THE BEHAVIOUR OF LEGENDRE AND ULTRASPHERICAL POLYNOMIALS IN $L_p$-SPACES

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ABSTRACT. We consider the analogue of the $\Lambda(p)$-problem for subsets of the Legendre polynomials or more general ultraspherical polynomials. We obtain the "best possible" result that if $2 < p < 4$ then a random subset of $N$ Legendre polynomials of size $N^{(p-1)/p}$ spans a Hilbertian subspace. We also answer a question of König concerning the structure of the space of polynomials of degree $n$ in various weighted $L_p$-spaces.

1. Introduction. Let $(P_n)$ denote the Legendre polynomials on $[-1, 1]$ and let $\varphi_n = c_n P_n$ be the corresponding polynomials normalized in $L_2[-1, 1]$. Then $(\varphi_n)_{n=1}^\infty$ is an orthonormal basis of $L_2[-1, 1]$. If we consider the same polynomials in $L_p[-1, 1]$ where $p > 2$ then $(\varphi_n)_{n=1}^\infty$ is a basis if and only if $\sup \|\varphi_n\|_p < \infty$ if and only if $p < 4$ [8], [9].

In this note our main result concerns the analogue of the $\Lambda(p)$-problem for the Legendre polynomials. In [2] Bourgain (answering a question of Rudin [12]) showed that for the trigonometric system $(e^{inx})_{n\in\mathbb{Z}}$ in $L_p(\mathbb{T})$ where $p > 2$ there is a constant $C$ so that for any $N$ there is a subset $A$ of $\{1, 2, \ldots, N\}$ with $|A| \geq N^{2/p}$ and such that for any $(\xi_n)_{n=1}^N$,

$$\left\| \sum_{n\in A} e^{inx} \right\|_p \leq C \left( \sum_{n\in A} \xi_n^2 \right)^{1/2}.$$ 

Actually Bourgain’s result is much stronger than this. He shows that if $(g_n)_{n=1}^\infty$ is a uniformly bounded orthonormal system in some $L_2(\mu)$ where $\mu$ is a finite measure, then there is a constant $C$ so that if $F$ is a finite subset of $\mathbb{N}$ then there is a further subset $A$ of $F$ with $|A| \geq |F|^{2/p}$ so that we have an estimate

$$\left( \sum_{n\in A} \xi_n g_n \right)_p \leq C \left( \sum_{n\in A} \xi_n^2 \right)^{1/2}. \quad (1.1)$$

In fact this estimate holds for a random subset of $F$. For an alternative approach to Bourgain’s results, see Talagrand [15].

It is natural to ask for a corresponding result for the Legendre polynomials. Since $(\varphi_n)_{n=1}^\infty$ is not bounded in $L_\infty[-1, 1]$ one cannot apply Bourgain’s result. However, Bourgain [2] states without proof the corresponding result for orthonormal systems which are bounded in some $L_r$ for $r > 2$. Suppose that $(g_n)$ is an orthonormal system...
which is uniformly bounded in \(L_p(\mu)\) for some \(2 < p < \infty\). Then he remarks that if \(2 < p < r < \infty\) there is a constant \(C\) so that for any subset \(F\) of \(\mathbb{N}\) there is a further subset \(A\) of \(F\) with \(|A| \geq |F|^{1/(r-p)}\) so that we have the estimate (1.1). Again this result holds for random subsets. It follows from this result that if \(2 < p < 4\) and \(\epsilon > 0\) \(\{1, 2, \ldots, N\}\) contains a subset \(A\) of size \(N^{4/p-1-\epsilon}\) so that we have the estimate

\[
\left\| \sum_{n \in A} \xi_n \varphi_n \right\|_p \leq C \left( \sum_{n \in A} |\xi_n|^2 \right)^{1/2}.
\]

As shown below in Proposition 3.1, there is an easy upper estimate \(|A| \leq CN^{4/p-1}\) for subsets obeying (1.2). The sharp estimate \(N^{4/p-1}\) cannot be obtained from Bourgain’s results since \((\varphi_n)_{n=1}^\infty\) is unbounded in \(L_4([-1, 1])\).

In this note we show that, nevertheless, if \(F\) is a finite subset of \(\mathbb{N}\) then there is a subset of \(A\) of \(F\) with \(|A| \geq |F|^{1/p-1}\) so that (1.2) holds, and again this holds for random subsets.

In fact we show the corresponding result for more general ultraspherical polynomials. Suppose \(0 < \lambda < \infty\). Let \((\varphi_n^{(\lambda)})_{n=1}^\infty\) be the orthonormal basis of \(L_p([-1, 1], (1-x^2)^{\lambda-\frac{1}{2}})\) obtained from \(\{1, x, x^2, \ldots\}\) by the Gram-Schmidt process. Then \((\varphi_n^{(\lambda)})\) is a basis in \(L_p([-1, 1], (1-x^2)^{\lambda-\frac{1}{2}})\) if \(2 < p < r = 2 + \lambda^{-1}\). We show in Theorem 3.6 that there is a constant \(C\) so that if \(F\) is a finite subset of \(\mathbb{N}\), there is a further subset \(A\) of \(F\) with \(|A| \geq |F|^{2\lambda/(\lambda-1)}\) so that we have the estimate

\[
\left\| \sum_{n \in A} \xi_n \varphi_n^{(\lambda)} \right\|_p \leq C \left( \sum_{n \in A} |\xi_n|^2 \right)^{1/2}.
\]

Here of course norms are computed with respect to the measure \((1-x^2)^{\lambda-\frac{1}{2}}\) \(dx\). Again this result is best possible as with the Legendre polynomials (the case \(\lambda = \frac{1}{2}\)) and holds for random subsets. Notice that if we set \(\lambda = 0\) we obtain the (normalized) Tchebicheff polynomials which after a change of variable reduce to the trigometric system on the circle. Thus Bourgain’s \(\Lambda(p)\)-theorem corresponds to the limiting case \(\lambda = 0\).

As will be seen we obtain our main result by using Bourgain’s theorem and an interpolation technique.

In Section 4 we answer a question of H. König by showing that the space \(P_n\) of polynomials is uniformly isomorphic to \(\ell_p^n\) in every space \(L_p([-1, 1], (1-x^2)^{\lambda-\frac{1}{2}})\) for \(\lambda > \frac{1}{2}\) and \(1 < p < \infty\).

2. Preliminaries. In this section, we collect together some preliminaries. A good general reference for most of the material we need is the book of Szegő [14].

For \(-\frac{1}{2} < \lambda < \infty\) with \(\lambda \neq 0\) we define the ultrasperical polynomials \(P_n^{(\lambda)}\) as in [14] by the generating function relation

\[
(1 - 2xw + w^2)^{-\lambda} = \sum_{n=0}^{\infty} P_n^{(\lambda)}(x)w^n.
\]

For \(\lambda = 0\) we define \(P_n^{(0)}(x) = \frac{2}{n}T_n(x)\) where \(T_n\) are the Tchebicheff polynomials defined by \(T_n(\cos \theta) = \cos n\theta\) for \(0 \leq \theta \leq \pi\). Then we have that if \(\lambda \neq 0\) \([14, p. 81 (4.7.16)\],

\[
\int_{-1}^{1} \left| P_n^{(\lambda)}(x) \right|^2 (1 - x^2)^{\lambda-\frac{1}{2}} dx = 2^{1-2\lambda} \pi \Gamma(\lambda)^{-2} \frac{\Gamma(n+2\lambda)}{(n+\lambda)\Gamma(n+1)}.
\]
It follows that we have
\[ \varphi_n^{(\lambda)} = 2^{\lambda - \frac{1}{2}} \pi^{-\frac{1}{2}} \Gamma(\lambda) \left( \frac{(n + \lambda) \Gamma(n + 1)}{\Gamma(n + 2\lambda)} \right)^{1/2} P_n^{(\lambda)}. \]

We now recall Theorem 8.21.11 of [14, p. 197].

**Proposition 2.1.** Suppose \( 0 < \lambda < 1 \). Then for \( 0 \leq \theta \leq \pi \) we have
\[
\left| P_n^{(\lambda)}(\cos \theta) - 2 \frac{\Gamma(n + 2\lambda)}{\Gamma(\lambda) \Gamma(n + \lambda + 1)} \cos \left( (n + \lambda) \theta - \lambda \pi / 2 \right) (2 \sin \theta)^{-\lambda} \right| \\
\leq \frac{4\lambda(1 - \lambda)\Gamma(n + 2\lambda)}{\Gamma(\lambda)(n + \lambda + 1)\Gamma(n + \lambda + 1)} (2 \sin \theta)^{-\lambda - 1}.
\]

**Remark.** Note we have used that \( \Gamma(\lambda)\Gamma(1 - \lambda) = \pi / \sin(\lambda \pi) \).

The next Proposition is a combination of results on p. 80 (4.7.14) and p. 168 (7.32.1) of [14].

**Proposition 2.2.** If \( 0 < \lambda < \infty \) then we have
\[
\max_{-1 \leq x \leq 1} |P_n^{(\lambda)}(x)| = P_n^{(\lambda)}(1) = \left( \frac{n + 2\lambda - 1}{n} \right).
\]

Here we write
\[
\binom{u}{v} = \frac{\Gamma(u + 1)}{\Gamma(u - v + 1)\Gamma(v + 1)}.
\]

For our purposes it will be useful to simplify the Gamma function replacing it by asymptotic estimates. For this purpose we note that
\[
\frac{\Gamma(n + \sigma)}{\Gamma(n)} = n^{\sigma} + O(n^{\sigma - 1}).
\]

**Proposition 2.3.** Suppose \( 0 < \lambda < \infty \). Then there exists a positive constant \( C = C(\lambda) \) such that
\[
|\varphi_n^{(\lambda)}(\cos \theta) - (2 / \pi)^{1/2} \cos \left( (n + \lambda) \theta - \lambda \pi / 2 \right) (\sin \theta)^{-\lambda}| \leq C(\sin \theta)^{-\lambda} \left( \min \left( (n \sin \theta)^{-1}, 1 \right) \right).
\]

**Proof.** Using the remark preceding the Proposition, we can deduce from Proposition 2.1 that
\[
(2.1) \quad \left| P_n^{(\lambda)}(\cos \theta) - 2^{1-\lambda}n^{\lambda-1}\Gamma(\lambda)^{-1} \cos \left( (n + \lambda) \theta - \lambda \pi / 2 \right) (\sin \theta)^{-\lambda}\right| \leq Cn^{\lambda-2}(\sin \theta)^{-1-\lambda}
\]
where \( C = C(\lambda) \), for \( 0 < \lambda < 1 \). This estimate also holds when \( \lambda = 1 \) trivially (with \( C = 0 \)).

We now prove the same estimate provided \( n \sin \theta \geq 1 \) for all \( \lambda > 0 \) by using the recurrence relation
\[
(2.2) \quad 2(\lambda - 1)(1 - x^2)P_n^{(\lambda)}(x) = (n + 2\lambda - 2)P_n^{(\lambda-1)}(x) - (n + 1)xP_n^{(\lambda-1)}(x)
\]
for which we refer to [14, p. 83 (4.7.27)].

Indeed assume the estimate (2.1) is known for $\lambda - 1$. Then with $x = \cos \theta$,

$$
|P_n^{(\lambda - 1)}(x) - xP_{n+1}^{(\lambda - 1)}(x)| - 2^{\lambda - 2} n^{\lambda - 2} \Gamma(\lambda - 1)^{-1} \cos((n + \lambda - 1)\theta - \lambda \pi / 2)(\sin \theta)^{1-\lambda} | \\
\leq Cn^{\lambda - 2}(\sin \theta)^{-\lambda}.
$$

We also have

$$
|P_n^{(\lambda - 1)}(x)| \leq Cn^{\lambda - 3}(\sin \theta)^{-\lambda} \leq Cn^{\lambda - 2}(\sin \theta)^{1-\lambda}
$$

provided $n \sin \theta \geq 1$. Now using the recurrence relation (2) we obtain an estimate of the form (2.1) provided $n \sin \theta \geq 1$.

Next we observe that for all $\lambda \geq 0$ we have by Proposition 2.2,

$$
|P_n^{(\lambda)}(x)| \leq P_n^{(\lambda)}(1) \leq Cn^{2\lambda - 1}
$$

where $C$ depends only on $\lambda$. Hence if $n \sin \theta < 1$ we have an estimate

$$
(2.3) \quad |P_n^{(\lambda)}(\cos \theta) - 2n^{\lambda - 1}\Gamma(\lambda)^{-1} \cos((n + \lambda)\theta - \lambda \pi / 2)(\sin \theta)^{-\lambda}| \leq Cn^{\lambda - 1}(\sin \theta)^{-\lambda}.
$$

Combining (2.2) and (2.3) gives us an estimate

$$
(2.4) \quad |P_n^{(\lambda)}(\cos \theta) - 2^{1-\lambda} n^{\lambda - 1}\Gamma(\lambda)^{-1} \cos((n + \lambda)\theta - \lambda \pi / 2)(\sin \theta)^{-\lambda}| \\
\leq C \min(n^{\lambda - 2}(\sin \theta)^{1-\lambda}, n^{\lambda - 1}(\sin \theta)^{-\lambda}).
$$

Recalling the relationship between $\varphi_n^{(\lambda)}$ and $P_n^{(\lambda)}$ we obtain the result. ■

**Proposition 2.4.** Suppose $-1/2 < \lambda, \mu < \infty$. Then the orthonormal system $(\varphi_n^{(\lambda)})_{n=0}^\infty$ is a basis of $L_r([-1, 1], (1 - x^2)^{\mu - 1})$ if and only if

$$
\left| \frac{2\mu + 1}{2r} - \frac{2\lambda + 1}{4} \right| < \min \left( \frac{1}{4}, \frac{2\lambda + 1}{4} \right).
$$

In particular, if $\lambda \geq 0$ and $r > 2$ then $(\varphi_n^{(\lambda)})_{n=0}^\infty$ is a basis of $L_r([-1, 1], (1 - x^2)^{\mu - 1/2})$ if and only if $r < 2 + \lambda^{-1}$.

**Proof.** This theorem is a special case of a very general result of Badkov [1, Theorem 5.1]. The second part is much older: see Pollard [9], [10] and [11], Newman-Rudin [8] and Muckenaupt [7].

We will also need some results on Gauss-Jacobi mechanical quadrature. To this end let $(\theta_n^{(\lambda)})_{n=0}^\infty$ be the zeros of the polynomial $\varphi_n^{(\lambda)}$ ordered so that $0 < \theta_n^{(\lambda)} < \theta_{n+1}^{(\lambda)} < \cdots < \theta_m^{(\lambda)} < \pi$. (We remark that the zeros are necessarily distinct and are all located in $(-1, 1)$; see Szegő [14, p. 44].)
PROPOSITION 2.5. Suppose \(-\frac{1}{2} < \lambda < \infty\). Then there exists a constant \(C\) depending only on \(\lambda\) so that
\[
\left| \theta_{nk}^{(\lambda)} - \frac{k\pi}{n} \right| \leq \frac{C}{n}.
\]
Furthermore, there exists \(c > 0\) so that
\[
|\theta_{nk}^{(\lambda)}| \geq \frac{ck}{n}
\]
if \(k < n/2\).

PROOF. The following result is contained in Theorem 8.9.1 of Szegő [14, p. 238]. The second part follows easily from the first and the fact that \(\lim_{n \to \infty} n\theta_{nk}^{(\lambda)}\) exists and is the first positive zero of the Bessel function \(J_{\lambda+\frac{1}{2}}(t)\) (see Szegő [14, Theorem 8.1.2, pp. 192–193]).

We will denote by \(\mathcal{P}_n\) the space of polynomials of degree at most \(n - 1\) so that \(\dim \mathcal{P}_n = n\).

PROPOSITION 2.6. Suppose that \(-\frac{1}{2} < \lambda < \infty\). Then there exist positive constants \((\alpha_{nk}^{(\lambda)})_{1 \leq k \leq n < \infty}\) such that if \(f \in \mathcal{P}_n\) then
\[
\int_{-1}^{1} f(x)(1 - x^2)^{\lambda - \frac{3}{4}} dx = \sum_{k=1}^{n} \alpha_{nk}^{(\lambda)} f(\theta_{nk}^{(\lambda)}).
\]
Furthermore there is a constant \(C\) depending only on \(\lambda\) such that
\[
\alpha_{nk}^{(\lambda)} \leq C (\sin \theta_{nk}^{(\lambda)})^{2\lambda} n^{-1}.
\]

PROOF. This is known as Gauss-Jacobi mechanical quadrature. See Szegő [14, pp. 47–50]. The estimate on the size of \((\alpha_{nk}^{(\lambda)})\) may be found on p. 354. However this estimate is perhaps most easily seen by combining the Tchebicheff-Markov-Stieltjes separation theorem (Szegő, p. 50) with the estimate on the zeros (Proposition 2.5). More precisely there exist \((y_k)_{k=0}^n\) such that \(1 = y_0 > \tau_{n,1}^{(\lambda)} > y_1 > \tau_{n,2}^{(\lambda)} > \cdots > \tau_{n,n}^{(\lambda)} = -1\) so that
\[
\alpha_{nk}^{(\lambda)} = \int_{y_{k-1}}^{y_k} (1 - x^2)^{\lambda - \frac{3}{4}} dx.
\]
The estimate follows from Proposition 2.5.

3. The \(\Lambda(p)\) problem. We first note that by Proposition 2.4, in order that \((\varphi_{n}^{(\lambda)})_{n=1}^\infty\) be a basis in \(L_p([-1, 1], (1 - x^2)^{\lambda - \frac{3}{4}})\), it is necessary and sufficient that \(2 < p < 2 + \lambda - 1\). Let us denote this critical index by \(r = r(\lambda) = 2 + \lambda - 1\).

Let \(\mathcal{A}\) be a subset of \(\mathbb{N}\), and \(2 < p < r\). We will say that \(\mathcal{A}\) is a \(\Lambda(p, \lambda)\)-set if there is a constant \(C\) so that for any finite-sequence \((\xi_n : n \in \mathcal{A})\) we have
\[
\left( \int_{-1}^{1} \left| \sum_{n \in \mathcal{A}} \xi_n \varphi_{n}^{(\lambda)}(x) \right|^p (1 - x^2)^{\lambda - \frac{3}{4}} dx \right)^{1/p} \leq C \left( \sum_{n \in \mathcal{A}} |\xi_n|^2 \right)^{1/2}.
\]
This means that the operator \( T: \ell_2(\mathbb{A}) \rightarrow L_p([-1, 1], (1 - x^2)^{-\frac{1}{2}}) \) defined by \( T\xi = \sum_{n \in \mathbb{A}} \xi_n \varphi_n^{(L)} \) is bounded, and indeed since there is an automatic lower bound, an isomorphic embedding. We denote the least constant \( C \) or equivalently \( \|T\| \) by \( \Lambda_{p, \lambda}(\mathbb{A}) \).

Note that if \( \lambda = 0 \) then \( \varphi_n^{(L)}(\cos \theta) = \cos n\theta \) and this definition reduces to the standard definition of a \( \Lambda(p) \)-set introduced by Rudin [12].

**Proposition 3.1.** For each \( \lambda > 0 \) there is a constant \( C = C(\lambda) \) depending on \( \lambda \) so that if \( \mathbb{A} \) is a \( \Lambda(p, \lambda) \)-set then

\[
|\mathbb{A} \cap [1, N]| \leq C \Lambda_{p, \lambda}(\mathbb{A})^2 N^{2\lambda(p-1)}.
\]

**Proof.** Observe first that

\[
\max_{-1 \leq k \leq 1} |\varphi_n^{(L)}(\lambda)| = |\varphi(1)| \geq c n^\lambda
\]

for some constant \( c > 0 \) depending only on \( \lambda \) by Proposition 2.2 and the remark thereafter. It follows from Bernstein’s inequality that if \( 0 \leq \theta \leq (2n)^{-1} \) then \( \varphi_n^{(L)}(\cos \theta) \geq c n^\lambda / 2 \).

In particular let \( J = \mathbb{A} \cap [N/2, N] \). Then for \( 0 \leq \theta \leq (2N)^{-1} \) we have

\[
\sum_{n \in J} \varphi_n^{(L)}(\cos \theta) \geq c N^{\lambda} |J|
\]

where \( c > 0 \) depends only on \( \lambda \). Since \( dx = (\sin \theta)^{2\Delta} d\theta \) we therefore have

\[
c N^{\lambda} |J| N^{-(2\lambda+1)/p} \leq C \Lambda(\mathbb{A}) |J|^{1/2}
\]

where \( 0 < c, C < \infty \) are again constants depending only on \( \lambda \). We thus have an estimate

\[
|J| \leq C \Lambda_{p, \lambda}(\mathbb{A})^2 N^{4\lambda(p-1)/p-2\lambda} = C \Lambda_{p, \lambda}(\mathbb{A})^2 N^{2\lambda(p-1)}.
\]

This clearly implies the result.

Our next Proposition uses the approximation of Proposition 2.3 to transfer the problem to a weighted problem on the circle \( T \) which we here identify with \([-\pi, \pi]\).

**Proposition 3.2.** Suppose \( \lambda > 0 \) and \( 2 < p < r(\lambda) \). Then \( \mathbb{A} \) is a \( \Lambda(p, \lambda) \)-set if and only if the operator \( S: \ell_2(\mathbb{A}) \rightarrow L_p(T, |\sin \theta|^{(2-p)}) \) is bounded where \( S \xi_n = e^{in\theta} \), where \( (e_n) \) is the canonical basis of \( \ell_2(\mathbb{A}) \). Furthermore there is a constant \( C = C(p, \lambda) \) so that \( C^{-1} ||S|| \leq \Lambda_{p, \lambda}(\mathbb{A}) \leq C ||S|| \).

**Proof.** Let us start by proving a similar estimate to Proposition 3.1 for the system \( \{e^{in\theta}\} \). Suppose \( S \) is bounded. If \( N \in \mathbb{N} \) then we note that for \( 1 \leq k \leq N \) we have \( \cos k\theta > 1/2 \) if \( \theta < \pi / 3N \). Hence if \( \theta < \pi / 3N \) we have \( \sum_{k \in J} \cos k\theta > \frac{1}{2} |J| \) where \( J = \mathbb{A} \cap [1, N] \). It follows that

\[
|J| N^{\lambda(p-2)-1/p} \leq C ||S|| |J|^{1/2}
\]

where \( C \) depends only on \( \lambda \). This yields an estimate

\[
|J| \leq C ||S||^2 N^{2\lambda(p-1)}
\]
where \( C \) depends only on \( \lambda \).

Now consider the map \( S_0: \ell_2(\mathbb{A}) \rightarrow L_q([0, \pi], |\sin \theta|^{2\lambda}) \) defined by \( S_0 e_n = \cos\left((n + \lambda)\theta - \frac{\lambda \pi}{2}\right) |\sin \theta|^{-\lambda} \). We will observe that \( S_0 \) is bounded if and only if \( S \) is bounded and indeed \( \|S_0\| \leq 2\|S\| \leq C\|S_0\| \) where \( C \) depends only on \( p \). In fact if \((\xi_n)_{n \in \mathbb{A}}\) are finitely non-zero and real then

\[
\|S_0 \xi\|^p \leq \int_0^\pi \left| \sum_{n \in \mathbb{A}} \xi_n e^{i\theta} \right|^p |\sin \theta|^{\lambda(2-p)} d\theta \leq \|\xi\|^p
\]

which leads easily to the first estimate \( \|S_0\| \leq 2\|S\| \). For the converse direction, we note that \( w(\theta) = |\sin \theta|^{\lambda(2-p)} \) is an \( A_p \)-weight in the sense of Muckenhaust (see [3], [4] or [7]), \( i.e. \), there is a constant \( C \) so that for every interval \( I \) on the circle we have

\[
\left( \int_I w(\theta) d\theta \right)^{1/p} \left( \int_I w(\theta)^{-p'/p} d\theta \right)^{1/p'} \leq C |I|
\]

where \(|I|\) denote the length of \( I \). It follows that the Hilbert-transform is bounded on the space \( L^p(T, w) \) so that there is a constant \( C = C(p, \lambda) \) such that if \((\xi_n)_{n \in \mathbb{A}}\) is finitely non-zero and real then

\[
\left( \int_0^\pi \left| \sum_{n \in \mathbb{A}} \xi_n \sin\left((n + \lambda)\theta - \frac{\lambda \pi}{2}\right) |\sin \theta|^{\lambda(2-p)} d\theta \right)^{1/p}
\]

\[
\leq C \left( \int_0^\pi \left| \sum_{n \in \mathbb{A}} \xi_n \cos\left((n + \lambda)\theta - \frac{\lambda \pi}{2}\right) |\sin \theta|^{\lambda(2-p)} d\theta \right)^{1/p}
\]

This quickly implies an estimate of the form \( \|S\xi\| \leq C\|S_0 \xi\| \).

Now consider the map \( T: \ell_2(\mathbb{A}) \rightarrow L_p([0, \pi], |\sin \theta|^2) \) defined by \( Te_n = \varphi_n^\lambda(\cos \theta) \).

Then for some constant \( C = C(\lambda) \) we have (using Proposition 2.3),

\[
|\psi_n(\theta)| \leq C (\sin \theta)^{-\lambda} \min\left( (\sin \theta)^{-1}, 1 \right)
\]

where

\[
\psi_n(\theta) = \varphi_n^\lambda(\cos \theta) - \cos\left((n + \lambda)\theta - \frac{\lambda \pi}{2}\right) |\sin \theta|^{-\lambda}.
\]

Now suppose \( \mathbb{A} \) satisfies an estimate \( |\mathbb{A} \cap [1, N]| \leq K N^{2\lambda(1/p-1)} \) for some constant \( K \).

We will let \( J_k = \mathbb{A} \cap [2^{k-1}, 2^k) \) and \( E_k = \{ \theta : 2^{-k} < \sin \theta < 2^{1-k} \} \). Then on \( E_k \) we have an estimate \( |\psi(\theta)| \leq C 2^{\lambda k} \) if \( n \leq 2^k \) and \( |\psi_n(\theta)| \leq C n^{-1} 2^{(1+\lambda)k} \) if \( n > 2^k \). Here \( C \) depends a constant depending only on \( p \) and \( \lambda \).

Let \((\xi_n)_{n \in \mathbb{A}}\) be any finitely non-zero sequence and set \( u_k = (\sum_{n \in J_k} |\xi_n|^2)^{1/2} \). Note that \( \sum_{n \in J_k} |\xi_n|^2 \leq |J_k|^{1/2} u_k \).

It follows that if \( 1 \leq l \leq k \) we have

\[
\left( \int_{E_k} \sum_{n \in J_l} |\xi_n \psi_n|^p |\sin \theta|^{2\lambda} d\theta \right)^{1/p} \leq C 2^{\lambda k} 2^{-(1+\lambda)k/p} |J_l|^{1/2} u_l
\]
while if $k + 1 \leq l < \infty$
\[
\left( \int_{E_l} \left| \sum_{n \in A} \xi_n \psi_n \right|^p (\sin \theta)^{2k} d\theta \right)^{1/p} \leq C 2^{2k+(k-1)2^{-\left(1+2\delta\right)k/p}} |J_l|^{1/2} u_l.
\]

Note that $\lambda = (1 + 2\lambda)/p = \lambda(1 - r/p)$. We also have $|J_l| \leq K 2^{2k(r/p - 1)}$. Hence we obtain an estimate
\[
\left\| \chi_{E_l} \sum_{n \in A} \xi_n \psi_n \right\| \leq CK^{1/2} \left( \sum_{l=1}^{\infty} 2^{k(r/p - 1)(l-k)} u_l + \sum_{l=k+1}^{\infty} 2^{k(r/p - 1)(l-k)} u_l \right).
\]

Let $\delta = \min\left(\lambda(r/p - 1), 1 - \lambda(r/p - 1)\right)$. Then the right-hand side may estimated by
\[
CK^{1/2} \left( \sum_{l=1}^{\infty} 2^{-\delta(|l-k|)} u_l \right) = CK^{1/2} \sum_{j \in \mathbb{Z}} 2^{-\delta j} u_{k+j}
\]
where $u_j = 0$ for $j \leq 0$. Since $p > 2$ we have
\[
\left\| \sum_{n \in A} \xi_n \psi_n \right\| \leq \left( \sum_{l=1}^{\infty} \left\| \chi_{E_l} \sum_{n \in A} \xi_n \psi_n \right\|^2 \right)^{1/2}.
\]
Hence by Minkowski’s inequality in $\ell_2$ we have
\[
\left\| \sum_{n \in A} \xi_n \psi_n \right\| \leq CK^{1/2} \sum_{j \in \mathbb{Z}} 2^{-\delta j} \left( \sum_{l=1}^{\infty} u_l^2 \right)^{1/2}.
\]

We conclude that $\|S \xi - T \xi\| \leq CK^{1/2}$. Now if $T$ is bounded then $K \leq C|T|^2$ while if $S$ is bounded then $K \leq C\|S\|^2$. This yields the estimates promised.

As remarked above, using Proposition 3.2 we can transfer the problem of identifying \(\Lambda(p, \lambda)\)-sets to a similar problem concerning the standard characters \(\{e^{i\theta}\}\) in a weighted $L_p$-space. We will now solve a corresponding problem in the case when $p = 2$ and then use the solution to obtain our main result in the case $p > 2$. To this end we will first prove a result concerning weighted norm inequalities for an operator on the sequence space $\ell_2(\mathbb{Z})$ which is the discrete analogue of a Riesz potential.

Suppose $0 < \alpha < 1/2$. For $m, n \in \mathbb{Z}$ we define $K(m, n) = |m - n|^{\alpha - 1}$ when $m \neq n$ and $K(m, n) = 1$ if $m = n$. Let $c_{00}(\mathbb{Z})$ be the space of finitely non-zero sequences. Then we can define a map $K: c_{00}(\mathbb{Z}) \to \ell_2(\mathbb{Z})$ by $K\xi(m) = \sum_{n \in \mathbb{Z}} K(m, n) \xi(n)$.

Now suppose $\nu \in \ell_\infty(\mathbb{Z})$. We define $L(\nu)$ to be the norm in $\ell_2(\mathbb{Z})$ of the operator $\xi \to \nu K \xi$ which we take to be $\infty$ if this operator is unbounded. Thus $L(\nu) = \sup\{\|\nu K \xi\| : \|\xi\| \leq 1\}$.

The following result can be derived from similar results in potential theory (for example, [13]). For more general results we refer to [5]. However we will give a self-contained exposition.
**Theorem 3.3.** Let $0 \leq M(v) \leq \infty$ be the least constant so that for every finite interval $I \subset \mathbb{Z}$ we have

$$\sum_{m,n \in I} v_m^2 v_n^2 \min(1, |m-n|^{2\alpha-1}) \leq M^2 \sum_{n \in I} v_n^2.$$ 

Then for a constant $C$ depending only on $\alpha$ we have $C^{-1} M(v) \leq L(v) \leq CM(v)$.

**Proof.** First suppose $L(v) < \infty$. Then by taking adjoints the map $\xi \mapsto K(v\xi)$ is bounded on $\mathcal{E}_2(\mathbb{Z})$ with norm $L(v)$. In particular we have for any interval $I$, $\|K(v^2\chi_I)\| \leq L(v)\|\chi_I\|$. Let us write $\langle \xi, \eta \rangle = \sum_{n \in I} \xi_n \eta_n$ where this is well-defined. Thus

$$\langle K^2(v^2\chi_I), v^2\chi_I \rangle \leq L(v)^2 \sum_{n \in I} v_n^2.$$ 

Now observe that $K^2(m, n) = \sum_{l=1}^\infty K(m, l)K(l, n) \geq c \min(1, |m-n|^{2\alpha-1})$ where $c > 0$ depends only on $\alpha$. Expanding out we obtain that $M(v) \leq CL(v)$ for some $C = C(\alpha)$.

We now turn to the opposite direction. By homogeneity it is only necessary to bound $L(v)$ when $M(v) = 1$. We therefore assume $M(v) = 1$. Notice that it follows from the definition of $M(v)$ that for any interval $I$, we have $|I|^{2\alpha-1} \sum_{m,n \in I} v_m^2 v_n^2 \leq \sum_{n \in I} v_n^2$ and so $\sum_{n \in I} v_n^2 \leq |I|^{1-2\alpha}$. 

Now let $u = K v^2$. This can be computed formally, with the possibility of some entries being infinite, but the calculations below will show that the entries of $u$ are finite; alternatively the estimate above leads quickly to the same conclusion. Suppose $m \in \mathbb{Z}$ and define sets $I_0 = \{m\}$ and then $I_k = \{n : 2^{k-1} \leq |m-n| < 2^k\}$ for $k \geq 1$. Note that if $k \geq 1 I_k$ is the union of two intervals of length $2^{k-1}$. Let $J_k = I_0 \cup \cdots \cup I_k$.

For any $k$ we have

$$u = K(v^2\chi_{J_k}) + \sum_{l=2^k+1}^{2^{k+1}+1} K(v^2\chi_{I_l}).$$

Let us write $u_1 = K(v^2\chi_{J_0})$ and $u_2 = u - u_1$.

Now if $l \geq k + 2$ and $j \in I_k$ we have

$$K(v^2\chi_{I_l})(j) \leq C2^{(\alpha-1)l} \sum_{n \in I_k} v_n^2,$$

Hence

$$u_2(j) \leq C \sum_{l=2^k+2}^{2^{k+1}+1} 2^{(\alpha-1)l} \sum_{n \in I_k} v_n^2.$$ 

Squaring and summing, and estimating $\sum_{n \in I_k} v_n^2$, we have

$$\sum_{j \in I_k} u_2(j)^2 \leq C \sum_{l=2^k+2}^{2^{k+1}+1} 2^{(\alpha-1)(l+1)} \sum_{n \in I_k} v_n^2.$$ 

Summing out over $i \geq l$ we have

$$\sum_{j \in I_k} u_2(j)^2 \leq C \sum_{l \geq k+2} 2^{-l} \sum_{n \in I_k} v_n^2.$$
On the other hand

\[ \sum_{j \in I_j} u_j^2(j) = \sum_{j \in I_j} \sum_{i \in I_{j+1}} \sum_{l \in I_{j+1}} K(j, i)K(l, j) v_i^2 v_j^2 \leq C \sum_{i \in I_{j+1}} \sum_{l \in I_{j+1}} \min(1, |i-l|^{2\alpha-1}) v_i^2 v_j^2 \leq C \sum_{n \in I_{j+1}} v_n^2 \]

where \( C \) depends only on \( \alpha \). In particular \( u(j) < \infty \) for all \( j \).

Hence

\[ \sum_{j \in I_j} u(j)^2 \leq C \left( \sum_{n \in I_{j+1}} v_n^2 + 2^k \left( \sum_{l=0}^{\infty} 2^{-l} \sum_{n \in I_l} v_n^2 \right) \right). \]

This can be written as

\[ \sum_{j \in I_j} u(j)^2 \leq C \sum_{l=0}^{\infty} \min(1, 2^{k-l}) \sum_{n \in I_l} v_n^2. \]

Let us use this to estimate \( Ku^2(m) \); we have (letting \( C \) be a constant which depends only on \( \alpha \) but may vary from line to line),

\[ Ku^2(m) \leq C \sum_{l=0}^{\infty} 2^{(\alpha-1)k} \sum_{n \in I_l} u_n^2 \]
\[ \leq C \sum_{l=0}^{\infty} 2^{(\alpha-1)k} \sum_{n \in I_l} \min(1, 2^{k-l}) \sum_{n \in I_l} v_n^2 \]
\[ \leq C \sum_{l=0}^{\infty} \sum_{n \in I_l} v_n^2 \sum_{k=0}^{\infty} 2^{(\alpha-1)k} \min(1, 2^{k-l}) \]
\[ \leq C \sum_{l=0}^{\infty} 2^{(\alpha-1)l} \sum_{n \in I_l} v_n^2 \]
\[ \leq CKv^2(m). \]

We thus have \( Ku^2 \leq CKv^2 \).

Now put \( w = v + Kv^2 \). Then \( Kw^2 \leq 2(Kv^2 + Ku^2) \leq CKv^2 \leq Cw \). We will show this implies an estimate on \( L(v) \).

Indeed if \( \xi \in c_{00}(Z) \) is positive then

\[ \langle wK\xi, w\xi \rangle = \langle w^2, (K\xi)^2 \rangle. \]

Now

\[ (K\xi)^2(m) = \sum_{i,j} K(m, i)K(m, j)\xi(i)\xi(j) \leq C \sum_{i,j} K(i, j)\left( K(m, i) + K(m, j) \right)\xi(i)\xi(j). \]

This implies \( (K\xi)^2 \leq CK(K\xi) \). Hence

\[ \|wK\xi\|^2 \leq C \langle w^2, K(K\xi) \rangle = C \langle Kw^2, K\xi \rangle \]
and hence as \( Kw^2 \leq Cw \)

\[
\|wK\xi\|^2 \leq C(w, \xi, wK\xi) = C(\xi, wK\xi) \leq C\|\xi\| \|wK\xi\|
\]

which leads to \( \|wK\xi\| \leq C\|\xi\| \) or \( L(v) \leq C(w) \) where \( C \) depends only on \( \alpha \). \( \blacksquare \)

**Theorem 3.4.** Suppose \( 0 < \alpha < 1/2 \). Let \( \mathcal{A} \) be a subset of \( \mathbb{Z} \). Let \( \kappa(\mathcal{A}) = \kappa_\alpha(\mathcal{A}) \) be the least constant (possibly infinite) such that for any finitely nonzero sequence \( (\xi_n)_{n \in \mathcal{A}} \) we have

\[
\left( \int_{\pi}^{\pi} \left| \sum_{n \in \mathcal{A}} \xi_n e^{in\theta} \right|^2 \frac{\sin \theta}{\theta} \frac{d\theta}{\theta} \right)^{1/2} \leq \kappa \left( \sum_{n \in \mathcal{A}} |\xi_n|^2 \right)^{1/2}.
\]

Let \( M = M(\mathcal{A}) = M(\chi_\mathcal{A}) \), be defined as the least constant \( M \) so that for any finite interval \( I \) we have, setting \( F = \mathcal{A} \cap I \),

\[
\sum_{m,n \in F} \min(1, |m-n|^{2\alpha-1}) \leq M^2 |F|.
\]

Then \( \kappa(\mathcal{A}) < \infty \) if and only if \( M(\mathcal{A}) < \infty \) and there is constant \( C \) depending only on \( \alpha \) such that \( C^{-1} M(\mathcal{A}) < \kappa(\mathcal{A}) \leq C M(\mathcal{A}) \).

**Proof.** First suppose \( M(\mathcal{A}) < \infty \). Note that \( \psi(\theta) = |\theta|^{-\alpha} \) is an \( L_2 \) function whose Fourier transform satisfies the property that \( \lim_{|n| \to \infty} n |n|^{-\alpha} \hat{\psi}(n) \) exists and is positive. Now suppose \( (\xi_n) \in c_{00}(\mathcal{A}) \) and let \( g = \sum_{n \in \mathcal{A}} \xi_n e^{in\theta} \). Suppose \( f \in L_2[-\pi, \pi] \). Then

\[
\langle |\theta|^{-\alpha} g, f \rangle = \langle \hat{\psi} \ast g, \hat{f} \rangle.
\]

Hence for a suitable \( C = C(\alpha) \) we have, using Plancherel’s theorem, with \( K \) as in Theorem 3.3,

\[
\langle |\theta|^{-\alpha} g, f \rangle \leq C \langle K|\hat{g}|, |\hat{f}| \rangle = C \langle |\hat{g}|, \chi_\mathcal{A} |\hat{f}| \rangle.
\]

We deduce

\[
\langle |\theta|^{-\alpha} g, f \rangle \leq CM(\mathcal{A}) ||g||_2 ||f||_2.
\]

Thus

\[
\int_{-\pi}^{\pi} |\hat{g}(\theta)|^2 |\theta|^{-2\alpha} d\theta \leq C^2 M^2 \left( \sum_{n \in \mathcal{A}} |\xi_n|^2 \right).
\]

By translation we also have

\[
\int_{-\pi}^{\pi} |\hat{g}(\theta)|^2 (\pi - |\theta|)^{-2\alpha} d\theta \leq C^2 M^2 \left( \sum_{n \in \mathcal{A}} |\xi_n|^2 \right).
\]

Since \( |\theta|^{-2\alpha} + (\pi - |\theta|)^{-2\alpha} \geq |\sin \theta|^{-2\alpha} \) we obtain immediately \( \kappa(\mathcal{A}) \leq CM(\mathcal{A}) \) where \( C \) depends only on \( \alpha \).

Conversely suppose \( \kappa(\mathcal{A}) < \infty \). Note first that there is positive-definite and non-negative trigonometric polynomial \( h \) so that \( h + \psi \) satisfies \( \hat{h}(n) + \hat{\psi}(n) \geq c \min(1, |n|^{\alpha-1}) \) where \( c > 0 \). Now clearly for \( (\xi_n) \in c_{00}(\mathcal{A}) \),

\[
\int_{-\pi}^{\pi} |\hat{g}|^2 (\psi + h)^2 d\theta \leq C \kappa \left( \sum_{n \in \mathcal{A}} |\xi_n|^2 \right)^{1/2}.
\]
Thus again by Plancherel’s theorem, if $\xi \geq 0$,

$$\|K\xi\|^2 \leq C\|\xi\|^2.$$  

A similar inequality then applies for general $\xi$.

It follows quickly by taking adjoints that $L(\chi_A) \leq C\xi$ and hence $M(\xi) \leq C\xi$.

**Theorem 3.5.** Suppose $F$ is a finite subset of $\mathbb{Z}$ and $|F| = N$. Let $(\eta_j)_{j \in \mathbb{Z}}$ be a sequence of independent $0 - 1$-valued random variables (or selectors) with $E(\eta_j) = \sigma = N^{-2\alpha}$ for $j \in F$. Let $A = \{j \in F : \eta_j = 1\}$ be the corresponding random subset of $F$. Then $E\left(M(\eta)^2\right) \leq C$ where $C$ depends only on $\alpha$.

**Proof.** It is easy to see that if this statement is proved for the set $F = \{1, 2, \ldots, N\}$ then it is true for every interval $F$ and then for every finite subset of $\mathbb{Z}$. It is also easy to see that it suffices to prove the result for $N = 2^n$ for some $n$.

Note next that

$$M(\eta)^2 \leq \sup_{1 \leq k \leq N} \sum_{n \in A} \min(|k - n|^{2\alpha - 1}, 1).$$

Hence

$$M(\eta)^2 \leq C \sum_{k=0}^n \max_{1 \leq k \leq 2^{k-1}} \lambda \left(2^{(2\alpha - 1)} \max_{1 \leq k \leq 2^{k-1}} |A \cap [(k-1)2^k + 1, j2^k]|\right),$$

where $C$ depends only on $\alpha$.

Fix an integer $s$. We estimate, for fixed $k$,

$$E\left(\max_{1 \leq k \leq 2^{k-1}} |A \cap [(k-1)2^k + 1, j2^k]|\right) \leq E\left(\sum_{j=1}^{2^n} \left(\sum_{t \geq |k-j|2^{k-1} + 1} \eta_j\right)^{s/2}\right) \leq E\left(\sum_{j=1}^{2^n} \left(\sum_{t \geq |k-j|2^{k-1} + 1} \eta_j\right)^{s/2}\right)^{1/2} \leq \left(E\left(\sum_{j=1}^{2^n} \eta_j\right)^{s/2}\right)^{1/2} \leq 2^{(n-\alpha)/2} \left(E\left(\sum_{j=1}^{2^n} \eta_j\right)^{s/2}\right)^{1/2}.$$  

Let us therefore estimate, setting $m = 2^k$,

$$E\left(\sum_{j=1}^{m} \eta_j\right)^s = \sum_{t \leq \min(s,m)} \sum_{\eta_{j_1} \cdots \eta_{j_t} = s} \sum_{j_1 \cdots j_t} \frac{s!}{j_1! \cdots j_t!} \sigma^{j_1 + \cdots + j_t} \leq \sum_{j=1}^{m} \left(\begin{array}{c} m \\ j \end{array}\right) \lambda^j \sigma^j \leq \sum_{j=1}^{s} \lambda^j (m \sigma)^j \leq s \max_{1 \leq \sigma \leq m} \lambda^j (m \sigma)^j.$$  

By maximizing the function $x^s e^{-x}$ we see that if $m \sigma \geq e^{-1}$ we can estimate this by

$$E\left(\sum_{j=1}^{m} \eta_j\right)^s \leq s^{s+1} (m \sigma)^s.$$  


On the other hand if \( m \sigma < e^{-1} \)

\[
E\left( \sum_{j=1}^{m} |\eta_j| \right)^r \leq s(s \log m \sigma^{-1} s/|\log m \sigma|) \leq s^{r+1} |\log m \sigma|^{-r}.
\]

Suppose \( k < n \). Put \( s = n - k \). We have

\[
E\left( \max_{1 \leq j \leq 2^{k}-1} |A \cap [(j-1)2^k + 1, j2^k]| \right) \leq C(n-k)2^k \sigma
\]
whenever \( 2^k \sigma \geq e^{-1} \) where \( C = C(\alpha) \). If \( 2^k \sigma < e^{-1} \),

\[
E\left( \max_{1 \leq j \leq 2^{k}-1} |A \cap [(j-1)2^k + 1, j2^k]| \right) \leq C \frac{n-k}{\log(\sigma 2^k)}
\]

Hence

\[
E(M(A)^2) \leq \sum_{2^k \sigma < e^{-1}} \frac{n-k}{\log(\sigma 2^k)^2} 2^{2k \sigma - 1}\sigma \leq \sum_{2^k \sigma \geq e^{-1}} (n-k+1) 2^{2k \sigma} \sigma.
\]

We can estimate this further by

\[
E(M(A)^2) \leq C(\sum_{2^k \sigma < e^{-1}} 2^{2k \sigma - 1}\sigma + n\sigma^{1-2\sigma} + 2^{2n} \sigma)
\]

where \( C = C(\alpha) \).

We now recall that \( \sigma = N^{-2\sigma} = 2^{-2m} \). We then obtain an estimate

\[
E(M(A)^2) \leq C(\alpha).
\]

**Theorem 3.6.** Suppose \( 0 < \lambda < \infty \) and that \( 2 < p < r = 2 + \lambda^{-1} \). Let \( \mathbb{F} \subset \mathbb{N} \) be a finite set with \( |\mathbb{F}| = N \). Let \( (\eta_j)_{j \in \mathbb{F}} \) be a sequence of independent \( 0-1 \)-valued random variables (or selectors) with \( \mathbb{E}(\eta_j) = \sigma = N^{(1/p-1/2)/(1/2-1/r)} \) for \( j \in \mathbb{F} \). Let \( A = \{ j \in \mathbb{F} : \eta_j = 1 \} \) be the corresponding random subset of \( \mathbb{F} \) (so that \( \mathbb{E}(|A|) = N^{(1/p-1)/(1/2-1/r)} \)).

Then \( \mathbb{E}(\max_{j \leq 1/\lambda}(A)^p) \leq C \) where \( C \) depends only on \( p \) and \( \lambda \).

**Proof.** Suppose \( (\xi_n)_{n \in A} \) are any (complex) scalars and let \( f = \sum_{n \in A} \xi_n e^{in\theta} \). Let \( \alpha = (1/2 - 1/p)/(1 - 2/r) \), and let \( \frac{1}{r} = \frac{1}{2} - \alpha \). Then by Holder's inequality, since \( \frac{1}{p} = (1 - 2/r)q + \frac{2}{r} \)

\[
\left( \int_{\pi}^{-\pi} |f| |\sin \theta|^{(2/p-1)} f \, d\theta \right)^{1/p} \leq \left( \int_{\pi}^{-\pi} |f|^q d\theta \right)^{(1-2/r)/q} \left( \int_{\pi}^{-\pi} |f|^{\lambda(1/p-1/2)} d\theta \right)^{1/r}.
\]

Note that \( r\lambda(1/p - 1/2) = (1/p - 1/2)/(1 - 2/r) = \alpha \). Hence

\[
\left( \int_{\pi}^{-\pi} |f| |\sin \theta|^{(2/p-1)} f \, d\theta \right)^{1/p} \leq \Lambda_{q,0}(A)^{1-2/r} \kappa_{\sigma}(A)^{2/r} \left( \sum_{n \in A} |\xi_n|^2 \right)^{1/2}.
\]
Thus we deduce

\[ \Lambda_{p,\lambda}(A_1) \leq \Lambda_{p,0}(A_1)^{1-2/r}\kappa(A_1)^{2/r}. \]

It follows further from Holder’s inequality that

\[ \left( E(\Lambda_{p,\lambda}(A_1))^p \right)^{1/p} \leq E(\Lambda_{q,0}(A_1)^{q(1-2/r)/q}) E(\kappa(A_1)^2)^{1/r}. \]

As \( E(|A|) = N^{2/q} \), we have by the \( \Lambda(p) \) theorem of Bourgain [2] that \( E(\Lambda_{q,0}(A_1)^{q})^{1/q} \leq C = C(q) \). By Theorem 3.5 above we obtain:

\[ \left( E(\Lambda_{p,\lambda}(A_1))^p \right)^{1/p} \leq C \]

where \( C = C(\lambda, p) \).

4. The structure of the space of polynomials. We recall that \( \tau_n^{(\lambda)} = \cos \theta_n^{(\lambda)} \) are the zeros of the polynomial \( \varphi_n^{(\lambda)} \) ordered so that \( 0 < \theta_1^{(\lambda)} < \theta_2^{(\lambda)} < \cdots < \theta_m^{(\lambda)} < \pi \).

**THEOREM 4.1.** Suppose \( 1 < p < \infty, -\frac{2}{3} < \lambda, \mu < \infty \) and that the ultraspherical polynomials \( (\varphi_n^{(\lambda)})_{n=0}^{\infty} \) form a basis of \( L_p((-1, 1), (1 - x^2)^{\mu-\frac{1}{2}}) \) or, equivalently that

\[ (4.1) \quad \frac{2\mu + 1}{2p} - \frac{2\lambda + 1}{4} < \min \left( \frac{1}{4}, \frac{2\lambda + 1}{4} \right). \]

Let \( \tau_n = \tau_n^{(\lambda)} \). Then there is a constant \( C = C(\lambda, \mu, p) \) independent of \( n \) so that if \( f \in \mathcal{P}_n \) then

\[ \frac{1}{C} \left( \frac{1}{n} \sum_{k=1}^{n} (1 - \tau_n^{2})^{\mu} |f(\tau_n)|^p \right)^{1/p} \leq \left( \int_{-1}^{1} |f(x)|^p (1 - x^2)^{\mu-\frac{1}{2}} \, dx \right)^{1/p} \]

\[ \leq C \left( \frac{1}{n} \sum_{k=1}^{n} (1 - \tau_n^{2})^{\mu} |f(\tau_n)|^p \right)^{1/p}. \]

In particular \( d(\mathcal{P}_n, \ell_2^p) \leq C^2 \).

**PROOF.** We will start by supposing that \( \mu \) is not of the form \( \frac{1}{2}(mp - 1) \) for \( m \in \mathbb{N} \) and that \(-\frac{1}{2} < \lambda \) is arbitrary (i.e., we do not assume (4.1)). In this case we can find \( m \in \mathbb{N} \) so that \(-\frac{1}{2} < \mu - \frac{1}{2}mp < \frac{1}{2}(p - 1)\). Then \( w(\theta) = (\sin \theta)^{2m-mp} \) is an \( A_p \)-weight. This implies (cf. [4]) that there is a constant \( C = C(\mu, p) \) so that for any trigonometric polynomial \( h(\theta) = \sum_{k=-N}^{N} \hat{h}(k)e^{ik\theta} \) of degree \( N \), and any \( 1 \leq l \leq N \) we have

\[ \left( \int_{-\pi}^{\pi} |\sum_{k \geq l} \hat{h}(k)e^{ik\theta} - i \sum_{k \leq -l} \hat{h}(k)e^{ik\theta}|^p w(\theta) \, d\theta \right)^{1/p} \]

\[ \leq C \left( \int_{-\pi}^{\pi} |h(\theta)|^p w(\theta) \, d\theta \right)^{1/p}. \]

Summing over \( l = 1, 2, \ldots, N \) we obtain

\[ \left( \int_{-\pi}^{\pi} \sum_{k=-N}^{N} |i \hat{h}(k)e^{ik\theta}|^p w(\theta) \, d\theta \right)^{1/p} \leq CN \left( \int_{-\pi}^{\pi} |h(\theta)|^p w(\theta) \, d\theta \right)^{1/p}. \]
\[ (\int_{-\pi}^{\pi} |h'(\theta)|^p w(\theta) \, d\theta)^{1/p} \leq CN \left( \int_{-\pi}^{\pi} |h(\theta)|^p w(\theta) \, d\theta \right)^{1/p}. \]

Now suppose \( f \in \mathcal{P}_n \) and let \( h(\theta) = (\sin \theta)^m f(\cos \theta) \) so that \( h \) is a trigonometric polynomial of degree at most \( m + n - 1 \leq C(\mu, p)n \). Let \( I_k \) be the interval \( [\theta - \theta_n, \theta] \leq \frac{\pi}{n} \) for \( 1 \leq k \leq n \). Then

\[
\int_{I_k} |h(\theta)| \, d\theta \leq \left( \int_{I_k} w(\theta)^{-1/p} \, d\theta \right)^{1/p} \left( \int_{I_k} |h(\theta)|^p w(\theta) \, d\theta \right)^{1/p} \\
\leq C \frac{1}{n^{1/p}} (\sin \theta_n)^{-2\mu/p} \left( \int_{I_k} |h|^p w \, d\theta \right)^{1/p}.
\]

Here we use the properties of \( (\tau_n) \) and \( (\theta_n) \) from Proposition 2.5. On the other hand,

\[
\int_{I_k} |h(\theta) - h(\theta_n)| \, d\theta \leq \frac{\pi}{n} \int_{I_k} |h'(\theta)| \, d\theta \\
\leq C \frac{1}{n^{1+1/p}} (\sin \theta_n)^{-2\mu/p} \left( \int_{I_k} |h'|^p w \, d\theta \right)^{1/p}.
\]

Putting these together we conclude that

\[
\frac{1}{n} |h(\theta_n)|^p (\sin \theta_n)^{2\mu - mp} \leq C \left( \int_{-\pi}^{\pi} |h|^p w(\theta) \, d\theta + \frac{1}{np} \int_{-\pi}^{\pi} |h'|^p w \, d\theta \right).
\]

On summing we obtain

\[
\frac{1}{n} \sum_{k=1}^{n} |f(\tau_n)|^p (1 - \tau_n^2)^\mu \leq C \left( \int_{-\pi}^{\pi} |h|^p w \, d\theta + \frac{1}{np} \int_{-\pi}^{\pi} |h'|^p w \, d\theta \right)
\]

since \( \sum_{k=1}^{n} \chi_{I_k} \) is uniformly bounded by Proposition 2.5. Now appealing to (4.2) we have

\[
\frac{1}{n} \sum_{k=1}^{n} |f(\tau_n)|^p (1 - \tau_n^2)^\mu \leq C \int_{-\pi}^{\pi} |h|^p w \, d\theta.
\]

Recalling the definition of \( w \) and \( h \) this implies

\[
\left( \frac{1}{n} \sum_{k=1}^{n} |f(\tau_n)|^p (1 - \tau_n^2)^\mu \right)^{1/p} \leq C \left( \int_{-\pi}^{\pi} |f(x)|^p (1 - x^2)^{-\delta} \, dx \right)^{1/p}.
\]

Note that we only have (4.3) when \( \mu \) is not of the form \( \frac{1}{2}(mp - 1) \). We now prove (4.3) for \( \mu \) in the exceptional case. We observe that if \( \nu = \frac{2}{3} + \frac{1}{3} - \frac{1}{2} \) then \( \nu > -\frac{1}{2} \) and (4.1) holds for \( \lambda = \nu \). In fact there exists \( 0 < \delta < \frac{\nu}{2} \) so that \( (\varphi_n^\nu) \) is a basis of both \( L_p([-1, 1], (1 - x^2)^{-\delta}) \) and of \( L_p([-1, 1], (1 - x^2)^{\nu/2}) \). Let

\[
S_n^\nu(f) = \sum_{k=0}^{n-1} \varphi_n^\nu(x) \int_{-1}^{1} f(x) \varphi_n^\nu(x)(1 - x^2)^{-\delta/2} \, dx.
\]
be the partial sum operator associated with this basis. Let us consider the map $T_n$: 
$L_p([-1, 1], (1 - x^2)^{\mu \pm \delta}) \to \mathbb{R}$ defined by 
$$T_n(f)_{\kappa} = (S^\mu_n f)(\tau_{n\kappa}).$$

Then there is a constant $C$ independent of $n$ so that 
$$\left(\frac{1}{n} \sum_{k=1}^{n} |T_n(f)_{\kappa}|^p (1 - \tau_{nk}^2)^{\mu \pm \delta}\right)^{1/p} \leq C \left(\int_{-1}^{1} |f(x)|^p (1 - x^2)^{\mu \pm \delta - \frac{1}{2}}\right)^{1/p}.$$  

It follows by interpolation that we obtain 
$$\left(\frac{1}{n} \sum_{k=1}^{n} |T_n(f)_{\kappa}|^p (1 - \tau_{nk}^2)^{\mu}\right)^{1/p} \leq C \left(\int_{-1}^{1} |f(x)|^p (1 - x^2)^{\mu - \frac{1}{p} - \frac{1}{2}}\right)^{1/p}$$
and on restricting to $P_n$ we have (4.3) for all $\mu$.

We now assume $\lambda$ satisfies (4.1) and complete the proof by duality. Let $\sigma$ be defined by $\frac{p}{\lambda} + \frac{p}{\mu} = \lambda$. Then (4.1) also holds if we replace $p, \mu$ by $p', \sigma$.

Suppose $f \in P_n$. Then there exists $h \in L_p([-1, 1], (1 - x^2)^{\sigma - \frac{1}{2}})$ so that 
$$\int_{-1}^{1} |h(x)|^{p'} (1 - x^2)^{\sigma - \frac{1}{2}} \, dx = 1$$
and 
$$\int_{-1}^{1} h(x)f(x)(1 - x^2)^{-\lambda - \frac{1}{2}} \, dx = \left(\int_{-1}^{1} |f(x)|^{p} (1 - x^2)^{\mu - \frac{1}{p} - \frac{1}{2}} \, dx\right)^{1/p}.$$ 

Let $g = S_{n}^{(\lambda)} f$. Then 
$$\int_{-1}^{1} |g(x)|^{p'} (1 - x^2)^{\sigma - \frac{1}{2}} \, dx \leq C'$$
where $C = C(p, \lambda, \mu)$ is independent of $n$. Now using Gauss-Jacobi quadrature (see Proposition 2.6) we have 
$$\frac{1}{n} \sum_{k=1}^{n} \alpha_{n\kappa}^{(\lambda)} f(\tau_{n\kappa}) g(\tau_{n\kappa}) = \int_{-1}^{1} f(x)h(x)(1 - x^2)^{\lambda - \frac{1}{2}} \, dx.$$ 

We recall that 
$$0 \leq \alpha_{n\kappa}^{(\lambda)} \leq C(1 - \tau_{nk}^2)^{\lambda} n^{-1}$$
where $C$ is again independent of $n$. It follows that 
$$\left(\int_{-1}^{1} |f(x)|^{p} (1 - x^2)^{\mu - \frac{1}{2}} \, dx\right)^{1/p} \leq C \left(\frac{1}{n} \sum_{k=1}^{n} |f(\tau_{n\kappa})|^p (1 - \tau_{nk}^2)^{\mu}\right)^{1/p} \left(\frac{1}{n} \sum_{k=1}^{n} |g(\tau_{n\kappa})|^{p'} (1 - \tau_{nk}^2)^{\sigma}\right)^{1/p'}.$$ 

Now applying (4.3) we can estimate the last term by a constant independent of $n$. Thus we have 
$$\left(\int_{-1}^{1} |f(x)|^{p} (1 - x^2)^{\mu - \frac{1}{2}} \, dx\right)^{1/p} \leq C \left(\frac{1}{n} \sum_{k=1}^{n} |f(\tau_{n\kappa})|^p (1 - \tau_{nk}^2)^{\mu}\right)^{1/p}.$$ 

This completes the proof. 

\hfill $\blacksquare$
REFERENCES


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