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BANACH ENVELOPES OF NON-LOCALLY CONVEX SPACES

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1. Introduction. Let X be a quasi-Banach space whose dual X^* separates the points of X. Then X^* is a Banach space under the norm

$$||x^*|| = \sup_{||x|| \leqq 1} |x^*(x)|.$$

From X we can construct the *Banach envelope* X_c of X by defining for $x \in X$, the norm

$$||x||_c = \sup_{||x^*|| \leqq 1} |x^*(x)|.$$

Then X_c is the completion of $(X, || \ ||_c)$. Alternatively $|| \ ||_c$ is the Minkowski functional of the convex hull of the unit ball. X_c has the property that any bounded linear operator $L:X \rightarrow Z$ into a Banach space extends with preservation of norm to an operator $\tilde{L}:X_c \rightarrow Z$.

The Banach envelope of l_p $(0 < p < 1)$ is, of course, l_1. In 1969, Duren, Romberg and Shields [3] identified the dual space of H_p $(0 < p < 1)$ and thus its Banach envelope (cf. [14]). The Banach envelope of H_p is a Bergman space which turns out to be isomorphic again to l_1 (see [18] for a recent direct proof of this). These examples and others prompted Joel Shapiro to ask what special properties a Banach envelope of a non-locally convex space (with separating dual) must have.

An example of Pelczynski, the space $l_2(l_p^n)$ $(0 < p < 1)$ has a reflexive Banach envelope $(l_2(l_1^n))$. This suggests that the answer to Shapiro's question lies with the finite-dimensional structure of the space.

We say that a Banach space Y *contains* l_1^n*'s uniformly* (or l_1 *is finitely representable in* Y) if for every $n \in \mathbf{N}$ and $\epsilon > 0$ there is a subspace F of Y with $\dim F = n$ and a linear isomorphism $T:l_1^n \rightarrow F$ with

$$||T|| \, ||T^{-1}|| < 1 + \epsilon.$$

We say that Y contains *uniformly complemented* l_1^n's if in addition F can be chosen so that there is a projection $P:Y \rightarrow F$ with $||P|| < 1 + \epsilon$.

Y fails to contain l_1^n's uniformly if and only if Y is of *type* p for some $p > 1$ i.e., for some $C < \infty$ and all $y_1, \ldots, y_n \in Y$

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$$\mathscr{E}\left(\|\sum_{i=1}^{n} \epsilon_i y_i\|^p\right) \leqq C \sum_{i=1}^{n} \|y_i\|^p$$

where $\epsilon_1, \ldots, \epsilon_n$ is a sequence of independent Bernoulli random variables on some probability space (with $P(\epsilon_i = 1) = P(\epsilon_i = -1) = 1/2$). Y fails to contain uniformly complemented l_1^n's if and only if Y^* fails to contain l_∞^n's uniformly; this in turn is equivalent to Y^* having *finite cotype p* for some $p < \infty$, i.e., for some $c < 0$ and all $y_1^*, \ldots, y_n^* \in Y^*$

$$\mathscr{E}\left(\|\sum_{i=1}^{n} \epsilon_i y_i^*\|^p\right) \geqq c \sum_{i=1}^{n} \|y_i^*\|^p.$$

See [**10**].

The author [**5**] has shown:

THEOREM 1.1. *Let X be a non-locally convex quasi-Banach space with separating dual. Then X_c contains l_1^n's uniformly.*

Thus l_p $(1 < p < \infty)$ and L_p $(1 < p < \infty)$ cannot be Banach envelopes of such spaces. However the case of c_0 (or l_∞) was left unresolved. Several examples have been suggested as possible spaces with Banach envelopes isomorphic to c_0. For example, Shapiro has recently studied the harmonic Hardy spaces h_p $(0 < p < 1)$, and these have subspaces h_p^0 which are hereditarily c_0 [**15**]. Similar examples have been studied by A. Matheson. The Banach space envelopes of these spaces have not been determined, however.

In this paper, we shall show first that these and similar examples can never have c_0 as a Banach envelope. Precisely, we introduce the concept of a natural quasi-Banach space. The idea is that all non-locally convex quasi-Banach spaces which commonly arise in analysis are natural (e.g. any subspace of an Orlicz function space or of a Lorentz function space). We then show that any non-locally convex subspace of a natural space with a basis (or the bounded approximation property) has a Banach envelope which contains uniformly complemented l_1^n's. The same conclusion applies to any subspace of the nonseparable space h_p (Theorems 3.4 and 3.5). The proofs of these facts depend on a theorem essentially due to Maurey [**8**] (cf. [**9**]) on the factorization of operators into L_p where $0 < p < 1$. As we require a slight modification of Maurey's theorem and, in any case, his proof lies deeply embedded in [**8**], we shall present an exposition of the result we need in Section 2, using ideas of Bennett ([**1**], [**2**]).

In Section 4, however, we construct a non-locally convex space whose Banach envelope is isomorphic to c_0; although this space has an unconditional basis, it cannot be natural. In Section 5 we show further that for any separable Banach space Z we can find a subspace of l_p $(0 < p < 1)$ whose Banach envelope is isomorphic to $l_1 \oplus Z$. Thus although c_0 is not the Banach envelope of any subspace of l_p, $l_1 \oplus c_0$ is such a Banach envelope. Note, of course, that l_p has no infinite-dimensional locally convex subspaces.

Finally in Section 6 we prove some special results for the Banach envelopes of spaces with unconditional bases.

All our results are stated for real quasi-Banach spaces with the understanding that they generalize to the complex case without difficulty.

We conclude the introduction by making a few remarks which we hope will assist the non-specialist reader.

First we remark that Banach spaces which do not contain l_1^n's uniformly have been studied in the past both from the point of view of the geometry of Banach spaces and from the point of view of probability in Banach spaces; in the latter context such spaces are called B-convex, and this terminology was used by the author in [**5**]. The reader is referred to [**12**] for several formulations of this notion. We note that every uniformly convex Banach space fails to contain l_1^n's uniformly, but, in the converse direction there do exist non-reflexive B-convex spaces ([**4**]). The most important observation for the reader is that c_0 does contain l_1^n's; indeed l_1^n embeds in l_∞^m if $m = 2^n$, and c_0 contains l_∞^n's. However c_0 does not contain uniformly complemented l_1^n's since $c_0^* = l_1$ does not contain l_∞^n's.

The second observation is that in Section 3 we introduce the notion of a natural space, building on ideas in [**6**]. However the reader who does not wish to delve into [**6**] may instead take the conclusion of Theorem 3.1 for the definition of a natural space. Using this it is easy enough to show that L_p, $0 < p < 1$, is natural since it is transitive (see [**13**]).

2. A theorem of Maurey. In this section we present a slight modification of a theorem due to Maurey [**8**]. This theorem is very difficult to trace in [**8**] however (cf. Theorem 93 and the remarks in [**1**] or [**2**]) and so we present a direct proof based on ideas of Bennett ([**1**], [**2**]).

Let (Ω, Σ, μ) be a measure space of total mass one. On $L_p(\Omega, \Sigma, \mu)$ we define for $E \in \Sigma$ the projection P_E by $P_E f = 1_E f$ where 1_E is the indicator function of E.

LEMMA 2.1. *Let X be a quasi-Banach space whose dual X^* has finite cotype, and suppose $0 < p < 1$. Then there is a constant C so that whenever*

$$T:X \to L_1(\Omega, \Sigma, \mu)$$

is an operator and $\epsilon > 0$ then there is $E \in \Sigma$ with $\mu(E) \geqq 1 - \epsilon$ and

$$\|P_E T\|_1 \leqq C\epsilon^{1-1/p} \|T\|_p.$$

Here $\|S\|_r$ denotes the norm of the operator $S:X \to L_r$.

Proof. Since X^* has finite cotype there exists an integer m so that whenever $x_1^*, \ldots, x_m^* \in X^*$ then

$$\max_{\eta_i = \pm 1} \|\Sigma \eta_i x_i^*\| \geqq 2^{1/p+1} \min_{1 \leqq i \leqq m} \|x_i^*\|.$$

Now suppose $S: X \to L_1(\Omega, \Sigma, \mu)$ is any operator and fix $\epsilon > 0$. We shall prove the existence of a set $E \in \Sigma$ with $\lambda(E) \geqq 1 - \epsilon$ so that for any $x \in X$

$$(*) \quad \|P_E Sx\|_1 \leqq 2^{-1/p}\|S\|_1\|x\| + \frac{p}{1-p}\left(\frac{m}{\epsilon}\right)^{1/p-1}\|Sx\|_p.$$

To prove this select inductively $x_1, \ldots, x_m \in X$ and disjoint sets $E_1, \ldots, E_m \in \Sigma$ so that

(a) $\|x_i\| = 1 \quad 1 \leqq i \leqq m$

(b) $\mu(E_i) = m^{-1}\epsilon \quad 1 \leqq i \leqq m$

(c) $\displaystyle\int_{E_i} |Tx| d\mu \geqq \frac{1}{2} \sup \int_F |Sy| d\mu$

where the supremum is taken over all $F \in \Sigma$ with

$$\mu(F) = m^{-1}\epsilon \quad \text{and} \quad F \subset \Omega \setminus \bigcup_{j<i} E_j$$

and all $y \in X$ with $\|y\| = 1$. Then let

$$E = \Omega \setminus \bigcup_{i=1}^{m} E_i;$$

clearly $\mu(E) = 1 - \epsilon$.

Now pick $u_i \in L_\infty$ with $\|u_i\|_\infty = 1$ and supp $u_i \subset E_i$ so that

$$\int_{E_i} u_i(\omega) Sx_i(\omega) d\mu = \int_{E_i} |Sx_i| d\mu.$$

Then if $S^*: L_\infty \to X^*$ is the adjoint of S,

$$\|S^* u_i\| \geqq \int_{E_i} |Sx_i| d\mu.$$

Now for $\eta_i = \pm 1$

$$\|\Sigma\, \eta_i S^* u_i\| \leqq \|S\|_1$$

so that

$$\min_{1 \leqq i \leqq m} \|S^* u_i\| \leqq 2^{-1/p-1}\|S\|_1$$

and hence

$$\min_{1 \leqq i \leqq m} \int_{E_i} |Sx_i| d\mu \leqq 2^{-1/p-1} \|S\|_1.$$

Now suppose $x \in X$ and let $h = 1_E Sx$. Let $f:(0, 1) \to \mathbf{R}$ be the decreasing rearrangement of h, i.e.,

$$f(t) = \inf_{\mu(A)=t} \sup_{\omega \notin A} |h(\omega)|.$$

Hence

$$f(t) \leqq t^{-1/p} \|Sx\|_p.$$

It follows that there exists $F \in \Sigma$ with $F \subset E$, $\mu(F) = m^{-1}\epsilon$ and

$$\int_{\Omega \setminus F} |h| d\mu \leqq \|Sx\|_p \int_{\epsilon/m}^{\infty} t^{-1/p} = \frac{p}{1-p} \left(\frac{m}{\epsilon}\right)^{1/p-1} \|Sx\|_p.$$

However

$$\int_F |h| d\mu \leqq \left(2 \min_{1 \leqq i \leqq m} \int_{E_i} |Sx_i| d\mu\right) \|x\|$$

$$\leqq 2^{-1/p} \|S\|_1 \|x\|.$$

Equation (*) now follows immediately.

To complete the proof of the lemma fix an operator $T:X \to L_1$ and define for $\epsilon > 0$

$$\phi(\epsilon) = \inf\{ \|P_E T\|_1 : \mu(E) \geqq 1 - \epsilon\}.$$

For convenience $\phi(\epsilon) = 0$ if $\epsilon \geqq 1$. Let

$$A = \sup_{\epsilon > 0} \epsilon^{1/p-1} \phi(\epsilon).$$

Clearly $A \leqq \|T\|_1$.

Now apply (*) to $S = P_E T$ for any E with $\mu(E) = 1 - \epsilon$. We decide that

$$\phi(2\epsilon) \leqq 2^{-1/p} \|S\|_1 + \frac{p}{1-p} m^{1/p-1} \epsilon^{1-1/p} \|T\|_p$$

and hence that

$$\phi(2\epsilon) \leqq 2^{-1/p} \phi(\epsilon) + \frac{p}{1-p} m^{1/p-1} \epsilon^{1-1/p} \|T\|_p.$$

We deduce that

$$(2\epsilon)^{1/p-1} \phi(2\epsilon) \leqq \frac{1}{2} \epsilon^{1/p-1} \phi(\epsilon) + \frac{p}{1-p} (2m)^{1/p-1} \|T\|_p$$

so that

$$A \leqq \frac{1}{2}A + \frac{p}{1-p}(2m)^{1/p-1}\|T\|_p.$$

Hence

$$A \leqq \frac{2^{1/p}}{1-p}m^{1/p-1}\|T\|_p$$

and the lemma is proved with

$$C = \frac{2^{1/p}}{1-p}m^{1/p-1}\|T\|_p.$$

If X and Y are two quasi-Banach spaces we shall define $\mathscr{A}(X, Y)$ to be the smallest linear subspace of $\mathscr{L}(X, Y)$ containing the finite-rank operators and closed under pointwise convergence of uniformly bounded nets. If $T \in \mathscr{A}(X, Y)$ we shall say that T is approximable.

THEOREM 2.2. *Let X be a quasi-Banach space such that X^* has finite cotype and suppose $0 < p < 1$. Then there is a constant $C = C(p, X)$ so that if $T:X \to L_p(\Omega, \Sigma, \mu)$ is a bounded operator then the following are equivalent*:

(i) *T is approximable*

(ii) *For $0 < \epsilon < 1$ there exists $E \in \Sigma$ with $\mu(E) \geqq 1 - \epsilon$ so that $P_E T \in \mathscr{L}(X, L_1)$ and*

$$\|P_E T\|_1 \leqq C\epsilon^{1-1/p}\|T\|_p.$$

(iii) *For $0 < \epsilon < 1$ there exists $E \in \Sigma$ with $\mu(E) \geqq 1 - \epsilon$ so that $P_E T$ is a bounded operator of X into L_1.*

Proof. (ii) $\Rightarrow$ (iii). This is immediate.

(iii) $\Rightarrow$ (ii). For $\epsilon > 0$ select E_0 so that

$$\mu(E) \geqq 1 - \frac{1}{2}\epsilon$$

and $P_{E_0}T$ maps X into L_1. Now apply Lemma 2.1 to $P_{E_0}T$ with ϵ replaced by $\frac{1}{2}\epsilon$. Condition (ii) follows with C replaced by $2^{1/p-1}C$.

(iii) $\Rightarrow$ (i). Select a sequence $E_n \in \Sigma$ so that

$$\mu(E_n) \geqq 1 - \frac{1}{n} \quad \text{and} \quad P_{E_n}T \in \mathscr{L}(X, L_1).$$

Now L_1 has the metric approximation property so that each $P_{E_n}T$ is approximable in $\mathscr{L}(X, L_1)$ and hence also in $\mathscr{L}(X, L_p)$. Thus T is approximable in $\mathscr{L}(X, L_p)$.

(i) $\Rightarrow$ (iii). Let $\mathscr{J}(X, L_p)$ be the set of operators $T:X \to L_p$ so that for every $\epsilon > 0$ we can find $E \in \Sigma$ with

$$P_E T \in \mathscr{L}(X, L_1) \quad \text{and} \quad \mu(E) \geqq 1 - \epsilon.$$

Hence $\mathscr{J}(X, L_p)$ is a linear subspace of $\mathscr{L}(X, L_p)$ and contains the finite-rank operators. Suppose $T \in \mathscr{A}(X, L_p)$; we will show that $T \in \mathscr{J}(X, L_p)$. Suppose T_α is a bounded net in $\mathscr{J}(X, L_p)$ so that

$$\lim_\alpha T_\alpha x = Tx \quad \text{for } x \in X.$$

Fix $\epsilon > 0$ and $E_\alpha \in \Sigma$ so that $\mu(E_\alpha) \geqq 1 - \epsilon$ and

$$\|P_{E_\alpha} T_\alpha\|_1 \leqq C\epsilon^{1-1/p}\|T_\alpha\|_p.$$

(using (iii) = (ii) proved above). The net 1_{E_α} in the unit ball of $L_\infty(\Omega, \Sigma, \mu)$ has a weak*-cluster point h with $0 \leqq h \leqq 1$ and

$$\int h d\mu \geqq 1 - \epsilon.$$

Now for each $x \in X$

$$\int_\Omega h|Tx|d\mu \leqq C\epsilon^{1-1/p} \sup_\alpha \|T_\alpha\|_p\|x\|.$$

Let $E = \{\omega:h(\omega) \geqq \frac{1}{2}\}$. Then

$$\mu(E) \geqq 1 - 2\epsilon \quad \text{and} \quad P_E T \in \mathscr{L}(X, L_1)$$

so that $T \in \mathscr{J}(X, L_p)$. It follows that

$$\mathscr{J}(X, L_p) \supset \mathscr{A}(X, L_p)$$

and hence (i) $\Rightarrow$ (iii).

3. Natural spaces and Banach envelopes. Let Y be a quasi-Banach lattice. Y is said to *L-convex* (or *lattice-convex*) **[6]** if there exists $0 < \delta < 1$ so that if $u \geqq 0$ and $0 \leqq x_i \leqq u$ $(1 \leqq i \leqq n)$ satisfy

$$\frac{1}{n}(x_1 + \dots + x_n) \geqq (1 - \delta)u$$

then

$$\max_{1 \leqq i \leqq n} \|x_i\| \geqq \delta\|u\|.$$

As pointed out in **[5]** those function spaces commonly studied in analysis (e.g. Orlicz spaces, Lorentz spaces, etc.) are automatically L-convex; one needs an 'artificial' construction to make non L-convex lattices. Motivated by this idea we shall define a (real) quasi-Banach space X to be *natural* if it is linearly isomorphic to a closed linear subspace of an L-convex lattice. This definition is rendered consistent by Theorem 4.2 of [**6**] which asserts that a natural quasi-Banach lattice is always L-convex.

We now characterize natural quasi-Banach spaces. As noted in the introduction, this theorem can be used to define natural spaces.

THEOREM 3.1. *A quasi-Banach space X is natural if and only if there exists $p > 0$ and $C < \infty$ so that if $x \in X$ with $x \neq 0$ there exists a measure space $(\Omega_x, \Sigma_x, \mu_x)$ with $\mu(\Omega_x) = 1$ and an operator*

$$T_x:X \to L_p(\Omega_x, \Sigma_x, \mu_x)$$

such that

(i) $\|T_x\| \leqq C\|x\|^{-1}$

(ii) $T_x x = 1_{\Omega_x}$.

Proof. For the "if" direction simply note that the map

$$x \mapsto \{ \|y\| T_y x\}_{y \in X, y \neq 0}$$

embeds X into an l^∞-product of spaces $L_p(\Omega_x, \Sigma_x, \mu_x)$ which is L-convex.

The "only if" direction is essentially proved in [**6**] Theorem 3.3. We can suppose that X is a closed linear subspace of an L-convex quasi-Banach lattice Y. By Theorem 2.2 of [**5**] Y is a p-convex lattice for some $p > 0$ i.e., for some constant B and $y_1, \dots, y_n \in Y$

$$\left\| \left(\sum_{i=1}^{n} |y_i|^p \right)^{1/p} \right\| \leqq B \left(\sum_{i=1}^{n} \|y_i\|^p \right)^{1/p}.$$

Fix $x \in X \subset Y$ with $x \neq 0$. Let $v = |x|$ and let V be the linear span of $[-v, v]$. Taking $[-v, v]$ as the unit ball, V is an abstract M-space and then may be isometrically and lattice isomorphically identified with a space $C(\Omega_x)$, where Ω_x is compact Hausdorff. Let $J:C(\Omega_x) \to V$ be the associated lattice isomorphism. Let Σ_x be the Borel subsets of Ω_x.

If $f_1, \dots, f_m \in C(\Omega_x)$ then

$$\|J(\Sigma |f_i|^p)^{1/p}\|^p \leqq B \sum_{i=1}^{n} \|f_i\|^p.$$

Hence it is impossible that

$$\sum_{i=1}^{m} |f_i(s)|^p > B^p \|v\|^{-p} \sum_{i=1}^{n} \|Jf_i\|^p$$

for all $s \in \Omega_x$, since this would imply that

$$J(\Sigma |f_i|^p)^{1/p} \geqq (1 + \epsilon) B \|v\|^{-1} \left(\sum_{i=1}^{n} \|Jf_i\|^p \right)^{1p} v$$

for some $\epsilon > 0$.

Now by a Hahn-Banach separation argument there is a probability measure μ_x on Ω_x so that

$$\int_{\Omega_x} |f|^p d\mu_x \leqq B^p \|v\|^{-p} \|Jf\|^p$$

for $f \in C(\Omega_x)$.

For $y \in Y_+$ define $Sy \in L_p(\Omega_x, \mu_x)$ by

$$Sy = \sup_n J^{-1}(y \wedge nv)$$

and extend by linearity. Then S is a bounded linear operator of Y into $L_p(\Omega_x, \mu_x)$ with $\|S\| \leqq B\|v\|^{-1}$.

Now define $T_x:X \to L_p(\Omega_x, \mu_x)$ by

$$T_x z = (Sx) \cdot Sz.$$

Note that $|Sx| = 1$, since S is a lattice isomorphism and $Sv = S|x| = 1$. Thus

$$\|T_x\| \leqq B\|v\|^{-1} = B\|x\|^{-1} \quad \text{and} \quad T_x x = 1_{\Omega_x}.$$

We now define a complex quasi-Banach space X to be *natural* if the underlying real space is natural. Then we observe that Theorem 3.1 holds also for complex spaces. One constructs the real-linear map T_x as in Theorem 3.1 and then complexifies by setting

$$\tilde{T}_x z = T_x(z) - iT_x(iz)$$

to obtain a complex linear map

$$\tilde{T}_x:X \to L_p(\Omega_x, \Sigma_x, \mu_x).$$

Finally one sets

$$S_x(z) = (\tilde{T}_x(x))^{-1}\tilde{T}_x(z)$$

(noting that $\operatorname{Re} \tilde{T}_x(x) = 1_{\Omega_x}$).

We shall need an observation before proceeding.

LEMMA 3.2. *Suppose X, Y, Z are quasi-Banach spaces and $S:X \to Y$, $T:Y \to Z$ are linear operators. If either S or T is approximable then so is TS.*

Proof. For example if S is approximable set $\mathcal{J} \subset \mathcal{L}(X, Y)$ to be the set of all R so that TR is approximable in $\mathcal{L}(X, Z)$. Then $\mathcal{J}$ contains the finite-rank operators and is closed under pointwise convergence of uniformly bounded nets. Thus $S \in \mathcal{J}$. The other case is similar.

THEOREM 3.3. *Let X and Y be quasi-Banach spaces and suppose X^* has finite cotype and Y is natural. Then there is a constant C so that if $T:X \to Y$*

is an approximable linear operator, then

$$\|Tx\| \leqq C\|T\|\,\|x\|_c \quad x \in X.$$

Proof. There exists B and $p > 0$ so that if $y \in Y$ and $y \neq 0$ there is a bounded linear operator

$$S_y: Y \to L_p(\Omega, \Sigma, \mu)$$

where

$$\mu(\Omega) = 1, \quad \|S_y\| \leqq B\|y\|^{-1} \quad \text{and} \quad S_y y = 1_\Omega.$$

We can use Theorem 2.2 to determine A (depending only on p) so that if

$$R: X \to L_p(\Omega, \Sigma, \mu)$$

then there exists $E \in \Sigma$ with $\mu(E) \geqq 1/2$ and

$$P_E R: X \to L_1(\Omega, \Sigma, \mu) \quad \text{with } \|P_E R\|_1 \leqq A\|R\|_p.$$

Now fix $x \in X$. If $Tx = 0$ there is nothing to prove. If $Tx \neq 0$, there exists

$$S: Y \to L_p(\Omega, \Sigma, \mu)$$

so that

$$\|S\| \leqq B\|Tx\|^{-1} \quad \text{and} \quad STx = 1_\Omega.$$

Choose $E \in \Sigma$ so that $P_E ST$ maps X into $L_1(\Omega, \Sigma, \mu)$,

$$\|P_E ST\|_1 \leqq A\|ST\|_p \quad \text{and} \quad \mu(E) \geqq 1/2.$$

Now

$$\begin{aligned}\frac{1}{2} &\leqq \|P_E STx\|_1 \leqq \|P_E ST\|_1\|x\|_c \\ &\leqq A\|ST\|_p\|x\|_c \\ &\leqq AB\|Tx\|^{-1}\|T\|\,\|x\|_c\end{aligned}$$

so that

$$\|Tx\| \leqq 2AB\|T\|\,\|x\|_c.$$

THEOREM 3.4. *Let Y be a natural quasi-Banach space with a basis (or more generally the identity on Y is approximable). Let X be any non-locally convex subspace of Y. Then the Banach envelope of X contains uniformly complemented l_1^n's (or equivalently X^* does not have finite cotype).*

Proof. The embedding $J: X \to Y$ is approximable and the result follows from Theorem 3.3.

Examples. If X is a subspace of l_p or of H_p $(0 < p < 1)$ then X_c contains uniformly complemented l_1^n's.

As a further example consider the space h_p $(0 < p < 1)$ studied by Shapiro [**12**]. This consists of all complex harmonic functions u on the open disc Δ in $\mathbf{C}$ so that

$$\|u\|_{h_p} = \sup_{0<r<1} \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} |u(re^{i\theta})|^p d\theta \right\}^{1/p} < \infty.$$

Shapiro shows that h_p contains many copies of c_0, and one might suspect that h_p has a non-locally convex subspace with a containing Banach space isomorphic to c_0. This is impossible however; in this case a slightly different proof is needed, since the identity on h_p is not known to be approximable.

THEOREM 3.5. *Let X be a closed linear subspace of h_p. If X is non-locally convex then X_c contains uniformly complemented l_1^n's.*

Proof. Suppose X_c does not contain uniformly complemented l_1^n's, and consider the map $T_r: X \to L_p(\mathbf{T})$ given by

$$T_r u(e^{i\theta}) = u(re^{i\theta}).$$

If $0 < r < 1$, T_r is approximable. In fact

$$T_r u = \lim_{n\to\infty} T_r^{[n]} u$$

where

$$T_r^{[n]} u = \sum_{|k| \leqq n} \hat{u}(k) r^{|k|} e^{ik\theta}$$

where $\{\hat{u}(k)\}_{k=-\infty}^{\infty}$ are the Fourier coefficients of u.

Thus for some constant C independent of r

$$\|T_r u\|_p \leqq C\|u\|_c$$

where $\| \ \|_c$ is the Banach envelope norm on X. Hence

$$\|u\|_{h_p} \leqq C\|u\|_c$$

i.e., X is locally convex.

4. A Banach envelope isomorphic to c_0. In this section we construct a non-locally convex quasi-Banach space with an unconditional basis, whose Banach envelope is isomorphic to c_0. Of course such a space cannot be natural by Theorem 3.4.

We start with an observation used by Talagrand [**16**] to construct a pathological submeasure. For each $n \in \mathbf{N}$ we can find a finite set Ω $(= \Omega_n)$ and a collection of subsets $\{A_1, \dots, A_{2n}\}$ of Ω so that

(i) $\bigcup_{i\in J} A_i \neq \Omega$

whenever $J \subset \{1, 2, \ldots, 2n\}$ and $|J| < n$, and

(ii) $(1_{A_1} + \ldots + 1_{A_{2n}}) = n1_\Omega$.

Indeed let Ω consist of all subsets of $\{1, 2, \ldots, 2n\}$ of cardinality n. Let

$$A_i = \{\omega \in \Omega : i \in \omega\}.$$

Fix p with $0 < p < 1$ and quasi-norm the finite-dimensional space $V_n = l^\infty(\Omega)$ by taking

$$\|f\|^p = \inf\left\{\sum |c_i|^p : |f| \leqq \sum_{i=1}^{2n} c_i 1_{A_i}\right\}.$$

Let

$$\|f\|_\infty = \max_{\omega \in \Omega} |f(\omega)|.$$

Then clearly $\|f\|_\infty \leqq \|f\|$. Suppose $c_i \geqq 0$ and

$$\sum c_i 1_{A_i} \geqq 1_\Omega.$$

Then let $J = \left\{i : c_i \geqq \frac{1}{2}n\right\}$. If $|J| < n$ then we may pick

$$\omega \in \Omega \setminus \bigcup_{i \in J} A_i$$

and then

$$\sum c_i 1_{A_i}(\omega) < 1.$$

Hence $|J| \geqq n$ and so

$$\|1_\Omega\|^p \geqq n \cdot \left(\frac{1}{2n}\right)^p = \left(\frac{1}{2}\right)^p n^{1-p}$$

i.e.,

$$\|1_\Omega\| \geqq \frac{1}{2} n^{1/p-1}.$$

Next we note that for the Banach envelope norm $\| \ \|_{c,n}$

$$\|1_\Omega\|_{c,n} \leqq \frac{1}{n} \sum_{i=1}^{2n} \|1_{A_i}\| \leqq 2.$$

Clearly $\| \ \|_{c,n}$ is a lattice norm on $l^\infty(\Omega)$ and $\| \ \|_{c,n} \geqq \| \ \|_\infty$. Hence for $f \in l^\infty(\Omega)$

$$\|f\|_\infty \leqq \|f\|_c \leqq 2\|f\|_\infty.$$

We repeat this construction for each n and then form the c_0-product $c_0(V_n)$ i.e., the space of all sequences f_n with $f_n \in V_n$, $\|f_n\| \to 0$ and

$$\|(f_n)\| = \max \|f_n\|.$$

The Banach envelope norm on $c_0(V_n)$ is clearly seen to be

$$\|(f_n)\|_c = \max \|f_n\|_{c,n}.$$

Thus the Banach envelope of $c_0(V_n)$ is isomorphic to c_0. However the space is non-locally convex since

$$\|(0, 0, \ldots, 1_{\Omega_n}, 0, \ldots)\| \geqq \frac{1}{2}n^{1/p-1}$$

but

$$\|(0, 0, \ldots, 1_{\Omega_n}, 0, \ldots)\|_c \leqq 2.$$

5. Subspaces of l_p. We shall need the following simple lemma.

LEMMA 5.1. *Let $(u_n:n \in \mathbf{N})$ be a normalized sequence in l_p $(0 < p < 1)$ with disjoint supports. Let $M = [u_n]$ be its closed linear span. Then*

(i) *l_p/M is linearly isomorphic to a subspace of l_p*

(ii) *The Banach envelope of l_p/M is isomorphic to l_1.*

Proof. We first note that (ii) is essentially trivial. The Banach envelope of l_p/M is isomorphic to $l_1/\hat{M}$ where $\hat{M}$ is the closure of M in l_1. However $\hat{M}$ is a complemented subspace of l_1 (by a projection of norm one) and $l_1/\hat{M} \cong l_1$.

To prove (i) first let (e_i) be the standard basis of l_p and

$$u_n = \sum_{i \in A_n} a_i e_i$$

where $|a_i| > 0$ for all $i \in A_n$ and

$$\sum_{i \in A_n} |a_i|^p = 1 \quad n = 1, 2, \ldots .$$

We shall define a linear map

$$S:l_p \to l_p(\mathbf{N} \times \mathbf{N}).$$

Let f_{ij} be the standard basis of $l_p(\mathbf{N} \times \mathbf{N})$ and define S so that

$$Se_i = f_{ii} \quad \text{if } i \notin \bigcup_{n \in \mathbf{N}} A_n$$

$$Se_i = \sum_{j \in A_n} a_j f_{ij} - \sum_{j \in A_n} a_j f_{ji} \quad i \in A_n.$$

Then S extends to a bounded linear operator with $\|S\|^p \leqq 2$. Note that $Su_n = 0$ for each n. We shall show that

$$\inf_{v \in M} \|x - v\| = d(x, M) \leqq \|Sx\|$$

for all $x \in l_p$.

Indeed if

$$x = \sum_{i=1}^{\infty} \xi_i e_i$$

in l_p then we may select for each $n \in \mathbf{N}$, η_n so that

$$\left\| \sum_{i \in A_n} (\xi_i - \eta_n a_i) e_i \right\| \leqq \left\| \sum_{i \in A_n} \xi_i e_i - \lambda u_n \right\|$$

for all $\lambda \in \mathbf{R}$.

Thus

$$\sum_{i \in A_n} |\xi_i a_j - \eta_n a_i a_j|^p \leqq \sum_{i \in A_n} |\xi_i a_j - \xi_j a_i|^p.$$

Hence

$$\sum_{n=1}^{\infty} \sum_{j \in A_n} \sum_{i \in A_n} |\xi_i a_j - \eta_n a_i a_j|^p \leqq \sum_{n=1}^{\infty} \sum_{j \in A_n} \sum_{i \in A_n} |\xi_i a_j - \xi_j a_i|^p.$$

Since the right-hand side is at most $\|Sx\|^p$ we conclude that $\sum |\eta_n|^p < \infty$ and if we set $y = \sum \eta_n u_n$ then

$$\|x - y\|^p \leqq \sum_{i \notin \cup A_n} |\xi_i|^p + \sum_{n=1}^{\infty} \sum_{j \in A_n} \sum_{i \in A_n} |\xi_i a_j - \xi_i a_i|^p$$

$$\leqq \|Sx\|^p.$$

Thus $d(x, M) \leqq \|Sx\|$ and S factors to an embedding of l_p/M into

$l_p(\mathbf{N} \times \mathbf{N}) \cong l_p$.

THEOREM 5.2. *Let Z be a separable Banach space and suppose* $0 < p < 1$. *Then there is a closed subspace X of l_p whose Banach envelope is linearly isomorphic to $l_1 \oplus Z$.*

Proof. Let (e_i) be the standard basis of $l_p = l_p(\mathbf{N})$ and let (f_{ij}) be the standard basis of $l_p(\mathbf{N} \times \mathbf{N})$ and, for convenience, we take the quasi-norm on the direct sum to be

$$\|(x, y)\| = \max(\|x\|, \|y\|).$$

Choose an integer α so that

$$\alpha\left(\frac{1}{p} - 1\right) > 3 \quad \text{and} \quad d = 2^{\alpha}.$$

Write $\mathbf{N}$ as a disjoint union $\bigcup_{n\in\mathbf{N}} A_n$ where $|A_n| = d^n$, and define the normalized sequence

$$u_n = d^{-n/p} \sum_{i\in A_n} e_i.$$

Select a total fundamental biorthogonal system (z_n, z_n^*) for Z so that $\|z_n\| = 1$ ($n \in \mathbf{N}$) and $\|z_n^*\| \leqq 2$ for $n \in \mathbf{N}$ (see **[11]**). We define $K:[u_n] \to Z$ to be a bounded linear map such that

$$Ku_n = 2^{-n}z_n.$$

Note that $\|K\| \leqq 1$ and that K is one-one. In fact if $x \in [u_n]$, say

$$x = \sum_{n=1}^{\infty} \xi_n u_n$$

then

$$|z_n^*(Kx)| = 2^{-n}|\xi_n| \leqq 2\|Kx\|$$

so that

$$|\xi_n| \leqq 2^{n+1}\|Kx\|.$$

Now let $\| \ \|_1$ be the l_1-norm on $l_p(\mathbf{N})$. Then

$$\begin{aligned}\|x\|_1 &= \sum_{n=1}^{\infty} |\xi_n| d^{n(1-1/p)} \\ &\leqq 2\|Kx\| \sum_{n=1}^{\infty} 2^n \cdot 2^{\alpha n(1-(1/p))} \\ &\leqq 2\|Kx\| \sum_{n=1}^{\infty} 4^{-n} \\ &\leqq \frac{2}{3}\|Kx\|.\end{aligned}$$

Now let $(y_i:i \in \mathbf{N})$ be any sequence dense in the unit sphere $\{x:\|x\| = 1\}$ of $[u_n]$. Then

$$y_i = \sum_{j=1}^{\infty} a_{ij}e_j$$

where

$$\sum_i |a_{ij}|^p = 1.$$

Let

$$v_i = \sum_{j=1}^{\infty} a_{ij} f_{ij}$$

in $l_p(\mathbf{N} \times \mathbf{N})$ and define for $(i, j) \in \mathbf{N} \times \mathbf{N}$

$$w_{ij} = f_{ij} + \beta_i e_j$$

where

$$\beta_i = \|y_i\|_1 / \|Ky_i\|.$$

Note $0 < \beta_i \leqq 2/3$ for all $i \in \mathbf{N}$ and hence that $[w_{ij}]$ is isometric to l_p.

Let $M = [v_n]$, and consider the quotient $l_p \oplus l_p(\mathbf{N} \times \mathbf{N})/M$. By Lemma 5.1 this quotient is isomorphic to a subspace of l_p. Let q be the quotient mapping and X be the closed linear span of

$$\{q(w_{ij}): i \in \mathbf{N}, j \in \mathbf{N}\}.$$

X will be our example; clearly X embeds into l_p.

Note first that for fixed i

$$\sum_j a_{ij} w_{ij} = \sum_j a_{ij}(f_{ij} + \beta_i e_j) = v_i + \beta_i y_i.$$

Thus X contains a subspace

$$X_0 = [q(y_n)] = [q(u_n)].$$

We first identify the quotient space X/X_0. To do this note that

$$q^{-1}(X_0) = [u_n] \oplus [y_n] \subset l_p \oplus l_p(\mathbf{N}^2).$$

Now suppose

$$x = \sum_{i,j} \xi_{ij} w_{ij} \in [w_{ij}]$$

and that for some c_n, $\sum |c_n|^p < \infty$

$$\left\| \sum_{i,j} \xi_{ij} f_{ij} - \sum_n c_n v_n \right\| < 1.$$

We shall show that

$$\left\| \sum_{i,j} \xi_{ij} w_{ij} - \sum_n c_n (v_n + \beta_n y_n) \right\| < 1.$$

Indeed we have

$$\sum_{i,j} |\xi_{ij} - c_i a_{ij}|^p < 1$$

and hence

$$\left\|\sum_{i,j} (\xi_{ij} - c_i a_{ij})\beta_i e_j\right\| < 1$$

i.e.,

$$\left\|\sum_{i,j} \xi_{ij}\beta_i e_j - \sum_n c_n\beta_n y_n\right\| < 1$$

as required.

Now suppose $x \in [w_{ij}]$ and $d(x, [u_n] \oplus [v_n]) < 1$. Then by the above

$$d(x, [v_n + \beta_n y_n]) < 1.$$

However $v_n + \beta_n y_n$ is a sequence with disjoint support with respect to (w_{ij}). Hence $[w_{ij}]/[v_n + \beta_n y_n]$ has a Banach envelope isomorphic to l_1. It follows that X/X_0 is isomorphic to $[w_{ij}]/[v_n + \beta_n y_n]$ and hence embeds into l_p and has a Banach envelope isomorphic to l_1. X_0 is then weakly closed in X (X/X_0 has a separating dual).

Let $\tilde{X}_0$ be the weak closure of X_0 in X_c. Then

$$X_c/\tilde{X}_0 \cong (X/X_0)_c \cong l_1,$$

and hence

$$X_c \cong l_1 \oplus \tilde{X}_0.$$

It remains to show that $\tilde{X}_0 \cong Z$.

To do this we define a bounded linear map $L_0:[w_{ij}] \to Z$ so that

$$L_0(w_{ij}) = (\operatorname{sgn} a_{ij})\frac{\beta i}{\|y_i\|_1}Ky_i.$$

To see L is bounded we need only note that

$$\|L_0(w_{ij})\| = \frac{\beta_i}{\|y_i\|_1}\|Ky_i\| = 1$$

for all i, j.

Suppose

$$x = \sum \xi_{ij}w_{ij} \in [w_{ij}] \quad \text{and} \quad d(x, M) < 1.$$

Then there exist c_i, $\sum |c_i|^p < 1$ so that

$$\|x - \sum c_i v_i\| < 1.$$

Hence

$$\left\|\sum_{i,j} \xi_{ij}f_{ij} - \sum c_i v_i\right\| < 1$$

and, as before,

$$\|x - \sum c_i(v_i + \beta_i y_i)\| < 1.$$

Thus

$$\|\sum c_i\beta_i K y_i\| < 2^{1/p}.$$

However

$$\begin{aligned} L_0(v_i + \beta_i y_i) &= \sum_j a_{ij} L_0(w_{ij}) \\ &= \left(\sum_j |a_{ij}|\right)\frac{\beta_i}{\|y_i\|_1} K y_i \\ &= \beta_i K y_i \end{aligned}$$

since $\|y_i\|_1 = \sum_j |a_{ij}|$.

Thus

$$\|L_0(\sum c_i(v_i + \beta_i y_i))\| < 2^{1/p}$$

and hence

$$\|L_0 x\| < 3^{1/p}.$$

In general for $x \in [w_{ij}]$,

$$\|L_0(x)\| \leqq 3^{1/p} \cdot d(x, M).$$

Hence L_0 factors to a linear operator $L:X \to Z$ with $\|L\| \leqq 3^{1/p}$. As Z is a Banach space

$$\|Lx\| \leqq 3^{1/p}\|x\|_c.$$

Now

$$\begin{aligned} Lq(y_i) &= Lq(y_i + \beta_i^{-1} v_i) \\ &= L_0(y_i + \beta_i^{-1} v_i) = K y_i. \end{aligned}$$

Hence

$$\|Kqy_i\| \leqq 3^{1/p}\|q(y_i)\|_c.$$

Extending by continuity, if $x \in [u_n]$

$$\|Kx\| \leqq 3^{1/p}\|q(x)\|_c.$$

Conversely

$$\begin{aligned} \|q(y_i)\|_c &= \beta_i^{-1}\|q(v_i + \beta_i y_i)\|_c \\ &\leqq \beta_i^{-1} \sum_j |a_{ij}| \, \|q(w_{ij})\|_c \\ &\leqq \beta_i^{-1}\|y_i\|_1 = \|K y_i\|. \end{aligned}$$

Hence for $x \in [u_n]$

$$\|q(x)\|_c \leqq \|Kx\|.$$

Thus

$$3^{-1/p}\|Kx\| \leqq \|q(x)\|_c \leqq \|Kx\| \quad \text{for } x \in X_0$$

so that $\widetilde{X}_0 \cong Z$ (since the range of K is dense).

6. Spaces with unconditional bases. In this section we give some special results for spaces with unconditional bases.

THEOREM 6.1. *Let X be a quasi-Banach space with an unconditional basis. Then X contains a complemented subspace isomorphic to c_0 if and only if X_c contains a copy of c_0.*

Proof. Suppose (e_n) is a normalized unconditional basis of X with unconditional basis constant K so that if $N \in \mathbf{N}$, $a_1, \dots, a_n \in \mathbf{R}$ and $|c_i| \leqq 1$ $(1 \leqq i \leqq N)$ then

$$\left\|\sum_{i=1}^{N} c_i a_i e_i\right\| \leqq K\left\|\sum_{i=1}^{N} a_i e_i\right\|.$$

Let e_n^* be the biorthogonal functionals in X^*. Then

$$\|c_n^*\| \leqq K \quad \text{and} \quad \|e_n\|_c \geqq K^{-1}.$$

Suppose X_c contains a copy of c_0. Then by a gliding hump argument we can find a block basic sequence (u_n) of the form

$$u_n = \sum_{k_{n-1}+1}^{k_n} b_i e_i$$

(where $k_0 = 0 < k_1 < k_2 \dots$) and so that for some $0 < C < \infty$ and all $a_1, \dots a_n \in \mathbf{R}$, $N \in \mathbf{N}$ we have

$$C^{-1} \max_{1 \leqq i \leqq N} |a_i| \leqq \left\|\sum_{i=1}^{n} a_i u_i\right\|_c \leqq C \max_{1 \leqq i \leqq N} |a_i|.$$

We may also determine

$$u_n^* = \sum_{k_{n-1}+1}^{k_n} c_i e_i^*$$

so that

$$\sup \|u_n^*\| = \beta < \infty \quad \text{and} \quad u_n^*(u_n) = 1.$$

Let $\Delta = [-2CK, 2CK]^{\mathbf{N}}$, Δ is a compact metrizable space. For $\xi = (\xi_i) \in \Delta$ define $\phi_n(\xi) \in X$ by

$$\phi_n(\xi) = \sum_{i=k_{n-1}+1}^{k_n} \xi_i e_i.$$

Let

$$\Delta_0 = \{\xi \in \Delta : \|\phi_1(\xi) + \ldots + \phi_n(\xi)\| \leqq 2CK, \ n \in \mathbf{N}\}.$$

Δ_0 is a closed subset of Δ.

For each $n \in \mathbf{N}$,

$$\|u_1 + \ldots + u_n\|_c \leqq C$$

and so we can express $u_1 + \ldots + u_n$ as a convex combination of elements of the unit ball of X. Identify $x \in X$ such that $\|x\| \leqq 1$ with $\xi \in \Delta$ so that $x = \sum \xi_i e_i$. Thus we can find a probability measure μ_n on Δ whose support is finite so that if $1 \leqq m \leqq n$

$$\int \phi_m(\xi) d\mu_n(\xi) = u_m.$$

(The integral is well-defined in the finite-dimensional space $[e_{k_{m-1}+1}, \ldots e_{k_m}]$). Furthermore μ_n is supported on Δ_0.

Let μ be any weak*-cluster point of $(\mu_n : n \in \mathbf{N})$ in $C(\Delta)^*$. μ is also supported on Δ_0 and

$$\int \phi_m(\xi) d\mu(\xi) = u_m \quad m \in \mathbf{N}.$$

Thus

$$\int u_m^*(\phi_m(\xi)) d\mu(\xi) = 1$$

and as

$$|u_m^*(\phi_m(\xi))| \leqq 2BCK,$$

we can conclude by the Bounded Convergence Theorem that there exists $m_k \uparrow \infty$ and $\xi \in \Delta_0$ so that

$$u_{m_k}^*(\phi_{m_k}(\xi)) = \beta_k > \gamma > 0.$$

Let

$$v_k = \beta_k^{-1} \phi_{m_k}(\xi).$$

Then v_k is an unconditional basic sequence in X.

We also have

$$\|a_1 v_1 + \ldots + a_k v_k\| \leqq \gamma^{-1} K \max |a_k| \left\| \sum_{i=1}^{m_k} \phi_i(\xi) \right\|$$

$$\leqq 2CK^2 \gamma^{-1} \max |a_k|.$$

Now $u^*_{m_k}(v_k) = 1$ so that $\|v_k\| \geqq B^{-1}$ so that (v_k) is equivalent to the usual c_0-basis and is complemented by the projection

$$Px = \sum_{k=1}^{\infty} u^*_{m_k}(x)v_k.$$

In fact

$$\|P\| \leqq 2BCK^2\gamma^{-1}.$$

The converse is trivial; if $X \cong c_0 \oplus Y$ then $X_c \cong c_0 \oplus Y_c$.

If we further assume $X_c \cong c_0$ then since (e_n) is also an unconditional basis for X_c with

$$K^{-1} \leqq \|e_n\|_c \leqq 1,$$

then by a theorem of Lindenstrauss and Pelczynski [7] (e_n) is actually equivalent to the usual c_0-basis in X_c. Thus in the preceding proof we can take $u_n = e_n$, and each v_k is a multiple of e_{m_k}. Thus we have also proved:

THEOREM 6.2. *Let X be a quasi-Banach space with normalized unconditional basis (e_n). If $X_c \cong c_0$, then some subsequence of (e_n) is equivalent (in X) to the standard basis of c_0.*

COROLLARY 6.3. *Let X be a quasi-Banach space with a symmetric basis. If $X_c \cong c_0$, then $X \cong c_0$.*

We remark that the example constructed in Section 4 has an unconditional basis and $X_c \cong c_0$, but is not locally convex. We also remark that it is easy to find spaces X with unconditional basis which contain c_0, but so that X_c does not contain c_0. For example let X be the "weak" l^p-sequence space $(0 < p < 1)$ of all sequences x_n so that the decreasing rearrangement x_n^* satisfies $n^{1/p}x_n^* \to 0$. This is quasi-normed by

$$\|x\| = \sup_n n^{1/p}x_n^*.$$

Then X has an unconditional basis and contains c_0, but $X_c \cong l_1$.

Finally we conclude with a theorem suggested by the example in Section 5.

THEOREM 6.4. *Let X be a quasi-Banach space with an unconditional basis. Suppose $X_c \cong l_1 \oplus Y$. Then*

(i) *If Y does not contain l_1^n's uniformly, then X contains a complemented subspace isomorphic to Y.*

(ii) *If Y does not contain uniformly complemented l_1^n's and X is natural, then X contains a complemented copy of Y.*

Proof. In either case every operator from Y into l_1 is compact. Let (e_n) be the unconditional basis of X. Then (e_n) is also an unconditional basis of $X_c \cong l_1 \oplus Y$ and also by a theorem of Wojtasczyk [**17**] there is an infinite subset $[e_n:n \in M]$ which is an unconditional basis of Y. Then $[e_n:n \in M]$ spans a subspace X_0 of X which is complemented and such that $(X_0)_c \cong Y$. Now apply either Theorem 1.1. or Theorem 3.4, to conclude that $X_0 \cong Y$.

Thus the examples in Theorem 6.2 do not have an unconditional basis in the case when Z does not contain uniformly complemented l_1^n's.

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