(\psi)^0 := \{f \in M(\psi) : f^*(\cdot) \le k\psi'\left(\frac{\cdot}{k}\right) \text{ for some } k \in \mathbb{N}\}.$$

Clearly, $N(\psi)$ is a Banach function space in its own right and is rearrangement invariant. However, under the present assumptions on ψ, the space $N(\psi)$ is not fully symmetric and is different from $M(\psi)$. For these facts, we refer the reader to [2], [20, II.5.7] and [25].

Let $M_+(\psi)$ (respectively, $m_+(\psi)$) denote the set of positive functions of $M(\psi)$ (respectively, $m(\psi)$). For every $x \in M(\psi)$, we write $x = x_+ - x_-$, where

$$x_+ := x\chi_{\{t: x(t)>0\}} \quad \text{and} \quad x_- := x - x_+.$$

2.2 Symmetric and Rearrangement Invariant Functionals on Marcinkiewicz Spaces

The main object of study in this article is described in the following definition (*cf.* [8], Definition 2.1)

Definition 2.1 A positive functional $f \in M(\psi)^*$ is said to be *symmetric* (respectively, RI) if $f(x) \le f(y)$ for all $x, y \in M_+(\psi)$ such that $x \prec\!\prec y$ (respectively, $x^* = y^*$). Such a functional is said to be *supported at infinity* (or *singular*) if $f(|x|) = 0$ for all $x \in M_1(\psi)$ (equivalently, $f(x^*\chi_{[0,s]}) = 0$, for every $x \in M(\psi)$ and the indicator function $\chi_{[0,s]}$ of the interval $[0,s]$ for all $s>0$).

Let $M_+(\psi)^*_{\mathrm{sym},\infty}$ denote the cone of symmetric functionals on $M(\psi)$ supported at infinity [9, §2]. Not every Marcinkiewicz space $M(\psi)$ admits non-trivial singular symmetric functionals on $M(\psi)$. The necessary and sufficient condition for the existence of such functionals on a Marcinkiewicz space $M(\psi)$, $\psi \in \Omega$ is that ψ satisfies the condition (2.1) [8, Theorem 3.4]. For various constructions of singular symmetric functionals on $M(\psi)$ (and more generally on fully symmetric spaces and their non-commutative counterparts) we refer to [5, 6, 8–10].

A priori it is not clear whether there exist RI functionals on $M(\psi)$ which are not necessarily symmetric. The main result of the present article asserts that if $\psi \in \Omega$ satisfies the condition

$$\lim_{t\to\infty} \frac{\psi(2t)}{\psi(t)} = 1, \tag{2.2}$$

then there exists a RI functional on $M(\psi)$ which is not symmetric.

3 Symmetric Functionals Involving Banach Limits

We recall that a positive linear functional $0 \le L \in \ell_\infty^*$ is called a Banach limit if L is translation invariant and $L(\mathbf{1}) = 1$ (here, $\mathbf{1} = (1, 1, 1 \ldots))$. A Banach limit L satisfies in particular $L(\xi) = 0$ for all $\xi \in c_0$. We denote the collection of all Banach limits on l_∞ by $\mathfrak{B}$. Note that $\|L\| = 1$ for all $L \in \mathfrak{B}$.

We recall that sequence $\xi = \{\xi_n\}_{n=1}^\infty \in \ell_\infty$ is said to be *almost convergent* to $\alpha \in \mathbb{R}$, denoted $F\text{-}\lim_{n\to\infty} \xi_n = \alpha$ if and only if $L(\xi) = \alpha$ for all $L \in \mathfrak{B}$. The notion of an almost convergent sequence is due to G. G. Lorentz [22], who showed that the $\{\xi_n\}_{n=1}^\infty$ is almost convergent to α if and only if the equality

$$\lim_{p\to\infty} \frac{\xi_n + \xi_{n+1} + \cdots + \xi_{n+p-1}}{p} = \alpha$$

holds uniformly for $n = 1, 2, \ldots$. We denote by ac (respectively, ac_0) the set of all almost convergent (respectively, all almost convergent to 0) sequences from ℓ_∞. Clearly, ac and ac_0 are closed subspaces in ℓ_∞.

It is convenient to use the following notation from [9]. Let $\mathbf{t} = \{t_n\}_{n=1}^\infty$ be a sequence in $(0, \infty)$ satisfying $0 < t_1 < t_2 < \cdots \nearrow \infty$, and let $\psi \in \Omega$. For $0 \le x \in M(\psi)$, we write

$$a(x, \mathbf{t}) = \{a(x, t_n)\}_{n=1}^\infty,$$

so that $a(x, \mathbf{t}) \in \ell_\infty$, for all $0 \le x \in M(\psi)$. It follows from a well-known submajorization inequality [6, Ch. II, (2.17)] that

$$a(x + y, \mathbf{t}) \le a(x, \mathbf{t}) + a(y, \mathbf{t})$$

for all $0 \le x, y \in M(\psi)$. Further if $0 \le x, y \in M(\psi)$ and $x \prec\!\!\prec y$, then it is clear that the inequality $0 \le a(x, \mathbf{t}) \le a(y, \mathbf{t})$ holds for the pointwise order on ℓ_∞.

We now describe the construction of symmetric functionals as follows. Suppose that $0 \le \varphi \in \ell_\infty^*$ and define $f_{\varphi,\mathbf{t}}(x) = \varphi(a(x, \mathbf{t}))$ for all $0 \le x \in M(\psi)$. It is clear from the above that the functional $f_{\varphi,\mathbf{t}}\colon M(\psi)^+ \to [0,\infty)$ is always subadditive and positively homogeneous. It now follows that $f_{\varphi,\mathbf{t}}$ extends to a symmetric functional on $M(\psi)$ if and only if $f_{\varphi,\mathbf{t}}$ is additive on $M(\psi)^+$, or equivalently

$$\varphi(a(x,\mathbf{t}) + a(y,\mathbf{t}) - a(x+y,\mathbf{t})) = 0 \quad \forall 0 \le x, y \in M(\psi).$$

In this case we will say that $f_{\varphi,\mathbf{t}}$ defines a symmetric functional on $M(\psi)$, and write $f_{\varphi,\mathbf{t}} \in M(\psi)^*_{\mathrm{sym},\infty}$. Additivity, and thus symmetry of $f_{\varphi,\mathbf{t}}$, depends on properties of the functional $\varphi \in \ell^*_\infty$ as well as on properties of the sequence $\mathbf{t}$.

For any sequence $\mathbf{t} = \{t_n\}_{n=1}^\infty$ as above, we define the cone

$$B^+_\psi(\mathbf{t}) = \{0 \le \varphi \in l^*_\infty : f_{\varphi,\mathbf{t}} \in M(\psi)^*_{\mathrm{sym},\infty}\}.$$

Let $\psi \in \Omega$ satisfy condition (2.2). It follows from [9, Theorem 3.8] that $\mathfrak{B} \subseteq B^+_\psi(\mathbf{p})$ where $\mathbf{p} := \{2^n\}_{n\ge 1}$.

4 The Main Result

This section contains the main result of the present article (Theorem 4.5 below). The main ingredients of the proof of Theorem 4.5 are contained in Lemmas 4.1–4.3.

Throughout this section we fix $\psi \in \Omega$ satisfying condition (2.2). For every pair of elements $0 \le x, y \in M(\psi)$ we have

$$B(a(x,\mathbf{p}) + a(y,\mathbf{p}) - a(x+y,\mathbf{p})) = 0, \qquad \forall B \in \mathfrak{B}.$$

In other words, $a(x,\mathbf{p}) + a(y,\mathbf{p}) - a(x+y,\mathbf{p}) \in ac_0$. We now extend the mapping $M(\psi)_+ \ni x \to a(x,\mathbf{p}) \in \ell_\infty$ to $M(\psi)$ by the following rule: if $x \in M(\psi)$ is represented by $x = x^+ - x^-$ with $x^+x^- = 0$, $x^+, x^- \in M(\psi)$, then we set

$$A(x) := a(x,\mathbf{p}) := a(x^+,\mathbf{p}) - a(x^-,\mathbf{p}).$$

Let Q denote the canonical homomorphism from ℓ_∞ onto the quotient ℓ_∞/ac_0.

Lemma 4.1 *The mapping $T := QA\colon M(\psi) \mapsto \ell_\infty/ac_0$ is a bounded linear operator with norm* 1.

Proof Since $f_{B,\mathbf{p}}$ is a singular symmetric functional on $M(\psi)$ for every $B \in \mathfrak{B}$, we infer that $a(x,\mathbf{p}) + a(y,\mathbf{p}) - a(x+y,\mathbf{p}) \in ac_0$ for any $x, y \in M(\psi)$. This shows that the operator T is additive on $M(\psi)$. The fact that $T(kx) = kT(x)$, $k \in \mathbb{R}$ is trivial and the inequality $\|T\|_{M(\psi)\to\ell_\infty/ac_0} \le 1$ follows immediately from the fact that $\|A(x)\|_{\ell_\infty} \le \|x\|_{M(\psi)}$ for all $x \in M(\psi)$. ■

We denote by $\rho(x, ac)$ the distance in ℓ_∞ from the element $x \in \ell_\infty$ to the subspace ac.

Lemma 4.2 *For every $x = \{x_n\}_{n=1}^{\infty} \in \ell_\infty$, we have*

$$\rho(x, ac) \geq \frac{1}{2}\Big(\lim_{k\to\infty} \sup_{n\in\mathbb{N}} \frac{1}{k} \sum_{i=n+1}^{n+k} x_i - \lim_{k\to\infty} \inf_{n\in\mathbb{N}} \frac{1}{k} \sum_{i=n+1}^{n+k} x_i \Big).$$

Proof By a result of Sucheston [27], we have

$$p(x) := \sup\{L(x) : L \in \mathfrak{B}\} = \lim_{k\to\infty} \sup_{n\in\mathbb{N}} \frac{1}{k} \sum_{i=n+1}^{n+k} x_i,$$

$$q(x) := \inf\{L(x) : L \in \mathfrak{B}\} = \lim_{k\to\infty} \inf_{n\in\mathbb{N}} \frac{1}{k} \sum_{i=n+1}^{n+k} x_i.$$

If $y = \{y_n\}_{n=1}^{\infty} \in ac$, then there exists $\alpha \in \mathbb{R}$ such that

$$\lim_{k\to\infty} \frac{1}{k} \sum_{i=n+1}^{n+k} y_i = \alpha$$

uniformly on $n \in \mathbb{N}$. Hence, $\|x - y\|_{\ell_\infty} \geq |p(x) - \alpha|$ and $\|x - y\|_{\ell_\infty} \geq |q(x) - \alpha|$. Since

$$\inf_{r\in\mathbb{R}} \max\{|p(x) - r|, |q(x) - r|\} = \frac{1}{2}(p(x) - q(x)),$$

we obtain $\rho(x, ac) = \inf_{y\in ac} \|x - y\|_{\ell_\infty} \geq \frac{1}{2}(p(x) - q(x))$. ■

We define a closed subspace M of $M(\psi)$ by setting $M := T^{-1}(ac/ac_0) = A^{-1}(ac)$. We denote $d(x, M)$ the distance between an element $x \in M(\psi)$ and the subspace M in the metric defined by the norm $\|\cdot\|_{M(\psi)}$.

Lemma 4.3 *If $\psi \in \Omega$ satisfies condition* (2.2)*, then for every $x \in M(\psi)$, we have*

$$d(x, M) \geq \frac{1}{2}\big(\limsup_{j\to\infty} a(x, p_j) - \liminf_{j\to\infty} a(x, p_j) \big).$$

Proof By Lemma 4.2, we have

$$\rho(Ax, ac) \geq \frac{1}{2}(\lim_{k\to\infty} \sup_{n\in\mathbb{N}} \frac{1}{k} \sum_{i=n+1}^{n+k} a(x, p_i) - \lim_{k\to\infty} \inf_{n\in\mathbb{N}} \frac{1}{k} \sum_{i=n+1}^{n+k} a(x, p_i)). \tag{4.1}$$

We show first that

$$\sup_{n\in\mathbb{N}} \frac{1}{k} \sum_{i=n+1}^{n+k} a(x, p_i) \geq \limsup_{j\to\infty} a(x, p_j), \qquad \forall\, k \geq 1. \tag{4.2}$$

Since ψ satisfies condition (2.2), for every $\epsilon > 0$ and $k \in \mathbb{N}$, there exists $s \in \mathbb{N}$ such that for all $j \geq s$ we have $\psi(p_j) \leq (1+\epsilon)^{\frac{1}{k}} \psi(p_{j-1})$ and

$$a(x, p_s) \geq \limsup_{j\to\infty} a(x, p_j) - \epsilon.$$

We now have

$$\begin{aligned}\frac{1}{k}\sum_{j=s}^{s+k-1} a(x, p_j) =& \frac{1}{k}\sum_{j=s}^{s+k-1} \frac{1}{\psi(p_j)} \int_0^{p_j} x^*(s)\, ds \\ &\geq \frac{\int_0^{p_s} x^*(s)\, ds}{\psi(p_s)} \frac{1}{k} \sum_{j=s}^{s+k-1} \frac{\psi(p_s)}{\psi(p_j)} \\ &\geq a(x, t_s) \frac{1}{k} \sum_{j=0}^{k-1} \frac{1}{(1+\epsilon)^{\frac{j}{k}}} \\ &\geq (\limsup_{j\to\infty} a(x, p_j) - \epsilon) \frac{1}{(1+\epsilon).}\end{aligned}$$

Since ϵ above is arbitrary, the estimate (4.2) now follows. A similar argument yields that

$$\inf_{n\in\mathbb{N}} \frac{1}{k} \sum_{i=n+1}^{n+k} a(x, p_i) \leq \liminf_{j\to\infty} a(x, p_j), \qquad \forall\, k \geq 1. \tag{4.3}$$

The assertion now follows from (4.1), (4.2), (4.3) and from the fact that $d(x, M) \geq \rho(Ax, ac)$ for every $x \in M(\psi)$ (see proof of Lemma 4.1). This completes proof of Lemma 4.3. ■

Corollary 4.4 *There exists an element* $0 \neq x_0 \in M(\psi)$*, such that*

$$d(x_0 + y, M) \geq \frac{1}{4}, \qquad \forall\, y \in M(\psi)_+.$$

Proof Since $\lim_{t\to\infty} \psi(t) = \infty$, there exists a subsequence $\{s_n\}_{n\geq 1}$ of $\mathbf{p}$ satisfying

$$\lim_{n\to\infty} \frac{s_n}{s_{n+1}} = 0, \quad \lim_{n\to\infty} \frac{\psi(s_n)}{\psi(s_{n+1})} = 0. \tag{4.4}$$

We define the function x_0 by setting

$$x_0 := \sum_{k=1}^{\infty} \frac{\psi(s_{k+1}) - \psi(s_k)}{s_{k+1} - s_k} \chi_{[s_k, s_{k+1})}.$$

Since ψ is concave, it is clear that x_0 is decreasing so that $x_0 = x_0^*$. Further,

$$\int_0^{s_k} x_0(t)\,dt = \psi(s_k)$$

for all $k = 1, 2, \ldots$, and $\int_0^s x_0(t)\,dt \leq \psi(s)$ for all $s \geq 0$. This implies in particular that $x_0 \in M(\psi)$, that $\|x_0\|_{M(\psi)} = 1$ and that $\limsup_{j\to\infty} a(x_0, p_j) = 1$.

Now fix $y \in M(\psi)_+$. We have

$$\begin{aligned}
d(x_0 + y, M) &\geq \frac{1}{2}\big(\limsup_{j\to\infty} a(x_0 + y, p_j) - \liminf_{j\to\infty} a(x_0 + y, p_j)\big) \\
&\geq \frac{1}{2}\limsup_{j\to\infty}(a(x_0 + y, p_j) - a(x_0 + y, p_{j-1})) \\
&= \frac{1}{2}\limsup_{j\to\infty}\Big(\int_0^{2^j} (x_0 + y)^*(s)\,ds/\psi(2^j) \\
&\qquad - \int_0^{2^{j-1}} (x_0 + y)^*(s)\,ds/\psi(2^{j-1})\Big) \\
&= \frac{1}{2}\limsup_{j\to\infty}\Big(\int_{2^{j-1}}^{2^j} (x_0 + y)^*(s)\,ds/\psi(2^j) \\
&\qquad - (1 - \frac{\psi(2^{j-1})}{\psi(2^j)})\int_0^{2^{j-1}} (x_0 + y)^*(s)\,ds/\psi(2^{j-1})\Big).
\end{aligned}$$

Hence, due to (2.2) we get

$$\begin{aligned}
d(x_0 + y, M) &\geq \frac{1}{2}\limsup_{j\to\infty}\int_{2^{j-1}}^{2^j} (x_0 + y)^*(s)\,ds/\psi(2^j) \\
&\geq \frac{1}{2}\limsup_{j\to\infty} 2^{j-1}\int_{2^j-1}^{2^j} (x_0 + y)^*(s)\,ds/\psi(2^j) \\
&\geq \frac{1}{2}\limsup_{j\to\infty} 2^{j-1}\int_{2^j-1}^{2^j} x_0^*(s)\,ds/\psi(2^j) \\
&= \frac{1}{2}\limsup_{j\to\infty} 2^{j-1}x_0^*(2^j - 1)/\psi(2^j).
\end{aligned}$$

In particular, it follows from (4.4) that

$$\begin{aligned}
d(x_0 + y, M) &\geq \frac{1}{2}\limsup_{k\to\infty}\frac{s_{k+1}}{2}x_0^*(s_{k+1} - 1)/\psi(s_{k+1}) \\
&= \frac{1}{2}\limsup_{k\to\infty}\frac{s_{k+1}}{2}\frac{\psi(s_{k+1}) - \psi(s_k)}{s_{k+1} - s_k}/\psi(s_{k+1}) \\
&= \frac{1}{4}\limsup_{k\to\infty}\frac{1 - \frac{\psi(s_k)}{\psi(s_{k+1})}}{1 - \frac{s_k}{s_{k+1}}} = \frac{1}{4}.
\end{aligned}$$

■

Theorem 4.5 *If $\psi \in \Omega$ satisfies condition* (2.2)*, then there exists an RI functional $0 \le \phi \in M(\psi)^*$ which does not belong to $M_+(\psi)^*_{\mathrm{sym},\infty}$.*

Proof Consider a convex closed set $\mathcal{F} := \overline{M - M_+(\psi)}^{\|\cdot\|_{M(\psi)}}$. Fix an element x_0 from Corollary 4.4 and note that $x_0 \notin \mathcal{F}$. Therefore, there exists a functional $\phi \in M(\psi)^*$ such that for some $\alpha \in \mathbb{R}$

$$\phi(x_0) > \alpha \ge \phi(x), \qquad \forall\, x \in \mathcal{F}.$$

The inequality above implies immediately that $\phi(x) = 0$ for every $x \in M$ and $\phi(y) \ge 0$ for every $y \in M_+(\psi)$, in particular $\phi \ge 0$. Furthermore, we see that $\alpha \ge 0$ and $\phi(x_0) > 0$.

To show that ϕ is RI, fix two elements $0 \le x, y \in M_+(\psi)$ such that $x^* = y^*$, and denote $z := x - y$. For every $B \in \mathfrak{B}$, the functional $f_{B,\mathbf{p}}$ belongs to $M_+(\psi)^*_{\mathrm{sym},\infty}$ (see Section 3 above). In particular, $f_{B,\mathbf{p}}(z) = f_{B,\mathbf{p}}(x) - f_{B,\mathbf{p}}(y) = 0$. This implies that $f_{B,\mathbf{p}}(z_+) = f_{B,\mathbf{p}}(z_-) = 0$, or equivalently, $B(a(z_+, \mathbf{p})) = B(a(z_-, \mathbf{p}))$. Since the latter equality holds for every Banach limit B, we infer that the sequence $A(z) = a(z, \mathbf{p}) = a(z_+, \mathbf{p}) - a(z_-, \mathbf{p})$ belongs to ac_0. This, in turn, implies that $z \in A^{-1}(ac) = M$ and therefore, $\phi(z) = 0$.

To complete the proof it remains to show that ϕ is not symmetric. Consider the element $\psi' \in M(\psi)$. Clearly $a(\psi', \mathbf{p}) = \mathbf{1} \in ac$ and therefore $\psi' \in A^{-1}(ac) = M$, and so $\phi(\psi') = 0$. On the other hand, we have $x_0 \prec\!\!\prec \psi'$ and $\phi(x_0) > 0$. This completes the proof of Theorem 4.5. ∎

The following corollary shows that ϕ is not continuous with respect to the weak*-topology on $M(\psi)$.

Corollary 4.6 *The functional ϕ defined in the proof of Theorem* 4.5 *vanishes on the set of all extreme points of the unit ball of the space $M(\psi)$*

Proof It follows from [26] that the set of all extreme points of the unit ball of the space $M(\psi)$ is given by $\mathrm{Extr} := \{x \in M(\psi) : x^* = \psi\}$. If $x \in \mathrm{Extr}$, then it follows from the proof of Theorem 4.5 that $\phi(x^*) = 0$, and since ϕ is RI, we obtain that $\phi(|x|) = 0$. Since $\phi \ge 0$, it follows further that $\phi(x) = 0$. ∎

It is obvious that by identifying a sequence $\alpha = \{\alpha_k\}_{k\ge1}$ with the function $x_\alpha := \sum_{k=1}^{\infty} \alpha_k \chi_{[k-1,k)}$, we may regard the Marcinkiewicz sequence space $m(\psi)$ as a closed subspace of the space $M(\psi)$. Note that the element x_0 built in the proof of Corollary 4.4 above belongs to this subspace. Now let π be a surjective and bijective mapping of $\mathbb{N}$ into itself. The next corollary now follows immediately.

Corollary 4.7 *The functional ϕ defined in the proof of Theorem* 4.5 *is a non-zero positive functional on $m(\psi)$, such that $\phi(\alpha) = \phi(\beta)$, for any $0 \le \{\alpha_k\}_{k\ge1}, \{\beta_k\}_{k\ge1} \in m(\psi)$, with $\beta_n = \alpha_{\pi(n)}$, $n \ge 1$ and which vanishes on the element*

$$\{\psi(n) - \psi(n-1)\}_{n\ge1}.$$

We complete this section with the remark that Marcinkiewicz spaces $M(\psi)$ and $m(\psi)$ can be regarded as examples of Marcinkiewicz spaces associated with von Neumann algebras $L_\infty(0,\infty)$ and ℓ_∞, respectively. In the next section, we shall explain that in fact our main result can be extended to Marcinkiewicz spaces associated with an arbitrary semifinite von Neumann algebras.

5 Application to Singular Traces

Dixmier [6] proved the existence of non-normal traces on the von Neumann algebra $\mathcal{L}(\mathcal{H})$. Dixmier's original construction involves singular dilation invariant positive linear functionals (see also an exposition of this construction in [5]). In [8] the traces of Dixmier were broadly generalized as singular symmetric functionals on Marcinkiewicz function (respectively, operator) spaces $M(\psi)$ on $[0,\infty)$ (respectively, on a semifinite von Neumann algebra).

5.1 Symmetric and RI functionals on Fully Symmetric Operator Spaces

Here, we will extend the ideas of the previous sections to the setting of (noncommutative) spaces of measurable operators. We denote by $\mathcal{M}$ a semifinite von Neumann algebra on the Hilbert space $\mathcal{H}$, with a fixed faithful and normal semifinite trace τ. The identity in $\mathcal{M}$ is denoted by $\mathbf{1}$. A linear operator $x\colon \operatorname{dom}(x) \to \mathcal{H}$, with domain $\operatorname{dom}(x) \subseteq \mathcal{H}$, is called affiliated with $\mathcal{M}$ if $ux = xu$ for all unitary u in the commutant $\mathcal{M}'$ of $\mathcal{M}$. The closed and densely defined operator x, affiliated with $\mathcal{M}$, is called τ-measurable if for every $\epsilon > 0$ there exists an orthogonal projection $p \in \mathcal{M}$ such the $p(\mathcal{H}) \subseteq \operatorname{dom}(x)$ and $\tau(1-p) < \epsilon$. The set of all τ-measurable operators is denoted by $\widetilde{\mathcal{M}}$.

We next recall the notion of generalized singular value function [14]. Given a self-adjoint operator x in $\mathcal{H}$, we denote by $e^x(\,\cdot\,)$ the spectral measure of x. Now assume that x is τ-measurable. Then $e^{|x|}(B) \in \mathcal{M}$ for all Borel sets $B \subseteq \mathbb{R}$, and there exists $s > 0$ such that $\tau(e^{|x|}(s,\infty)) < \infty$. For $t \geq 0$, we define

$$\mu_t(x) = \inf\{s \geq 0 : \tau(e^{|x|}(s,\infty)) \leq t\}.$$

The function $\mu(x)\colon [0,\infty) \to [0,\infty]$ is called the *generalized singular value function* (or decreasing rearrangement) of x; note that $\mu_t(x) < \infty$ for all $t > 0$. For the basic properties of this singular value function we refer the reader to [14].

Considering $\mathcal{M} = L^\infty([0,\infty), m)$ (where m denotes Lebesgue measure on $[0,\infty)$) as an abelian von Neumann algebra acting via multiplication on the Hilbert space $\mathcal{H} = L^2([0,\infty), m)$ with the trace given by integration with respect to m, it is easy to see that the set of all τ-measurable operators affiliated with $\mathcal{M}$ consists of all measurable functions on $[0,\infty)$ which are bounded except on a set of finite measure and that the generalized singular value function $\mu(f)$ is precisely the decreasing rearrangement f^*.

If $\mathcal{M} = \mathcal{L}(\mathcal{H})$ (respectively, $\ell_\infty(\mathbb{N})$) and τ is the standard trace Tr (respectively, the counting measure on $\mathbb{N}$), then it is not difficult to see that $\widetilde{\mathcal{M}} = \mathcal{M}$. In this case,

$x \in \mathcal{M}$ is compact if and only if $\lim_{t\to\infty} \mu_t(x) = 0$; moreover,

$$\mu_n(x) = \mu_t(x), \quad t \in [n, n+1), \quad n = 0, 1, 2, \ldots,$$

and $\{\mu_n(x)\}_{n=0}^{\infty}$ is just the sequence of eigenvalues of $|x|$ in non-increasing order and counted according to multiplicity.

Given a semifinite von Neumann algebra $(\mathcal{M}, \tau)$ and a fully symmetric Banach function space $(E, \|\cdot\|_E)$ on $([0,\infty), m)$, we define the corresponding non-commutative space $E(\mathcal{M}, \tau)$ by setting $E(\mathcal{M}, \tau) = \{x \in \widetilde{\mathcal{M}} : \mu(x) \in E\}$. Equipped with the norm $\|x\|_{E(\mathcal{M},\tau)} := \|\mu(x)\|_E$, the space $(E(\mathcal{M}, \tau), \|\cdot\|_{E(\mathcal{M},\tau)})$ is a Banach space and is called the (non-commutative) symmetric operator space associated with $(\mathcal{M}, \tau)$ corresponding to $(E, \|\cdot\|_E)$. If $\mathcal{M} = \ell_\infty(\mathbb{N})$, then the space $E(\mathcal{M}, \tau)$ is simply the (fully) symmetric sequence space ℓ_E, which may be viewed as the linear span in E of the vectors $e_n = \chi_{[n-1,n)}$, $n \geq 1$ (see [23]). If $\mathcal{M} = \mathcal{L}(\mathcal{H})$, then the space $E(\mathcal{M}, \tau)$ is simply the Schatten ideal $\mathcal{C}_E$ associated with ℓ_E (see [18]).

Further references to the theory of symmetric operator spaces can be found in [8].

Definition 5.1 A linear functional $\phi \in E(\mathcal{M}, \tau)^*$ is called symmetric (respectively, rearrangement invariant) if ϕ is positive, (that is, $\phi(x) \geq 0$ whenever $0 \leq x \in E(\mathcal{M}, \tau)$) and $\phi(x) \leq \phi(y)$ whenever $\mu(x) \prec\!\!\prec \mu(y)$ (respectively, $\phi(x) = \phi(y)$ whenever $x, y \geq 0$ and $\mu(x) = \mu(y)$).

The next theorem simply states that we can transfer a symmetric or RI functional from E to $E(\mathcal{M}, \tau)$ to have the same properties. This result gives an alternative proof of [8, Theorem 4.2] and resolves a question raised in [17, p. 395], for example, where RI functionals are referred to as dilation-invariant.

Theorem 5.2 *Suppose ϕ is a symmetric (respectively, rearrangement invariant) linear functional on E. Then there is a unique symmetric (respectively, rearrangement invariant) functional $\hat{\phi} \in E(\mathcal{M}, \tau)^*$ such that*

$$\hat{\phi}(x) = \phi(\mu(x)) \qquad 0 \leq x \in E(\mathcal{M}, \tau). \tag{5.1}$$

Proof First of all, we note that it is sufficient to prove the assertion for non-atomic algebras $\mathcal{M}$. Indeed, setting

$$\mathcal{N} := \mathcal{M} \otimes L_\infty([0,1]), \quad \tau_{\mathcal{N}} := \tau \otimes m,$$

we see that $\mathcal{M} \otimes \chi_{[0,1]}$ is a von Neumann subalgebra of $(\mathcal{N}, \tau_{\mathcal{N}})$ and $\tau_{\mathcal{N}}|_{\mathcal{M}\otimes\chi_{[0,1]}} = \tau$. In particular, for every $x \in \widetilde{M}$, the function $\mu(x)$ coincides with the generalized singular value function of the element $x \otimes \chi_{[0,1]}$ computed with respect to the trace $\tau_{\mathcal{N}}$. Thus, as soon as the assertion of the theorem is established for all non-atomic von Neumann algebras, it would automatically hold also for an arbitrary semifinite $\mathcal{M}$.

Assuming that $\mathcal{M}$ is non-atomic, we consider the tensor product $\mathcal{L}(\mathcal{H}) \otimes \mathcal{M}$ with the induced trace, still denoted by τ and note that the latter algebra is again non-atomic. We use (5.1) as the definition of $\hat{\phi}$ for positive $x \in E(\mathcal{L}(\mathcal{H}) \otimes \mathcal{M}, \tau)$. In order to show that $\hat{\phi}$ extends to a linear functional on $E(\mathcal{M}, \tau)$ with the required properties we need only show that $\hat{\phi}(x+y) = \hat{\phi}(x) + \hat{\phi}(y)$ $x, y \geq 0$.

Let us we recall the generalized K^*-function defined in [15]. If $f \in L_0(0,\infty)$ has the property that $\lambda(f > t) < \infty$ for every $t > 0$, we define

$$K^*(r, s; f) = \int_{r \leq |f(t)| < s} f(t)\, dt.$$

Now suppose $x, y \in E(\mathcal{M}, \tau)$ with $x, y \geq 0$ and consider the matrix (as an element of $E(\mathcal{L}(\mathcal{H}) \otimes \mathcal{M}, \tau)$)

$$v = \begin{bmatrix} x+y & 0 & 0 \\ 0 & -x & 0 \\ 0 & 0 & -y \end{bmatrix}.$$

We can regard v as a member of $E(\mathcal{M}, \tau)$. Note that (see [13] where this trick is also used):

$$\begin{bmatrix} x & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -y \end{bmatrix} = \begin{bmatrix} 0 & x & 0 \\ 0 & 0 & 0 \\ -y & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & I \\ I & 0 & 0 \\ 0 & I & -0 \end{bmatrix}$$

while

$$\begin{bmatrix} -y & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & I \\ I & 0 & 0 \\ 0 & I & -0 \end{bmatrix} \begin{bmatrix} 0 & x & 0 \\ 0 & 0 & 0 \\ -y & 0 & 0 \end{bmatrix}.$$

Thus v considered as an element of $E(\mathcal{L}(\mathcal{H}) \otimes \mathcal{M}, \tau)$ can be expressed as a commutator $v = uw - wu$ with $u \in \mathcal{L}(\mathcal{H}) \otimes \mathcal{M},\ w \in E(\mathcal{L}(\mathcal{H}) \otimes \mathcal{M}, \tau)$. Using the fact that the algebra $\mathcal{L}(\mathcal{H}) \otimes \mathcal{M}$ is non-atomic and that u is bounded, we infer from [12, Lemma 4.1] (in the case when $\mathcal{M} = \mathcal{L}(\mathcal{H})$, one can instead use the corresponding Lemma in [11]), that

$$|\tau(ve^{|v|}(\mu_{t_1}(|v|), \mu_{t_2}(|v|)))| \leq t_1 h(t_1) + t_2 h(t_2), \quad 0 < t_1 < t_2 < \infty,$$

where $0 \leq h \in E$ is a decreasing positive function on $(0, \infty)$; we can assume h is continuous (otherwise replace it by $h_1(t) = 2t^{-1} \int_{t/2}^{t} h(s)ds$). For any $0 < r < s < \infty$ define $t_1 = \sup\{t : \mu_t(|v|) \leq r\}$ and $t_2 = \inf\{t : \mu_t(|v|) > s\}$. Then we have

$$|\tau(ve^{|v|}([r, s)))| \leq r\lambda(h > r) + s\lambda(h > s), \quad 0 < r < s < \infty.$$

Now let f, g, F be nonnegative functions in E with disjoint supports such that

$$f^*(t) = \mu_t(x),\ g^*(t) = \mu_t(y), F^*(t) = \mu_t(x+y).$$

The above inequality gives

$$|K^*(r, s; F - f - g)| \leq r\lambda(h > r) + s\lambda(h > s) \quad 0 < r < s < \infty.$$

We can now appeal to [15, Theorem 2.8, Lemma 2.4]. Note that in [15] the term symmetric functional applies to any linear functional ψ (not necessarily positive or bounded) satisfying $\psi(f) = \psi(g)$ whenever $f, g \geq 0$ and $f^* = g^*$. We conclude that $\phi(F - f - g) = 0$ or equivalently $\hat{\phi}(x+y) = \hat{\phi}(x) + \hat{\phi}(y)$. ■

The first example of a singular trace on $\mathcal{L}(\mathcal{H})$ was built in [6] (see the introduction). In many subsequent publications (see [1, 3, 8, 16, 19, 21, 28] and references therein) many examples of singular traces on Marcinkiewicz ideals (and more general operator spaces) have been manufactured. However in all the papers cited above, as well as in the original Dixmier paper, singular traces were also symmetric functionals. The following theorem is now immediate by Theorem 5.2.

Theorem 5.3 *If $\psi \in \Omega$ satisfies condition* (2.2) *and* $(\mathcal{M}, \tau)$ *as above, then there exists a non-trivial RI functional* $0 \leq \phi \in M(\psi)(\mathcal{M}, \tau)^*$ *which is not symmetric.*

This theorem answers implicit questions raised by Guido and Isola [17, p. 395] who commented that it was not known whether an arbitrary (commutative) RI functional gives rise to a corresponding singular trace and that all known examples for singular traces are given by a "monotone", *i.e.,* symmetric, functional. It also strengthens and complements [16, Theorems 4.1.6, 4.2.3, 4.2.5].

Corollary 5.4 *There exists a singular trace on $\mathcal{L}(\mathcal{H})$, whose restriction to $\mathcal{L}^{(1,\infty)}$ is a positive continuous linear functional on the latter space and which vanishes on every element $x \in \mathcal{L}^{(1,\infty)}$ whose sequence of s-numbers is given by $\{\frac{1}{n}\}_{n\geq 1}$.*

One might be tempted to pose a question asking whether the concept of operators measurable with respect to the set of all Dixmier traces introduced in [5, 21] can be sensibly formulated for the set of all singular traces. The corollary above answers this in the negative.

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