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SUMS OF IDEMPOTENTS IN BANACH ALGEBRAS

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ABSTRACT. We prove that the sum of two idempotents is in a Banach algebra is itself an idempotent if and only if it is power-bounded.

Let A be a Banach algebra and let p and q be idempotents in A. It is very easy to show that $p + q$ will be an idempotent if and only if $pq = qp = 0$. This note is motivated by the observation that these conditions are also equivalent to the condition $(p + q)^3 = p + q$; this may be established by easy algebraic arguments.

In fact if $n \geqq 3$ then $p + q$ is an idempotent if $(p + q)^n = p + q$. The author first established a proof for the case when A is finite-dimensional, which is essentially reproduced below in the first proof of our main theorem. Subsequently M. Hochster pointed out to the author that if $(p + q)^n = p + q$ then the algebra generated by p and q is always finite-dimensional, so that this also establishes the general case.

In this note we extend these ideas by replacing the condition $(p + q)^n = p + q$ by the weaker hypothesis that $((p + q)^m : m \in \mathbf{N})$ is bounded. We give first the proof for finite-dimensional algebras which suggested the result and then give a proof for arbitrary Banach algebras. We shall assume, without loss of generality, that all algebras are over the complex numbers and have identities.

THEOREM. *Let A be a Banach algebra and that $p, q \in A$ are idempotents. Then $p + q$ is an idempotent if (and only if)*

$$\sup_m \|(p + q)^m\| < \infty.$$

PROOF FOR A FINITE-DIMENSIONAL. We may suppose that p and q are $(n \times n)$-matrices. The hypothesis on $p + q$ implies that every eigenvalue λ of $(p + q)$ satisfies $|\lambda| \leqq 1$. However the trace of $p + q$, $\tau(p + q) = \tau(p) + \tau(q)$ and the rank of $p + q$ satisfies $r(p + q) \leqq r(p) + r(q)$. Hence $r(p + q) \leqq \tau(p + q)$ so that every eigenvalue of $p + q$ is either one or zero.

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Now consider the Jordan normal form for $p + q$. We can write $p + q = d + h$ where d is an idempotent, h is nilpotent and $dh = hd$. Note that if $h^s = 0$

$$(d + h)^m = d\left(1 + \binom{m}{1}h + \binom{m}{2}h^2 + \dots + \binom{m}{s-1}h^{s-1}\right)$$

for $m \geqq s$. Hence, as this sequence is bounded, $dh = hd = 0$. But then $r(h) + r(d) = r(h + d) = r(p + q) \leqq r(p) + r(q) = \tau(p) + \tau(q) = \tau(p + q) = \tau(d) = r(d)$. Thus $h = 0$ and $p + q$ is an idempotent.

Proof for the General Case. We shall show that $pq = 0$. It will then follow by the same argument that $qp = 0$ and hence $p + q$ is an idempotent.

Let B be the commutative Banach algebra generated by the identity and pq. We adjoin to B a square-root ξ for pq to form B_0. Precisely let B_0 be the commutative algebra of all formal sums $(b_1 + b_2\xi)$ where $b_1, b_2 \in B$. We choose any $\theta > \sqrt{||pq||}$ and norm B_0 by

$$||b_1 + b_2\xi|| = ||b_1|| + \theta||b_2||.$$

It is readily verified that B_0 is then a Banach algebra.

Now by hypothesis there exists M so that for $m = 1, 2, \dots$

$$||p(p + q)^m q|| \leqq M.$$

In fact $p(p + q)^m q$ is a polynomial in pq. Let

$$p(p + q)^m q = \sum_{r=1}^{m} \alpha_{mr}(pq)^r.$$

Then α_{mr} is the number of integer solutions for

$$S_1 + S_2 + \dots S_{2r} = m$$

where $S_1 \geqq 0$, $S_{2r} \geqq 0$ and $S_i \geqq 1$ for $i = 2, 3, \dots, 2r - 1$. This in turn is the coefficient of x^m in the expansion of $x^{2r-2}(1 - x)^{-2r}$.

Thus $\alpha_{mr} = 0$ if $2r > m + 2$ and otherwise

$$\alpha_{mr} = \binom{m+1}{m-2r+2} = \binom{m+1}{2r-1}.$$

It follows that

$$p(p + q)^m q = \sum_{r=1}^{[m/2]+1} \binom{m+1}{2r-1}\xi^{2r}$$

$$= \frac{1}{2}\xi((1 + \xi)^{m+1} - (1 - \xi)^{m+1}).$$

We conclude that

$$\left\|\frac{1}{2}\xi((1+\xi)^m-(1-\xi)^m)\right\| \leqq M \quad m=0,1,2\ldots.$$

Let ψ be any multiplicative linear functional on B_0. Suppose $\psi(\xi)=\lambda\in\mathbf{C}$. Then $\{\lambda((1+\lambda)^m-(1-\lambda)^m)\}_{m=0}^{\infty}$ is bounded and hence $\lambda=0$. Hence

$$\lim_{m\to\infty}\|\xi^m\|^{1/m}=0$$

and from this we have that if $\delta>0$ there is a constant C_δ so that for all $z\in\mathbf{C}$

$$\|\exp(z\xi)\| \leqq C_\delta e^{\delta|z|}.$$

If $t\in\mathbf{R}$ and $t\geqq 0$ then

$$\left\|\sum_{m=0}^{\infty}\frac{t^m}{m!}\left(\frac{1}{2}\xi(1+\xi)^m-\frac{1}{2}\xi(1-\xi)^m\right)\right\| \leqq Me^t$$

i.e.

$$\|\xi e^t\sinh(t\xi)\| \leqq Me^t$$

and hence if $-\infty<t<\infty$,

$$\|\xi\sinh(t\xi)\| \leqq M.$$

We also note that for $\delta>0$

$$\|\xi\sinh(z\xi)\| \leqq C_\delta e^{\delta|z|}\|\xi\|.$$

Thus if $\phi\in B_0^*$ the entire function

$$F(z)=\phi(\xi\sinh(z\xi))$$

is constant by Theorem 6.2.14 of Boas [1], since it is of exponential type zero and is bounded on the real axis. In fact, in our circumstances, this is also a simple consequence of Theorem 1.4.3 of [1], which will imply F is bounded in both the upper and lower half-planes. In particular, we conclude that $\phi(\xi^2)=0$. Hence by the Hahn-Banach theorem, $\xi^2=0$, i.e. $pq=0$ as required.

Concluding Remarks. We observe that it is impossible to replace the hypothesis $\sup\|(p+q)^n\|<\infty$ by the weaker hypothesis that the spectral radius $\lim\|(p+q)^n\|^{1/n}\leqq 1$. To see this simply take p and q as the 2×2-matrices

$$p=\begin{pmatrix}1&1\\0&0\end{pmatrix}\ q=\begin{pmatrix}0&0\\0&1\end{pmatrix}.$$

Also let us note that in the proof given above, we borrowed some ideas from the theory of numerical ranges (cf. [2], p. 51).

We would like to thank several mathematicians for enlightening discussions on this question over the years, including J. Duncan, who pointed out a simplification of our argument using [1], R. J. Hindley, G. V. Wood, I. J. Papick and M. Hochster.

References

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