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THE BEHAVIOUR OF LEGENDRE AND ULTRASPHERICAL POLYNOMIALS IN L_p-SPACES

N. J. KALTON AND L. TZAFRIRI

ABSTRACT. We consider the analogue of the $\Lambda(p)-$problem for subsets of the Legendre polynomials or more general ultraspherical polynomials. We obtain the "best possible" result that if $2 < p < 4$ then a random subset of N Legendre polynomials of size $N^{4/p-1}$ spans an Hilbertian subspace. We also answer a question of König concerning the structure of the space of polynomials of degree n in various weighted L_p-spaces.

1. **Introduction.** Let (P_n) denote the Legendre polynomials on $[-1,1]$ and let $\varphi_n = c_nP_n$ be the corresponding polynomials normalized in $L_2[-1,1]$. Then $(\varphi_n)_{n=0}^{\infty}$ is an orthonormal basis of $L_2[-1,1]$. If we consider the same polynomials in $L_p[-1,1]$ where $p > 2$ then $(\varphi_n)_{n=0}^{\infty}$ is a basis if and only if $\sup \|\varphi_n\|_p < \infty$ if and only if $p < 4$ [8], [9].

In this note our main result concerns the analogue of the $\Lambda(p)$-problem for the Legendre polynomials. In [2] Bourgain (answering a question of Rudin [12]) showed that for the trigonometric system $(e^{in\theta})_{n\in\mathbb{Z}}$ in $L_p(\mathbf{T})$ where $p > 2$ there is a constant C so that for any N there is a subset $\mathbb{A}$ of $\{1,2,\ldots,N\}$ with $|\mathbb{A}| \geq N^{2/p}$ and such that for any $(\xi_n)_{n\in\mathbb{A}}$,

$$\Big\|\sum_{n\in\mathbb{A}} \xi_n e^{in\theta}\Big\|_p \leq C\Big(\sum_{n\in\mathbb{A}} |\xi_n|^2\Big)^{1/2}.$$

Actually Bourgain's result is much stronger than this. He shows that if $(g_n)_{n=1}^{\infty}$ is a uniformly bounded orthonormal system in some $L_2(\mu)$ where μ is a finite measure, then there is a constant C so that if $\mathbb{F}$ is finite subset of $\mathbb{N}$ then there is a further subset $\mathbb{A}$ of $\mathbb{F}$ with $|\mathbb{A}| \geq |\mathbb{F}|^{2/p}$ so that we have an estimate

$$\Big\|\sum_{n\in\mathbb{A}} \xi_n g_n\Big\|_p \leq C\Big(\sum_{n\in\mathbb{A}} |\xi_n|^2\Big)^{1/2}. \tag{1.1}$$

In fact this estimate holds for a random subset of $\mathbb{F}$. For an alternative approach to Bourgain's results, see Talagrand [15].

It is natural to ask for a corresponding result for the Legendre polynomials. Since $(\varphi_n)_{n=1}^{\infty}$ is not bounded in $L_\infty[-1,1]$ one cannot apply Bourgain's result. However, Bourgain [2] states without proof the corresponding result for orthonormal systems which are bounded in some L_r for $r > 2$. Suppose that (g_n) is an orthonormal system

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which is uniformly bounded in $L_r(\mu)$ for some $2 < r < \infty$. Then he remarks that if $2 < p < r$ there is a constant C so that for any subset $\mathbb{F}$ of $\mathbb{N}$ there is a further subset $\mathbb{A}$ of $\mathbb{F}$ with $|\mathbb{A}| \geq |\mathbb{F}|^{(\frac{1}{p}-\frac{1}{r})/(\frac{1}{2}-\frac{1}{r})}$ so that we have the estimate (1.1). Again this result holds for random subsets. It follows from this result that if $2 < p < 4$ and $\epsilon > 0$ $\{1, 2, \ldots, N\}$ contains a subset $\mathbb{A}$ of size $N^{4/p-1-\epsilon}$ so that we have the estimate

$$\Big\|\sum_{n\in\mathbb{A}} \xi_n\varphi_n\Big\|_p \leq C\Big(\sum_{n\in\mathbb{A}} |\xi_n|^2\Big)^{1/2}. \tag{1.2}$$

As shown below in Proposition 3.1, there is an easy upper estimate $|\mathbb{A}| \leq CN^{4/p-1}$ for subsets obeying (1.2). The sharp estimate $N^{4/p-1}$ cannot be obtained from Bourgain's results since $(\varphi_n)_{n=1}^{\infty}$ is unbounded in $L_4[-1, 1]$.

In this note we show that, nevertheless, if $\mathbb{F}$ is a finite subset of $\mathbb{N}$ then there is a subset of $\mathbb{A}$ of $\mathbb{F}$ with $|\mathbb{A}| \geq |\mathbb{F}|^{4/p-1}$ so that (1.2) holds, and again this holds for random subsets.

In fact we show the corresponding result for more general ultraspherical polynomials. Suppose $0 < \lambda < \infty$. Let $(\varphi_n^{(\lambda)})_{n=0}^{\infty}$ be the orthonormal basis of $L_2\Big([-1, 1], (1-x^2)^{\lambda-\frac{1}{2}}\Big)$ obtained from $\{1, x, x^2, \ldots\}$ by the Gram-Schmidt process. Then $(\varphi_n^{(\lambda)})$ is a basis in $L_p\Big([-1, 1], (1-x^2)^{\lambda-\frac{1}{2}}\Big)$ if $2 < p < r = 2 + \lambda^{-1}$. We show in Theorem 3.6 that there is a constant C so that if $\mathbb{F}$ is a finite subset of $\mathbb{N}$, there is a further subset $\mathbb{A}$ of $\mathbb{F}$ with $|\mathbb{A}| \geq |\mathbb{F}|^{2\lambda(\frac{r}{p}-1)}$ so that we have the estimate

$$\Big\|\sum_{n\in\mathbb{A}} \xi_n\varphi_n^{(\lambda)}\Big\|_p \leq C\Big(\sum_{n\in\mathbb{A}} |\xi_n|^2\Big)^{1/2}.$$

Here of course norms are computed with respect to the measure $(1-x^2)^{\lambda-\frac{1}{2}}\, dx$. Again this result is best possible as with the Legendre polynomials (the case $\lambda = \frac{1}{2}$) and holds for random subsets. Notice that if we set $\lambda = 0$ we obtain the (normalized) Tchebicheff polynomials which after a change of variable reduce to the trigometric system on the circle. Thus Bourgain's $\Lambda(p)-$theorem corresponds to the limiting case $\lambda = 0$.

As will be seen we obtain our main result by using Bourgain's theorem and an interpolation technique.

In Section 4 we answer a question of H. König by showing that the space P_n of polynomials is uniformly isomorphic to ℓ_p^n in every space $L_p\Big([-1, 1], (1-x^2)^{\lambda-\frac{1}{2}}\Big)$ for $\lambda > \frac{1}{2}$ and $1 < p < \infty$.

2. **Preliminaries.** In this section, we collect together some preliminaries. A good general reference for most of the material we need is the book of Szegö [14].

For $-\frac{1}{2} < \lambda < \infty$ with $\lambda \neq 0$ we define the *ultraspherical polynomials* $P_n^{(\lambda)}$ as in [14] by the generating function relation

$$(1 - 2xw + w^2)^{-\lambda} = \sum_{n=0}^{\infty} P_n^{(\lambda)}(x)w^n.$$

For $\lambda = 0$ we define $P_n^{(0)}(x) = \frac{2}{n}T_n(x)$ where T_n are the Tchebicheff polynomials defined by $T_n(\cos\theta) = \cos n\theta$ for $0 \leq \theta \leq \pi$. Then we have that if $\lambda \neq 0$ [14, p. 81 (4.7.16)],

$$\int_{-1}^{+1} |P_n^{(\lambda)}(x)|^2(1-x^2)^{\lambda-\frac{1}{2}}\, dx = 2^{1-2\lambda}\pi\Gamma(\lambda)^{-2}\,\frac{\Gamma(n+2\lambda)}{(n+\lambda)\Gamma(n+1)}.$$

It follows that we have

$$\varphi_n^{(\lambda)} = 2^{\lambda-\frac{1}{2}}\pi^{-\frac{1}{2}}\Gamma(\lambda)\left(\frac{(n+\lambda)\Gamma(n+1)}{\Gamma(n+2\lambda)}\right)^{1/2} P_n^{(\lambda)}.$$

We now recall Theorem 8.21.11 of [14, p. 197].

PROPOSITION 2.1. *Suppose* $0 < \lambda < 1$. *Then for* $0 \leq \theta \leq \pi$ *we have*

$$\left|P_n^{(\lambda)}(\cos\theta) - 2\frac{\Gamma(n+2\lambda)}{\Gamma(\lambda)\Gamma(n+\lambda+1)}\cos\Big((n+\lambda)\theta - \lambda\pi/2\Big)(2\sin\theta)^{-\lambda}\right| \leq \frac{4\lambda(1-\lambda)\Gamma(n+2\lambda)}{\Gamma(\lambda)(n+\lambda+1)\Gamma(n+\lambda+1)}(2\sin\theta)^{-\lambda-1}.$$

REMARK. Note we have used that $\Gamma(\lambda)\Gamma(1-\lambda) = \pi/\sin(\lambda\pi)$.

The next Proposition is a combination of results on p. 80 (4.7.14) and p. 168 (7.32.1) of [14].

PROPOSITION 2.2. *If* $0 < \lambda < \infty$ *then we have*

$$\max_{-1\leq x\leq 1}|P_n^{(\lambda)}(x)| = P_n^{(\lambda)}(1) = \binom{n+2\lambda-1}{n}.$$

Here we write

$$\binom{u}{v} = \frac{\Gamma(u+1)}{\Gamma(u-v+1)\Gamma(v+1)}.$$

For our purposes it will be useful to simplify the Gamma function replacing it by asymptotic estimates. For this purpose we note that

$$\frac{\Gamma(n+\sigma)}{\Gamma(n)} = n^{\sigma} + O(n^{\sigma-1}).$$

PROPOSITION 2.3. *Suppose* $0 < \lambda < \infty$. *Then there exists a positive constant* $C = C(\lambda)$ *such that*

$$\left|\varphi_n^{(\lambda)}(\cos\theta) - (2/\pi)^{1/2}\cos\Big((n+\lambda)\theta - \lambda\pi/2\Big)(\sin\theta)^{-\lambda}\right| \leq C(\sin\theta)^{-\lambda}(\min\Big((n\sin\theta)^{-1}, 1\Big).$$

PROOF. Using the remark preceding the Proposition, we can deduce from Proposition 2.1 that

$$\left|P_n^{(\lambda)}(\cos\theta) - 2^{1-\lambda}n^{\lambda-1}\Gamma(\lambda)^{-1}\cos\Big((n+\lambda)\theta - \lambda\pi/2\Big)(\sin\theta)^{-\lambda}\right| \leq Cn^{\lambda-2}(\sin\theta)^{-1-\lambda} \tag{2.1}$$

where $C = C(\lambda)$, for $0 < \lambda < 1$. This estimate also holds when $\lambda = 1$ trivially (with $C = 0$).

We now prove the same estimate provided $n\sin\theta \geq 1$ for all $\lambda > 0$ by using the recurrence relation

$$2(\lambda-1)(1-x^2)P_n^{(\lambda)}(x) = (n+2\lambda-2)P_n^{(\lambda-1)}(x) - (n+1)xP_{n+1}^{(\lambda-1)}(x) \tag{2.2}$$

for which we refer to [14, p. 83 (4.7.27)].

Indeed assume the estimate (2.1) is known for $\lambda - 1$. Then with $x = \cos\theta$,

$$\left|P_n^{(\lambda-1)}(x) - xP_{n+1}^{(\lambda-1)}(x) - 2^{-\lambda}n^{\lambda-2}\Gamma(\lambda-1)^{-1}\cos\big((n+\lambda-1)\theta - \lambda\pi/2\big)(\sin\theta)^{1-\lambda}\right| \\ \leq Cn^{\lambda-3}(\sin\theta)^{-\lambda}.$$

We also have

$$|P_n^{(\lambda-1)}(x)| \leq Cn^{\lambda-3}(\sin\theta)^{-\lambda} \leq Cn^{\lambda-2}(\sin\theta)^{1-\lambda}$$

provided $n\sin\theta \geq 1$. Now using the recurrence relation (2) we obtain an estimate of the form (2.1) provided $n\sin\theta \geq 1$.

Next we observe that for all $\lambda > 0$ we have by Proposition 2.2,

$$|P_n^{(\lambda)}(x)| \leq P_n^{(\lambda)}(1) \leq Cn^{2\lambda-1}$$

where C depends only on λ. Hence if $n\sin\theta < 1$ we have an estimate

$$\left|P_n^{(\lambda)}(\cos\theta) - 2n^{\lambda-1}\Gamma(\lambda)^{-1}\cos\big((n+\lambda)\theta - \lambda\pi/2\big)(\sin\theta)^{-\lambda}\right| \leq Cn^{\lambda-1}(\sin\theta)^{-\lambda}. \tag{2.3}$$

Combining (2.2) and (2.3) gives us an estimate

$$\left|P_n^{(\lambda)}(\cos\theta) - 2^{1-\lambda}n^{\lambda-1}\Gamma(\lambda)^{-1}\ \cos\big((n+\lambda)\theta - \lambda\pi/2\big)(\sin\theta)^{-\lambda}\right| \\ \leq C\min\big(n^{\lambda-2}(\sin\theta)^{-1-\lambda}, n^{\lambda-1}(\sin\theta)^{-\lambda}\big) \tag{2.4}$$

Recalling the relationship between $\varphi_n^{(\lambda)}$ and $P_n^{(\lambda)}$ we obtain the result. ■

PROPOSITION 2.4. *Suppose $-1/2 < \lambda, \mu < \infty$. Then the orthonormal system $(\varphi_n^{(\lambda)})_{n=0}^{\infty}$ is a basis of $L_r\big([-1,1], (1-x^2)^{\mu-\frac{1}{2}}\big)$ if and only if*

$$\left|\frac{2\mu+1}{2r} - \frac{2\lambda+1}{4}\right| < \min\left(\frac{1}{4}, \frac{2\lambda+1}{4}\right).$$

In particular, if $\lambda \geq 0$ and $r > 2$ then $(\varphi^{(\lambda)})_{n=0}^{\infty}$ is a basis of $L_r\big([-1,1], (1-x^2)^{\lambda-\frac{1}{2}}\big)$ if and only if $r < 2 + \lambda^{-1}$.

PROOF. This theorem is a special case of a very general result of Badkov [1, Theorem 5.1]. The second part is much older: see Pollard [9], [10] and [11], Newman-Rudin [8] and Muckenhaupt [7].

We will also need some results on Gauss-Jacobi mechanical quadrature. To this end let $(\tau_{nk}^{(\lambda)} = \cos\theta_{nk}^{(\lambda)})_{k=1}^{n}$ be the zeros of the polynomial $\varphi_n^{(\lambda)}$ ordered so that $0 < \theta_{n,1}^{(\lambda)} < \theta_{n,2}^{\lambda)} < \cdots < \theta_{nn}^{(\lambda)} < \pi$. (We remark that the zeros are necessarily distinct and are all located in $(-1,1)$; see Szegö [14, p. 44].)

PROPOSITION 2.5. *Suppose* $-\frac{1}{2} < \lambda < \infty$. *Then there exists a constant* C *depending only on* λ *so that*

$$\left|\theta_{nk}^{(\lambda)} - \frac{k\pi}{n}\right| \le \frac{C}{n}.$$

Furthermore, there exists $c > 0$ *so that*

$$|\theta_{nk}^{(\lambda)}| \ge \frac{ck}{n}$$

if $k < n/2$.

PROOF. The following result is contained in Theorem 8.9.1 of Szegö [14, p. 238]. The second part follows easily from the first and the fact that $\lim_{n\to\infty} n\theta_{n1}^{(\lambda)}$ exists and is the first positive zero of the Bessel function $J_{\lambda+\frac{1}{2}}(t)$ (see Szegö [14, Theorem 8.1.2, pp. 192–193]). ∎

We will denote by $\mathcal{P}_n$ the space of polynomials of degree at most $n-1$ so that $\dim \mathcal{P}_n = n$.

PROPOSITION 2.6. *Suppose that* $-\frac{1}{2} < \lambda < \infty$. *Then there exist positive constants* $(\alpha_{nk}^{(\lambda)})_{1\le k\le n<\infty}$ *such that if* $f \in \mathcal{P}_{2n}$ *then*

$$\int_{-1}^{1} f(x)(1-x^2)^{\lambda-\frac{1}{2}}\,dx = \sum_{k=1}^{n} \alpha_{nk}^{(\lambda)} f(\tau_{nk}^{(\lambda)}).$$

Furthermore there is a constant C *depending only on* λ *such that*

$$\alpha_{nk}^{(\lambda)} \le C(\sin\theta_{nk})^{2\lambda} n^{-1}.$$

PROOF. This is known as Gauss-Jacobi mechanical quadrature. See Szegö [14, pp. 47–50]. The estimate on the size of $(\alpha_{nk}^{(\lambda)})$ may be found on p. 354. However this estimate is perhaps most easily seen by combining the Tchebicheff-Markov-Stieltjes separation theorem (Szegö, p. 50) with the estimate on the zeros (Proposition 2.5). More precisely there exist $(y_k)_{k=0}^{n}$ such that $1 = y_0 > \tau_{n,1}^{(\lambda)} > y_1 > \tau_{n,2}^{(\lambda)} > \cdots > \tau_{nn}^{(\lambda)} > y_n = -1$ so that

$$\alpha_{nk}^{(\lambda)} = \int_{y_{k-1}}^{y_k} (1-x^2)^{\lambda-\frac{1}{2}}\,dx.$$

The estimate follows from Proposition 2.5. ∎

3. **The** $\Lambda(p)$ **problem.** We first note that by Proposition 2.4, in order that $(\varphi_n^{(\lambda)})_{n=1}^{\infty}$ be a basis in $L_p\big([-1,1],(1-x^2)^{\lambda-\frac{1}{2}}\big)$, it is necessary and sufficient that $2 < p < 2+\lambda^{-1}$. Let us denote this critical index by $r = r(\lambda) = 2 + \lambda^{-1}$.

Let $\mathbb{A}$ be a subset of $\mathbb{N}$, and $2 < p < r$. We will say that $\mathbb{A}$ is a $\Lambda(p,\lambda)$-set if there is a constant C so that for any finite-sequence $(\xi_n : n \in \mathbb{A})$ we have

$$\left(\int_{-1}^{+1} \Big|\sum_{n\in\mathbb{A}} \xi_n \varphi_n^{(\lambda)}(x)\Big|^p (1-x^2)^{\lambda-\frac{1}{2}}\,dx\right)^{1/p} \le C\Big(\sum_{n\in A} |\xi_n|^2\Big)^{1/2}.$$

This means that the operator $T\colon \ell_2(\mathbb{A}) \longrightarrow L_p\big([-1,1],(1-x^2)^{\lambda-\frac{1}{2}}\big)$ defined by $T\xi = \sum_{n\in\mathbb{A}} \xi_n\varphi_n^{(\lambda)}$ is bounded, and indeed since there is an automatic lower bound, an isomorphic embedding. We denote the least constant C or equivalently $\|T\|$ by $\Lambda_{p,\lambda}(\mathbb{A})$. Note that if $\lambda = 0$ then $\varphi_n^{(\lambda)}(\cos\theta) = \cos n\theta$ and this definition reduces to the standard definition of a $\Lambda(p)$-set introduced by Rudin [12].

PROPOSITION 3.1. *For each $\lambda > 0$ there is a constant $C = C(\lambda)$ depending on λ so that if $\mathbb{A}$ is a $\Lambda(p,\lambda)$-set then*

$$|\mathbb{A}\cap[1,N]| \leq C\Lambda_{p,\lambda}(\mathbb{A})^2 N^{2\lambda(r/p-1)}.$$

PROOF. Observe first that

$$\max_{-1\leq x\leq 1} |\varphi_n^{(\lambda)}(x)| = \varphi(1) \geq cn^\lambda$$

for some constant $c > 0$ depending only on λ by Proposition 2.2 and the remark thereafter. It follows from Bernstein's inequality that if $0 \leq \theta \leq (2n)^{-1}$ then $\varphi_n^{(\lambda)}(\cos\theta) \geq cn^\lambda/2$.

In particular let $J = \mathbb{A}\cap[N/2,N]$. Then for $0 \leq \theta \leq (2N)^{-1}$ we have

$$\sum_{n\in J} \varphi_n^{(\lambda)}(\cos\theta) \geq cN^\lambda|J|$$

where $c > 0$ depends only on λ. Since $dx = (\sin\theta)^{2\lambda}d\theta$ we therefore have

$$cN^\lambda|J|N^{-(2\lambda+1)/p} \leq C\Lambda(\mathbb{A})|J|^{1/2}$$

where $0 < c, C < \infty$ are again constants depending only on λ. We thus have an estimate $|J| \leq C\Lambda(\mathbb{A})^2N^{(4\lambda+2)/p-2\lambda)} = C\Lambda(\mathbb{A})^2N^{2\lambda(r/p-1)}$. This clearly implies the result. ∎

Our next Proposition uses the approximation of Proposition 2.3 to transfer the problem to a weighted problem on the circle $\mathbf{T}$ which we here identify with $[-\pi,\pi]$.

PROPOSITION 3.2. *Suppose $\lambda > 0$ and $2 < p < r(\lambda)$. Then $\mathbb{A}$ is a $\Lambda(p,\lambda)-$set if and only if the operator $S\colon \ell_2(\mathbb{A}) \longrightarrow L_p(\mathbf{T},|\sin\theta|^{\lambda(2-p)})$ is bounded where $Se_n = e^{in\theta}$, where (e_n) is the canonical basis of $\ell_2(\mathbb{A})$. Furthermore there is a constant $C = C(p,\lambda)$ so that $C^{-1}\|S\| \leq \Lambda_{p,\lambda}(\mathbb{A}) \leq C\|S\|$.*

PROOF. Let us start by proving a similar estimate to Proposition 3.1 for the system $\{e^{in\theta}\}$. Suppose S is bounded. If $N \in \mathbb{N}$ then we note that for $1 \leq k \leq N$ we have $\cos k\theta > 1/2$ if $|\theta| < \pi/3N$. Hence if $|\theta| < \pi/3N$ we have $\sum_{k\in J}\cos k\theta > \frac{1}{2}|J|$ where $J = \mathbb{A}\cap[1,N]$. It follows that

$$|J|N^{(\lambda(p-2)-1)/p} \leq C\|S\|\,|J|^{1/2}$$

where C depends only on λ. This yields an estimate

$$|J| \leq C\|S\|^2N^{2\lambda(r/p-1)}$$

where C depends only on λ.

Now consider the map $S_0\colon \ell_2(\mathbb{A}) \longrightarrow L_p([0,\pi], |\sin\theta|^{2\lambda})$ defined by $S_0 e_n = \cos\big((n+\lambda)\theta - \lambda\pi/2\big)(\sin\theta)^{-\lambda}$. We will observe that S_0 is bounded if and only if S is bounded and indeed $\|S_0\| \le 2\|S\| \le C\|S_0\|$ where C depends only on p. In fact if $(\xi_n)_{n\in\mathbb{A}}$ are finitely non-zero and real then

$$\|S_0\xi\|^p \le \int_0^\pi \Big|\sum_{n\in\mathbb{A}} \xi_n e^{in\theta}\Big|^p |\sin\theta|^{\lambda(2-p)}\,d\theta \le \|S\xi\|^p$$

which leads easily to the first estimate $\|S_0\| \le 2\|S\|$. For the converse direction, we note that $w(\theta) = |\sin\theta|^{\lambda(2-p)}$ is an A_p-weight in the sense of Muckenhaupt (see [3], [4] or [7]), *i.e.*, there is a constant C so that for every interval I on the circle we have

$$\left(\int_I w(\theta)\,d\theta\right)^{1/p}\left(\int_I w(\theta)^{-p/p'}\,d\theta\right)^{1/p'} \le C|I|$$

where $|I|$ denote the length of I. It follows that the Hilbert-transform is bounded on the space $L_p(\mathbf{T}, w)$ so that there is a constant $C = C(p,\lambda)$ such that if $(\xi_n)_{n\in\mathbb{A}}$ is finitely non-zero and real then

$$\left(\int_{-\pi}^{\pi}\Big|\sum_{n\in\mathbb{A}}\xi_n \sin\big((n+\lambda)\theta - \lambda\pi/2\big)\Big|^p |\sin\theta|^{\lambda(2-p)}\,d\theta\right)^{1/p}$$
$$\le C\left(\int_{-\pi}^{\pi}\Big|\sum_{n\in\mathbb{A}}\xi_n \cos\big((n+\lambda)\theta - \lambda\pi/2\big)\Big|^p |\sin\theta|^{\lambda(2-p)}\,d\theta\right)^{1/p}.$$

This quickly implies an estimate of the form $\|S\xi\| \le C\|S_0\xi\|$.

Now consider the map $T\colon \ell_2(\mathbb{A}) \longrightarrow L_p([0,\pi], |\sin\theta|^{2\lambda})$ defined by $Te_n = \varphi_n^{(\lambda)}(\cos\theta)$. Then for some constant $C = C(\lambda)$ we have (using Proposition 2.3),

$$|\psi_n(\theta)| \le C(\sin\theta)^{-\lambda}\min\big((n\sin\theta)^{-1}, 1\big)$$

where

$$\psi_n(\theta) = \varphi_n^{\lambda}(\cos\theta) - \cos\big((n+\lambda)\theta - \lambda\pi/2\big)(\sin\theta)^{-\lambda}.$$

Now suppose $\mathbb{A}$ satisfies an estimate $|\mathbb{A}\cap[1,N]| \le KN^{2\lambda(r/p-1)}$ for some constant K.

We will let $J_k = \mathbb{A}\cap[2^{k-1}, 2^k)$ and $E_k = \{\theta : 2^{-k} < \sin\theta < 2^{1-k}\}$. Then on E_k we have an estimate $|\psi(\theta)| \le C2^{\lambda k}$ if $n \le 2^k$ and $|\psi_n(\theta)| \le Cn^{-1}2^{(1+\lambda)k}$ if $n > 2^k$. Here C depends a constant depending only on p and λ.

Let $(\xi_n)_{n\in\mathbb{A}}$ be any finitely non-zero sequence and set $u_k = (\sum_{n\in J_k}|\xi_n|^2)^{1/2}$. Note that $\sum_{n\in J_k}|\xi_n| \le |J_k|^{1/2}u_k$.

It follows that if $1 \le l \le k$ we have

$$\left(\int_{E_k}\Big|\sum_{n\in J_l}\xi_n\psi_n\Big|^p(\sin\theta)^{2\lambda}\,d\theta\right)^{1/p} \le C2^{\lambda k}2^{-(1+2\lambda)k/p}|J_l|^{1/2}u_l$$

while if $k+1 \le l < \infty$

$$\left(\int_{E_k} \Big|\sum_{n\in J_l} \xi_n\psi_n\Big|^p (\sin\theta)^{2\lambda}\,d\theta\right)^{1/p} \le C2^{\lambda k+(k-l)}2^{-(1+2\lambda)k/p}|J_l|^{1/2}u_l.$$

Note that $\lambda-(1+2\lambda)/p = \lambda(1-r/p)$. We also have $|J_l| \le K2^{2\lambda l(r/p-1)}$. Hence we obtain an estimate

$$\Big\|\chi_{E_k}\sum_{n\in\mathbb{A}}\xi_n\psi_n\Big\| \le CK^{1/2}\Big(\sum_{l=1}^{k}2^{\lambda(r/p-1)(l-k)}u_l + \sum_{l=k+1}^{\infty}2^{(\lambda(r/p-1)-1)(l-k)}u_l\Big).$$

Let $\delta = \min\big(\lambda(r/p-1), 1-\lambda(r/p-1)\big)$. Then the right-hand side may estimated by

$$CK^{1/2}\Big(\sum_{l=1}^{\infty}2^{-\delta|l-k|}u_l\Big) = CK^{1/2}\sum_{j\in\mathbb{Z}}2^{-\delta|j|}u_{k+j}$$

where $u_j = 0$ for $j \le 0$. Since $p > 2$ we have

$$\Big\|\sum_{n\in\mathbb{A}}\xi_n\psi_n\Big\| \le \Big(\sum_{k=1}^{\infty}\Big\|\chi_{E_k}\sum_{n\in\mathbb{A}}\xi_n\psi_n\Big\|^2\Big)^{1/2}.$$

Hence by Minkowski's inequality in ℓ_2 we have

$$\Big\|\sum_{n\in\mathbb{A}}\xi_n\psi_n\Big\| \le CK^{1/2}\sum_{j\in\mathbb{Z}}2^{-\delta|j|}\Big(\sum_{l=1}^{\infty}u_l^2\Big)^{1/2}.$$

We conclude that $\|S_0\xi - T\xi\| \le CK^{1/2}$. Now if T is bounded then $K \le C\|T\|^2$ while if S is bounded then $K \le C\|S\|^2$. This yields the estimates promised. ∎

As remarked above, using Proposition 3.2 we can transfer the problem of identifying $\Lambda(p,\lambda)$-sets to a similar problem concerning the standard characters $\{e^{in\theta}\}$ in a weighted L_p-space. We will now solve a corresponding problem in the case when $p = 2$ and then use the solution to obtain our main result in the case $p > 2$. To this end we will first prove a result concerning weighted norm inequalities for an operator on the sequence space $\ell_2(\mathbb{Z})$ which is the discrete analogue of a Riesz potential.

Suppose $0 < \alpha < 1/2$. For $m, n \in \mathbb{Z}$ we define $K(m,n) = |m-n|^{\alpha-1}$ when $m \neq n$ and $K(m,n) = 1$ if $m = n$. Let $c_{00}(\mathbb{Z})$ be the space of finitely non-zero sequences. Then we can define a map $K\colon c_{00}(\mathbb{Z}) \longrightarrow \ell_2(\mathbb{Z})$ by $K\xi(m) = \sum_{n\in\mathbb{Z}} K(m,n)\xi(n)$.

Now suppose $v \in \ell_\infty(\mathbb{Z})$. We define $L(v)$ to be the norm in $\ell_2(\mathbb{Z})$ of the operator $\xi \longrightarrow vK\xi$ which we take to be ∞ if this operator is unbounded. Thus $L(v) = \sup\{\|vK\xi\| : \|\xi\| \le 1\}$.

The following result can be derived from similar results in potential theory (for example, [13]). For more general results we refer to [5]. However we will give a self-contained exposition.

THEOREM 3.3. *Let $0 \leq M(v) \leq \infty$ be the least constant so that for every finite interval $I \subset \mathbb{Z}$ we have*

$$\sum_{m,n\in I} v_m^2 v_n^2 \min(1, |m-n|^{2\alpha-1}) \leq M^2 \sum_{n\in I} v_n^2.$$

Then for a constant C depending only on α we have $C^{-1}M(v) \leq L(v) \leq CM(v)$.

PROOF. First suppose $L(v) < \infty$. Then by taking adjoints the map $\xi \longrightarrow K(v\xi)$ is bounded on $\ell_2(\mathbb{Z})$ with norm $L(v)$. In particular we have for any interval I, $\|K(v^2\chi_I)\| \leq L(v)\|v\chi_I\|$. Let us write $\langle \xi, \eta\rangle = \sum_{n\in\mathbb{Z}} \xi_n\eta_n$ where this is well-defined. Thus

$$\left\langle K^2(v^2\chi_I), v^2\chi_I\right\rangle \leq L(v)^2 \sum_{n\in I} v_n^2.$$

Now observe that $K^2(m,n) = \sum_{l=1}^{\infty} K(m,l)K(l,n) \geq c\min(1, |m-n|^{2\alpha-1})$ where $c > 0$ depends only on α. Expanding out we obtain that $M(v) \leq CL(v)$ for some $C = C(\alpha)$.

We now turn to the opposite direction. By homogeneity it is only necessary to bound $L(v)$ when $M(v) = 1$. We therefore assume $M(v) = 1$. Notice that it follows from the definition of $M(v)$ that for any interval I, we have $|I|^{2\alpha-1}\sum_{m,n\in I} v_m^2 v_n^2 \leq \sum_{n\in I} v_n^2$ and so $\sum_{n\in I} v_n^2 \leq |I|^{1-2\alpha}$.

Now let $u = Kv^2$. This can be computed formally, with the possibility of some entries being infinite, but the calculations below will show that the entries of u are finite; alternatively the estimate above leads quickly to the same conclusion. Suppose $m \in \mathbb{Z}$ and define sets $I_0 = \{m\}$ and then $I_k = \{n : 2^{k-1} \leq |m-n| < 2^k\}$ for $k \geq 1$. Note that if $k \geq 1$ I_k is the union of two intervals of length 2^{k-1}. Let $J_k = I_0 \cup \cdots \cup I_k$.

For any k we have

$$u = K(v^2\chi_{J_{k+1}}) + \sum_{l=k+2} K(v^2\chi_{I_l}).$$

Let us write $u_1 = K(v^2\chi_{J_{k+1}})$ and $u_2 = u - u_1$.

Now if $l \geq k+2$ and $j \in I_k$ we have

$$K(v^2\chi_{I_l})(j) \leq C2^{(\alpha-1)l}\sum_{n\in I_l} v_n^2.$$

Hence

$$u_2(j) \leq C\sum_{l=k+2}^{\infty} 2^{(\alpha-1)l}\sum_{n\in I_l} v_n^2.$$

Squaring and summing, and estimating $\sum_{n\in I_i} v_n^2$, we have

$$\sum_{j\in I_k} u_2(j)^2 \leq C2^k \sum_{i\geq l\geq k+2} 2^{(\alpha-1)(i+l)}2^{i(1-2\alpha)}\sum_{n\in I_l} v_n^2.$$

Summing out over $i \geq l$ we have

$$\sum_{j\in I_k} u_2(j)^2 \leq C2^k \sum_{l\geq k+2} 2^{-l}\sum_{n\in I_l} v_n^2.$$

On the other hand

$$\begin{aligned}\sum_{j\in I_k} u_1^2(j) &= \sum_{j\in I_k}\sum_{i\in J_{k+1}}\sum_{l\in J_{k+1}} K(j,i)K(j,l)v_i^2v_l^2\\ &\leq C\sum_{i\in J_{k+1}}\sum_{l\in J_{k+1}}\min(1,|i-l|^{2\alpha-1})v_i^2v_l^2\\ &\leq C\sum_{n\in J_{k+1}} v_n^2\end{aligned}$$

where C depends only on α. In particular $u(j)<\infty$ for all j.

Hence

$$\sum_{j\in I_k} u(j)^2 \leq C\Big(\sum_{n\in J_{k+1}} v_n^2 + 2^k\Big(\sum_{l=k+2}^{\infty} 2^{-l}\sum_{n\in I_l} v_n^2\Big)\Big).$$

This can be written as

$$\sum_{j\in I_k} u(j)^2 \leq C\sum_{l=0}^{\infty}\min(1,2^{k-l})\sum_{n\in I_l} v_n^2.$$

Let us use this to estimate $Ku^2(m)$; we have (letting C be a constant which depends only on α but may vary from line to line),

$$\begin{aligned}Ku^2(m) &\leq C\sum_{k=0}^{\infty} 2^{(\alpha-1)k}\sum_{n\in I_k} u_n^2\\ &\leq C\sum_{k=0}^{\infty} 2^{(\alpha-1)k}\sum_{l=0}^{\infty}\min(1,2^{k-l})\sum_{n\in I_l} v_n^2\\ &\leq C\sum_{l=0}^{\infty}\sum_{n\in I_l} v_n^2\sum_{k=0}^{\infty} 2^{(\alpha-1)k}\min(1,2^{k-l})\\ &\leq C\sum_{l=0}^{\infty} 2^{(\alpha-1)l}\sum_{n\in I_l} v_n^2\\ &\leq CKv^2(m).\end{aligned}$$

We thus have $Ku^2 \leq CKv^2$.

Now put $w = v + Kv^2$. Then $Kw^2 \leq 2(Kv^2 + Ku^2) \leq CKv^2 \leq Cw$. We will show this implies an estimate on $L(v)$.

Indeed if $\xi \in c_{00}(\mathbb{Z})$ is positive then

$$\langle wK\xi, wK\xi\rangle = \big\langle w^2, (K\xi)^2\big\rangle.$$

Now

$$(K\xi)^2(m) = \sum_{i,j} K(m,i)K(m,j)\xi(i)\xi(j) \leq C\sum_{i,j} K(i,j)\big(K(m,i)+K(m,j)\big)\xi(i)\xi(j).$$

This implies $(K\xi)^2 \leq CK(\xi K\xi)$. Hence

$$\|wK\xi\|^2 \leq C\big\langle w^2, K(\xi K\xi)\big\rangle = C\langle Kw^2, \xi K\xi\rangle$$

and hence as $Kw^2 \le Cw$

$$\|wK\xi\|^2 \le C\langle w, \xi K\xi\rangle = C\langle \xi, wK\xi\rangle \le C\|\xi\|\,\|wK\xi\|$$

which leads to $\|wK\xi\| \le C\|\xi\|$ or $L(v) \le L(w) \le C$ where C depends only on α. ∎

THEOREM 3.4. *Suppose $0 < \alpha < 1/2$. Let $\mathbb{A}$ be a subset of $\mathbb{Z}$. Let $\kappa(\mathbb{A}) = \kappa_\alpha(\mathbb{A})$ be the least constant (possibly infinite) such that for any finitely nonzero sequence $(\xi_n)_{n\in\mathbb{A}}$ we have*

$$\left(\int_{-\pi}^{\pi}\Big|\sum_{n\in\mathbb{A}}\xi_n e^{in\theta}\Big|^2\,|\sin\theta|^{-2\alpha}\,d\theta\right)^{1/2} \le \kappa\Big(\sum_{n\in\mathbb{A}}|\xi_n|^2\Big)^{1/2}.$$

Let $M = M(\mathbb{A}) = M(\chi_A)$, be defined as the least constant M so that for any finite interval I we have, setting $F = \mathbb{A}\cap I$,

$$\sum_{m,n\in F}\min(1, |m-n|^{2\alpha-1}) \le M^2|F|.$$

Then $\kappa(\mathbb{A}) < \infty$ if and only if $M(\mathbb{A}) < \infty$ and there is constant C depending only on α such that $C^{-1}M(\mathbb{A}) \le \kappa(\mathbb{A}) \le CM(\mathbb{A})$. ∎

PROOF. First suppose $M(\mathbb{A}) < \infty$. Note that $\psi(\theta) = |\theta|^{-\alpha}$ is an L_2-function whose Fourier transform satisfies the property that $\lim_{|n|\to\infty}|n|^{1-\alpha}\hat\psi(n)$ exists and is positive. Now suppose $(\xi_n) \in c_{00}(\mathbb{A})$ and let $g = \sum_{n\in\mathbb{A}}\xi_n e^{in\theta}$. Suppose $f \in L_2[-\pi,\pi]$. Then

$$\langle |\theta|^{-\alpha}g, f\rangle = \langle \hat\psi * \hat g, \hat f\rangle.$$

Hence for a suitable $C = C(\alpha)$ we have, using Plancherel's theorem, with K as in Theorem 3.3,

$$\langle |\theta|^{-\alpha}g, f\rangle \le C\langle K|\hat g|, |\hat f|\rangle = C\langle |\hat g|, \chi_{\mathbb{A}}K|\hat f|\rangle.$$

We deduce

$$\langle |\theta|^{-\alpha}g, f\rangle \le CM(\mathbb{A})\|g\|_2\|f\|_2.$$

Thus

$$\int_{-\pi}^{\pi}|g(\theta)|^2\,|\theta|^{-2\alpha}\,d\theta \le C^2M^2\Big(\sum_{n\in\mathbb{A}}|\xi_n|^2\Big).$$

By translation we also have

$$\int_{-\pi}^{\pi}|g(\theta)|^2(\pi-|\theta|)^{-2\alpha}\,d\theta \le C^2M^2\Big(\sum_{n\in\mathbb{A}}|\xi_n|^2\Big).$$

Since $|\theta|^{-2\alpha} + (\pi-|\theta|)^{-2\alpha} \ge |\sin\theta|^{-2\alpha}$ we obtain immediately $\kappa(\mathbb{A}) \le CM(\mathbb{A})$ where C depends only on α.

Conversely suppose $\kappa(\mathbb{A}) < \infty$. Note first that there is positive-definite and non-negative trigonometric polynomial h so that $h+\psi$ satisfies $\hat h(n) + \hat\psi(n) \ge c\min(1, |n|^{\alpha-1})$ where $c > 0$. Now clearly for $(\xi_n) \in c_{00}(\mathbb{A})$,

$$\int_{-\pi}^{\pi}|g|^2(\psi+h)^2\,d\theta \le C\kappa\Big(\sum_{n\in\mathbb{A}}|\xi_n|^2\Big)^{1/2}.$$

Thus again by Plancherel's theorem, if $\xi \geq 0$,

$$\|K\xi\|_2^2 \leq C\kappa\|\xi\|_2^2.$$

A similar inequality then applies for general ξ.

It follows quickly by taking adjoints that $L(\chi_{\mathbb{A}}) \leq C\kappa$ and hence $M(\mathbb{A}) \leq C\kappa(\mathbb{A})$. ∎

THEOREM 3.5. *Suppose $\mathbb{F}$ is a finite subset of $\mathbb{Z}$ and $|\mathbb{F}| = N$. Let $(\eta_j)_{j\in\mathbb{F}}$ be a sequence of independent $0-1$-valued random variables (or selectors) with $\mathbf{E}(\eta_j) = \sigma = N^{-2\alpha}$ for $j \in \mathbb{F}$. Let $\mathbb{A} = \{j \in \mathbb{F} : \eta_j = 1\}$ be the corresponding random subset of $\mathbb{F}$. Then $\mathbf{E}\big(M(\mathbb{A})^2\big) \leq C$ where C depends only on α.*

PROOF. It is easy to see that if this statement is proved for the set $\mathbb{F} = \{1, 2, \ldots, N\}$ then it is true for every interval $\mathbb{F}$ and then for every finite subset of $\mathbb{Z}$. It is also easy to see that it suffices to prove the result for $N = 2^n$ for some n.

Note next that

$$M^2(\mathbb{A}) \leq \sup_{1\leq k\leq N} \sum_{n\in\mathbb{A}} \min(|k-n|^{2\alpha-1}, 1).$$

Hence

$$M^2(\mathbb{A}) \leq C\sum_{k=0}^{n} \max_{1\leq j\leq 2^{n-k}} 2^{k(2\alpha-1)}\big|\mathbb{A}\cap\big[(j-1)2^k+1, j2^k\big]\big|,$$

where C depends only on α.

Fix an integer s. We estimate, for fixed k,

$$\begin{aligned}
\mathbf{E}\Big(\max_{1\leq j\leq 2^{n-k}}\big|\mathbb{A}\cap\big[(j-1)2^k+1, j2^k\big]\big|\Big) &\leq \mathbf{E}\Big(\sum_{j=1}^{2^{n-k}}\Big(\sum_{l=(j-1)2^k+1}^{j2^k}\eta_l\Big)^s\Big)^{1/s}\\
&\leq \Big(\mathbf{E}\Big(\sum_{j=1}^{2^{n-k}}\Big(\sum_{l=(j-1)2^k+1}^{j2^k}\eta_l\Big)^s\Big)\Big)^{1/s}\\
&\leq 2^{(n-k)/s}\Big(\mathbf{E}\Big(\sum_{j=1}^{2^k}\eta_j\Big)^s\Big)^{1/s}.
\end{aligned}$$

Let us therefore estimate, setting $m = 2^k$,

$$\begin{aligned}
\mathbf{E}\Big(\sum_{j=1}^{m}\eta_j\Big)^s &= \sum_{l\leq\min(s,m)}\ \sum_{j_1+\cdots+j_l=s}\frac{s!}{j_1!\ldots j_l!}\sigma^l\\
&\leq \sum_{l=1}^{s}\binom{m}{l}l^s\sigma^l\\
&\leq \sum_{l=1}^{s} l^s(m\sigma)^l\\
&\leq s\max_{1\leq l\leq m}\big(l^s(m\sigma)^l\big).
\end{aligned}$$

By maximizing the function $x^s e^{-ax}$ we see that if $m\sigma \geq e^{-1}$ we can estimate this by

$$\mathbf{E}\Big(\sum_{j=1}^{m}\eta_j\Big)^s \leq s^{s+1}(m\sigma)^s.$$

On the other hand if $m\sigma < e^{-1}$

$$\mathbf{E}\Big(\sum_{j=1}^m \eta_j\Big)^s \le s(s|\log m\sigma|^{-1})^{s/|\log m\sigma|} \le s^{s+1}|\log m\sigma|^{-s}.$$

Suppose $k < n$. Put $s = n-k$. We have

$$\mathbf{E}\Big(\max_{1\le j\le 2^{n-k}}\big|\mathbb{A}\cap\big[(j-1)2^k+1, j2^k\big]\big|\Big) \le C(n-k)2^k\sigma$$

whenever $2^k\sigma \ge e^{-1}$ where $C = C(\alpha)$. If $2^k\sigma < e^{-1}$,

$$\mathbf{E}\Big(\max_{1\le j\le 2^{n-k}}\big|\mathbb{A}\cap\big[(j-1)2^k+1, j2^k\big]\big|\Big) \le C\frac{n-k}{|\log(\sigma 2^k)|}.$$

Hence

$$\mathbf{E}\Big(M(\mathbb{A})^2\Big) \le \sum_{2^k\sigma<e^{-1}} \frac{n-k}{|\log(\sigma 2^k)|}2^{(2\alpha-1)k} + \sum_{2^k\sigma\ge e^{-1}}(n-k+1)2^{2\alpha k}\sigma.$$

We can estimate this further by

$$\mathbf{E}\Big(M(\mathbb{A})^2\Big) \le C\Big(\sum_{2^k\sigma<e^{-n}} 2^{(2\alpha-1)k} + n\sigma^{1-2\alpha} + 2^{2\alpha n}\sigma\Big)$$

where $C = C(\alpha)$.

We now recall that $\sigma = N^{-2\alpha} = 2^{-2\alpha n}$. We then obtain an estimate

$$\mathbf{E}\Big(M(\mathbb{A})^2\Big) \le C(\alpha). \qquad \blacksquare$$

THEOREM 3.6. *Suppose* $0 < \lambda < \infty$ *and that* $2 < p < r = 2 + \lambda^{-1}$. *Let* $\mathbb{F} \subset \mathbb{N}$ *be a finite set with* $|\mathbb{F}| = N$. *Let* $(\eta_j)_{j\in\mathbb{F}}$ *be a sequence of independent* $0-1$*-valued random variables (or selectors) with* $\mathbf{E}(\eta_j) = \sigma = N^{(1/p-1/2)/(1/2-1/r)}$ *for* $j \in \mathbb{F}$. *Let* $\mathbb{A} = \{j \in \mathbb{F} : \eta_j = 1\}$ *be the corresponding random subset of* $\mathbb{F}$ *(so that* $\mathbf{E}(|\mathbb{A}|) = N^{(1/p-1/r)/(1/2-1/r)}$*). Then* $\mathbf{E}\Big(\Lambda_{p,\lambda}(\mathbb{A})^p\Big) \le C$ *where* C *depends only on* p *and* λ.

PROOF. Suppose $(\xi_n)_{n\in\mathbb{A}}$ are any (complex) scalars and let $f = \sum_{n\in\mathbb{A}}\xi_n e^{in\theta}$. Let $\alpha = (1/2-1/p)/(1-2/r)$, and let $\frac{1}{q} = \frac{1}{2} - \alpha$. Then by Holder's inequality, since $\frac{1}{p} = (1-\frac{2}{r})\frac{1}{q} + \frac{2}{r}\frac{1}{2}$

$$\begin{aligned}\Big(\int_{-\pi}^{\pi}&(|f|\,|\sin\theta|^{\lambda(2/p-1)})^p\,d\theta\Big)^{1/p}\\ &\le \Big(\int_{-\pi}^{\pi}|f|^q\,d\theta\Big)^{(1-2/r)/q}\Big(\int_{-\pi}^{\pi}(|f|\,|\sin\theta|^{r\lambda(1/p-1/2)})^2\,d\theta\Big)^{1/r}.\end{aligned}$$

Note that $r\lambda(1/p-1/2) = (1/p-1/2)/(1-2/r) = \alpha$. Hence

$$\Big(\int_{-\pi}^{\pi}(|f|\,|\sin\theta|^{\lambda(2/p-1)})^p\,d\theta\Big)^{1/p} \le \Lambda_{q,0}(\mathbb{A})^{1-2/r}\kappa_\alpha(\mathbb{A})^{2/r}\Big(\sum_{n\in\mathbb{A}}|\xi_n|^2\Big)^{1/2}.$$

Thus we deduce

$$\Lambda_{p,\lambda}(\mathbb{A}) \le \Lambda_{q,0}(\mathbb{A})^{1-2/r}\kappa_\alpha(\mathbb{A})^{2/r}.$$

It follows further from Holder's inequality that

$$\Big(\mathbf{E}\Big(\Lambda_{p,\lambda}(\mathbb{A})\Big)^p\Big)^{1/p} \le \mathbf{E}\Big(\Lambda_{q,0}(\mathbb{A})^q\Big)^{(1-2/r)/q}\mathbf{E}\Big(\kappa_\alpha(\mathbb{A})^2\Big)^{1/r}.$$

As $\mathbf{E}(|\mathbb{A}|) = N^{2/q}$, we have by the $\Lambda(p)$ theorem of Bourgain [2] that $\mathbf{E}\Big(\Lambda_{q,0}(\mathbb{A})^q\Big)^{1/q} \le C = C(q)$. By Theorem 3.5 above we obtain:

$$\Big(\mathbf{E}\Big(\Lambda_{p,\lambda}(\mathbb{A})\Big)^p\Big)^{1/p} \le C$$

where $C = C(\lambda, p)$. ∎

4. **The structure of the space of polynomials.** We recall that $(\tau_{nk}^{(\lambda)} = \cos\theta_{nk}^{(\lambda)})_{k=1}^n$ are the zeros of the polynomial $\varphi_n^{(\lambda)}$ ordered so that $0 < \theta_{n,1}^{(\lambda)} < \theta_{n,2}^{(\lambda)} < \cdots < \theta_{nn}^{(\lambda)} < \pi$.

THEOREM 4.1. *Suppose $1 < p < \infty$, $-\frac{1}{2} < \lambda$, $\mu < \infty$ and that the ultraspherical polynomials $(\varphi_n^{(\lambda)})_{n=0}^\infty$ form a basis of $L_p\Big([-1,1],(1-x^2)^{\mu-\frac{1}{2}}\Big)$ or, equivalently that*

$$\left|\frac{2\mu+1}{2p} - \frac{2\lambda+1}{4}\right| < \min\left(\frac{1}{4}, \frac{2\lambda+1}{4}\right). \tag{4.1}$$

Let $\tau_{nk} = \tau_{nk}^{(\lambda)}$. Then there is a constant $C = C(\lambda, \mu, p)$ independent of n so that if $f \in \mathcal{P}_n$ then

$$\begin{aligned}\frac{1}{C}\left(\frac{1}{n}\sum_{k=1}^n(1-\tau_{nk}^2)^\mu|f(\tau_{nk})|^p\right)^{1/p} &\le \left(\int_{-1}^1|f(x)|^p(1-x^2)^{\mu-\frac{1}{2}}\,dx\right)^{1/p}\\ &\le C\left(\frac{1}{n}\sum_{k=1}^n(1-\tau_{nk}^2)^\mu|f(\tau_{nk})|^p\right)^{1/p}.\end{aligned}$$

In particular $d(\mathcal{P}_n, \ell_p^n) \le C^2$.

PROOF. We will start by supposing that μ is not of the form $\frac{1}{2}(mp-1)$ for $m \in \mathbb{N}$ and that $-\frac{1}{2} < \lambda$ is arbitrary (*i.e.*, we do not assume (4.1)). In this case we can find $m \in \mathbb{N}$ so that $-\frac{1}{2} < \mu - \frac{1}{2}mp < \frac{1}{2}(p-1)$. Then $w(\theta) = (\sin\theta)^{2\mu-mp}$ is an A_p-weight. This implies (*cf.* [4]) that there is a constant $C = C(\mu, p)$ so that for any trigonometric polynomial $h(\theta) = \sum_{k=-N}^N \hat{h}(k)e^{ik\theta}$ of degree N, and any $1 \le l \le N$ we have

$$\left(\int_{-\pi}^{\pi}\Big|i\sum_{k\ge l}\hat{h}(k)e^{ik\theta} - i\sum_{k\le -l}\hat{h}(k)e^{ik\theta}\Big|^p w(\theta)\,d\theta\right)^{1/p} \le C\left(\int_{-\pi}^{\pi}|h(\theta)|^p w(\theta)\,d\theta\right)^{1/p}.$$

Summing over $l = 1, 2, \ldots, N$ we obtain

$$\left(\int_{-\pi}^{\pi}\Big|\sum_{k=-N}^N ik\hat{h}(k)e^{ik\theta}\Big|^p w(\theta)\,d\theta\right)^{1/p} \le CN\left(\int_{-\pi}^{\pi}|h(\theta)|^p w(\theta)\,d\theta\right)^{1/p},$$

i.e.,

$$\left(\int_{-\pi}^{\pi}|h'(\theta)|^p w(\theta)\,d\theta\right)^{1/p} \le CN\left(\int_{-\pi}^{\pi}|h(\theta)|^p w(\theta)\,d\theta\right)^{1/p}. \tag{4.2}$$

Now suppose $f \in P_n$ and let $h(\theta) = (\sin\theta)^m f(\cos\theta)$ so that h is a trigonometric polynomial of degree at most $m+n-1 \le C(\mu,p)n$. Let I_k be the interval $|\theta-\theta_{nk}| \le \frac{\pi}{n}$ for $1 \le k \le n$. Then

$$\begin{aligned}\int_{I_k}|h(\theta)|\,d\theta &\le \left(\int_{I_k} w(\theta)^{-p'/p}\,d\theta\right)^{1/p'}\left(\int_{I_k}|h(\theta)|^p w(\theta)\,d\theta\right)^{1/p}\\ &\le C\frac{1}{n^{1/p'}}(\sin\theta_{nk})^{m-2\mu/p}\left(\int_{I_k}|h|^p w(\theta)\,d\theta\right)^{1/p}.\end{aligned}$$

Here we use the properties of (τ_{nk}) and (θ_{nk}) from Proposition 2.5. On the other hand,

$$\begin{aligned}\int_{I_k}|h(\theta)-h(\theta_{nk})|\,d\theta &\le \frac{\pi}{n}\int_{I_k}|h'(\theta)|\,d\theta\\ &\le C\frac{1}{n^{1+1/p'}}(\sin\theta_{nk})^{m-2\mu/p}\left(\int_{I_k}|h'|^p w\,d\theta\right)^{1/p}.\end{aligned}$$

Putting these together we conclude that

$$\frac{1}{n}|h(\theta_{nk})|^p(\sin\theta_{nk})^{2\mu-mp} \le C^p\left(\int_{I_k}|h|^p w(\theta)\,d\theta + \frac{1}{n^p}\int_{I_k}|h'|^p w\,d\theta\right).$$

On summing we obtain

$$\frac{1}{n}\sum_{k=1}^{n}|f(\tau_{nk})|^p(1-\tau_{nk}^2)^{\mu} \le C^p\left(\int_{-\pi}^{\pi}|h|^p w\,d\theta + \frac{1}{n^p}\int_{-\pi}^{\pi}|h'|^p w\,d\theta\right)$$

since $\sum_{k=1}^{n}\chi_{I_k}$ is uniformly bounded by Proposition 2.5. Now appealing to (4.2) we have

$$\frac{1}{n}\sum_{k=1}^{n}|f(\tau_{nk})|^p(1-\tau_{nk}^2)^{\mu} \le C^p\int_{-\pi}^{\pi}|h|^p w\,d\theta.$$

Recalling the definition of w and h this implies

$$\left(\frac{1}{n}\sum_{k=1}^{n}|f(\tau_{nk})|^p(1-\tau_{nk}^2)^{\mu}\right)^{1/p} \le C\left(\int_{-1}^{+1}|f(x)|^p(1-x^2)^{\mu-\frac{1}{2}}\,dx\right)^{1/p}. \tag{4.3}$$

Note that we only have (4.3) when μ is not of the form $\frac{1}{2}(mp-1)$. We now prove (4.3) for μ in the exceptional case. We observe that if $\nu = \frac{2}{r}\mu + \frac{1}{r} - \frac{1}{2}$ then $\nu > -\frac{1}{2}$ and (4.1) holds for $\lambda = \nu$. In fact there exists $0 < \delta < \frac{p}{2}$ so that $(\varphi_n^{(\nu)})$ is a basis of both $L_p\big([-1,1],(1-x^2)^{\mu-\delta}\big)$ and of $L_p\big([-1,1],(1-x^2)^{\mu+\delta}\big)$. Let

$$S_n^{\nu}(f) = \sum_{k=0}^{n-1}\varphi_n^{(\lambda)}\int_{-1}^{+1}f(x)\varphi_n^{\nu}(x)(1-x^2)^{\nu-\frac{1}{2}}\,dx$$

be the partial sum operator associated with this basis. Let us consider the map T_n: $L_p\big([-1,1],(1-x^2)^{\mu\pm\delta}\big)\longrightarrow \mathbf{R}^n$ defined by

$$T_n(f)_k = (S_n^{(\nu)}f)(\tau_{nk}).$$

Then there is a constant C independent of n so that

$$\left(\frac{1}{n}\sum_{k=1}^{n}|T_n(f)_k|^p(1-\tau_{nk}^2)^{\mu\pm\delta}\right)^{1/p} \le C\left(\int_{-1}^{+1}|f(x)|^p(1-x^2)^{\mu\pm\delta-\frac{1}{2}}\right)^{1/p}.$$

It follows by interpolation that we obtain

$$\left(\frac{1}{n}\sum_{k=1}^{n}|T_n(f)_k|^p(1-\tau_{nk}^2)^{\mu}\right)^{1/p} \le C\left(\int_{-1}^{+1}|f(x)|^p(1-x^2)^{\mu-\frac{1}{2}}\right)^{1/p}$$

and on restricting to P_n we have (4.3) for all μ.

We now assume λ satisfies (4.1) and complete the proof by duality. Let σ be defined by $\frac{\sigma}{p'}+\frac{\mu}{p}=\lambda$. Then (4.1) also holds if we replace p,μ by p',σ.

Suppose $f\in P_n$. Then there exists $h\in L_p\big([-1,1],(1-x^2)^{\sigma-\frac{1}{2}}\big)$ so that

$$\int_{-1}^{+1}|h(x)|^{p'}(1-x^2)^{\sigma-\frac{1}{2}}\,dx = 1$$

and

$$\int_{-1}^{+1}h(x)f(x)(1-x^2)^{\lambda-\frac{1}{2}}\,dx = \left(\int_{-1}^{+1}|f(x)|^p(1-x^2)^{\mu-\frac{1}{2}}\,dx\right)^{1/p}.$$

Let $g=S_n^{(\lambda)}f$. Then

$$\int_{-1}^{+1}|g(x)|^{p'}(1-x^2)^{\sigma-\frac{1}{2}}\,dx \le C^p$$

where $C=C(p,\lambda,\mu)$ is independent of n. Now using Gauss-Jacobi quadrature (see Proposition 2.6) we have

$$\frac{1}{n}\sum_{k=1}^{n}\alpha_{nk}^{(\lambda)}f(\tau_{nk})g(\tau_{nk}) = \int_{-1}^{+1}f(x)h(x)(1-x^2)^{\lambda-\frac{1}{2}}\,dx.$$

We recall that

$$0\le \alpha_{nk}^{(\lambda)} \le C(1-\tau_{nk}^2)^{\lambda}n^{-1}$$

where C is again independent of n. It follows that

$$\begin{aligned}&\left(\int_{1}^{+1}|f(x)|^p(1-x^2)^{\mu-\frac{1}{2}}\,dx\right)^{1/p}\\ &\qquad\le C\left(\frac{1}{n}\sum_{k=1}^{n}|f(\tau_{nk})|^p(1-\tau_{nk}^2)^{\mu}\right)^{1/p}\left(\frac{1}{n}\sum_{k=1}^{n}|g(\tau_{nk})|^{p'}(1-\tau_{nk}^2)^{\sigma}\right)^{1/p'}.\end{aligned}$$

Now applying (4.3) we can estimate the last term by a constant independent of n. Thus we have

$$\left(\int_{1}^{+1}|f(x)|^p(1-x^2)^{\mu-\frac{1}{2}}\,dx\right)^{1/p} \le C\left(\frac{1}{n}\sum_{k=1}^{n}|f(\tau_{nk})|^p(1-\tau_{nk}^2)^{\mu}\right)^{1/p}.$$

This completes the proof. ∎

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Department of Mathematics
University of Missouri
Columbia, MO 65211
USA
email: nigel@math.missouri.edu

Department of Mathematics
The Hebrew University
Jerusalem
Israel
email: liortz@math.huji.ac.il