Can. J. Math., Vol. XXV, No. 3, 1973, pp. 511-524

INCLUSION THEOREMS FOR K-SPACES

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1. Notation and preliminary ideas. A *sequence space* is a vector subspace of the space ω of all real (or complex) sequences. A sequence space E with a locally convex topology τ is called a *K-space* if the inclusion map $E \to \omega$ is continuous, when ω is endowed with the product topology ($\omega = \prod_{i=1}^{\infty} (\mathbf{R})_i$). A K-space E with a Frechet (i.e., complete, metrizable and locally convex) topology is called an *FK-space*; if the topology is a Banach topology, then E is called a *BK-space*. The following familiar BK-spaces will be important in the sequel:

m, the space of all bounded sequences;
c, the space of all convergent sequences;
c_0, the space of all null sequences;
l^p, $1 \leqq p < \infty$, the space of all absolutely p-summable sequences.

We shall also consider the space ϕ of all finite sequences and the linear span, m_0, of all sequences of zeros and ones; these provide examples of spaces having no FK-topology.

A sequence space is *solid* (respectively *monotone*) provided that $xy \in E$ whenever $x \in m$ (respectively m_0) and $y \in E$. For $x \in \omega$, we denote by $P_n(x)$ the sequence $(x_1, x_2, \ldots x_n, 0, \ldots)$; if a K-space (E, τ) has the property that $P_n(x) \to x$ in τ for each $x \in E$, then we say that (E, τ) is an *AK-space*. We also write

$$W_E = \{x \in E: P_n(x) \to x \text{ weakly}\}$$

and

$$S_E = \{x \in E: P_n(x) \to x \text{ in } \tau\}.$$

One of the most natural ways of defining K-space topologies is by considering dual pairs of sequence spaces. If E is a sequence space, we write

$$E^\beta = \left\{y \in \omega: \sum_{j=1}^{\infty} x_j y_j \text{ converges, for each } x \in E\right\}$$

so that, if F is a vector subspace of E^β, E and F form a dual pair under the natural bilinear form

$$\langle x, y\rangle = \sum_{j=1}^{\infty} x_j y_j.$$

We may then consider topologies of this dual pair, for example, the weak topology $\sigma(E, F)$ and the Mackey topology $\tau(E, F)$. (We follow the notation

Received February 2, 1972.

of Schaefer [**23**].) If $F \supseteq \phi$, then E is a K-space in any $\langle E, F\rangle$ dual topology; topologies of this type have been considered in some detail by Garling [**6**; **7**].

We shall also mention the *α-dual* of a sequence space E, defined by

$$E^\alpha = \left\{ y \in \omega : \sum_{j=1}^{\infty} |x_j y_j| < \infty, \text{ for each } x \in E \right\}.$$

It is clear that E is always contained in $E^{\alpha\alpha} = (E^\alpha)^\alpha$; if $E = E^{\alpha\alpha}$, then E is called a *perfect sequence space.*

In this paper, we investigate the properties of a K-space E by considering various properties of the inclusion maps $E \to F$, where F is an arbitrary FK-space. In particular, we give criteria for E to be barrelled and to have a weakly sequentially complete dual space. We also study some inclusion properties for summability domains.

2. Barrelled spaces.

PROPOSITION 1. *Let E be a Banach (respectively Frechet) space and let E_0 be a dense linear subspace of E; then the following statements are equivalent:*

(i) *E_0 is barrelled;*

(ii) *if F is a Banach (respectively Frechet) space and $T : F \to E$ is a continuous linear map with $T(F) \supseteq E_0$, then $T(F) = E$.*

Proof. We shall give the proof only in the case when E is a Banach space; the same method then gives the corresponding result for Frechet spaces.

(i) $\Rightarrow$ (ii). Let $N = T^{-1}(\{0\})$ and consider the induced injective map

$$S : F/N \to E.$$

Then $S(F/N) = T(F) \supseteq E_0$, so that there is an inverse map $R : E_0 \to F/N$. We now apply the closed graph theorem for barrelled spaces [**21**, p. 116] to deduce that, since R has closed graph (S is continuous), R is continuous. If $x \in E$, there exists a sequence $\{x_n\}_{n=1}^{\infty}$ of elements of E_0 with $x_n \to x$. $\{Rx_n\}_{n=1}^{\infty}$ is then a Cauchy sequence in F/N and so, since F/N is complete, there exists $y \in F/N$ with $Rx_n \to y$. It follows that

$$x_n = SRx_n \to Sy$$

so that $x = Sy$ and $T(F) = S(F/N) = E$.

(ii) $\Rightarrow$ (i). We use Mahowald's theorem [**17**, Theorem 2.2]; let G be a Banach space and suppose that $T : E_0 \to G$ has closed graph. Define a new norm on E_0 by

$$|||x||| = ||x|| + ||Tx||,$$

so that $|||x||| \geqq ||x||$. Let F be any completion of $(E_0, |||\cdot|||)$ and let J be the unique continuous extension of the identity map

$$(E_0, |||\cdot|||) \to (E_0, ||\cdot||)$$

where $J : F \to E$. By hypothesis, since $J(F) \supseteq E_0$, we have $J(F) = E$.

We next show that J is injective: let $x \in F$ be such that $Jx = 0$. There exists a sequence $\{x_n\}_{n=1}^{\infty}$ of elements of E_0 with

$$|||x - x_n||| \to 0.$$

It follows that $\{x_n\}_{n=1}^{\infty}$ is Cauchy in E and $\{Tx_n\}_{n=1}^{\infty}$ is Cauchy in G; but $Jx = 0$ so that $||x_n|| \to 0$, and, since T has closed graph, $||Tx_n|| \to 0$. Thus

$$|||x_n||| \to 0,$$

giving $x = 0$.

Thus J is bijective; by the closed graph theorem, J^{-1} is continuous and we have, for $x \in E_0$,

$$|||x||| \leqq ||J^{-1}|| \, ||x||,$$

i.e.,

$$||Tx|| \leqq (||J^{-1}|| - 1)||x||,$$

so that T is continuous. We now apply Mahowald's theorem to deduce that E_0 is barrelled.

As a direct consequence of a result of Seever [**24**, Theorem 3.3] we have

COROLLARY. *m_0 is a barrelled subspace of m.*

Most barrelled spaces occurring in analysis are known to be barrelled because for some formally stronger property (for example, second category), or because of some permance result concerning barrelled spaces. In view of this, the following observation should be of some interest to vector space pathologists:

m_0 is of the first category.

To see this, let m_n denote the space of all sequences taking only n values; then m_n is closed in m, has empty interior, and $\bigcup_{n=1}^{\infty} m_n = m_0$.

THEOREM 1. *Let E be a BK-(respectively FK-) space and let E_0 be a dense subspace of E; then the following statements are equivalent:*

(i) *E_0 is barrelled;*

(ii) *if $E_0 \subseteq F$, where F is a BK-(respectively FK-space), then $E \subseteq F$;*

(iii) *if $E_0 \subseteq F \subseteq E$, where F is a BK-(respectively FK-) space, then $F = E$.*

Proof. We again restrict attention to the Banach space case.

(i) $\Rightarrow$ (ii). By the closed graph theorem for barrelled spaces [**21**, p. 116] the inclusion mapping $E_0 \to F$ is continuous. Now if $x \in E$, there exists a sequence $\{x_n\}_{n=1}^{\infty}$ of elements of E_0 such that $x^n \to x$. $\{x^n\}_{n=1}^{\infty}$ is then a Cauchy sequence in F and so $x^n \to y$ (say) in F. However, for every j, we have

$$y_j = \lim_{n \to \infty} x_j^n = x_j$$

so that $x = y \in F$.

(ii) $\Rightarrow$ (iii). This is immediate.

(iii) $\Rightarrow$ (i). We use Proposition 1; suppose that G is a Banach space and T

is a continuous linear mapping from G into E with $T(G) \supseteq E_0$. Now $T(G)$ may be identified with the quotient space $G/T^{-1}(\{0\})$, and it is easy to check that $T(G)$, with the quotient topology, becomes a BK-space. Then $E_0 \subseteq T(G) \subseteq E$ so that $T(G) = E$ and the result follows.

COROLLARY. *An FK-space contains m if and only if it contains* m_0.

This last result has several interesting applications, and we begin by showing how it may be applied to complex function theory.

For $1 \leqq p \leqq \infty$, H^p denotes the usual Hardy class (see [9]). A sequence of points $\{z_n\}_{n=1}^{\infty}$ in the disk $\{z : |z| < 1\}$ is said to be *p-interpolating* if

(*) for each $w = \{w_n\}_{n=1}^{\infty} \in m$, there exists $f \in H^p$ such that
$$f(z_n) = w_n \text{ for } n = 1, 2, \ldots .$$

Hayman [8] (see also [25]) has shown that, for $p = \infty$, a sequence $\{z_n\}_{n=1}^{\infty}$ is p-interpolating if and only if (*) holds for each $w \in m_0$. Following Snyder [26], we denote by $H^p(\{z_n\})$ the set
$$\{\{f(z_n)\}_{n=1}^{\infty} : f \in H^p\}.$$
It is clear that $H^p(\{z_n\})$ can be identified with
$$H^p/\{f \in H^p : f(z_n) = 0, n = 1, 2, \ldots\},$$
and that $H^p(\{z_n\})$ becomes a BK-space when endowed with the quotient topology [26, Theorem 3.1(i)]. As a direct consequence of the corollary to Theorem 1 we now have

THEOREM 2. *For* $1 \leqq p \leqq \infty$, *a sequence* $\{z_n\}_{n=1}^{\infty}$ *is p-interpolating if and only if* (*) *holds for each* $w \in m_0$.

Following the proof of Theorem 1 of [1] we obtain the following result, which settles a conjecture of Kadec and Pelczynski [10]. (See [5] for notation.)

THEOREM 3. *Let E be a separable Frechet space with a total biorthogonal sequence* $\{\langle x_i, f_i\rangle\}_{i=1}^{\infty}$; *then the following conditions are equivalent for each* $x \in E$:

(i) $\sum_{i=1}^{\infty} f_i(x)x_i$ *converges unconditionally to x;*

(ii) *given* $\lambda \in m_0$, *there exists* $y \in E$ *such that* $f_i(y) = \lambda_i f_i(x)$, $i = 1, 2, \ldots$.

We note that Theorem 3 has been obtained for Banach spaces by Bachelis and Rosenthal in [1, Theorem 1]. A different proof of Theorem 3, not involving the results of Seever, is given in [5, Theorem 26].

Further applications of the corollary to Theorem 1 are postponed until Section 4.

3. Inclusion theorems for $(E, \tau(E, E^\beta))$.

In this section we consider a sequence space E endowed with the associated Mackey topology $\tau(E, E^\beta)$. $A = \{a_{ij}\}_{i,j=1}^{\infty}$ denotes an infinite matrix with complex entries; for $x \in \omega$, we define $y = Ax$ where

$$y_i = \sum_{j=1}^{\infty} a_{ij}x_j \quad (i = 1, 2, \ldots),$$

whenever each sum converges. If F is an FK-space, then the summability domain F_A, defined by

$$F_A = \{x \in \omega \colon Ax \text{ exists and } Ax \in F\},$$

can be topologized in a natural way so that it too becomes an FK-space [**29**, Theorem 4.10(a)].

THEOREM 4. *The following conditions are equivalent for any sequence space E containing ϕ:*

(i) *$(E, \tau(E, E^\beta))$ is barrelled;*

(ii) *if F is an FK-space and A is a matrix mapping E into F, then $A : (E, \tau(E, E^\beta)) \to F$ is continuous;*

(iii) *if F is an FK-space containing E, then the inclusion mapping $(E, \tau(E, E^\beta)) \to F$ is continuous;*

(iv) *if F is an FK-space containing E, then $E \subseteq W_F$;*

(v) *if F is an FK-space containing E, then $E \subseteq S_F$.*

Proof. We shall prove the theorem according to the logical scheme:

$$\text{(i)} \Rightarrow \text{(ii)} \Rightarrow \text{(iii)} \Rightarrow \text{(iv)} \Rightarrow \text{(i)}; \ \text{(i)} \wedge \text{(iii)} \Rightarrow \text{(v)} \Rightarrow \text{(iv)}.$$

(i) $\Rightarrow$ (ii). If A maps E into F, the rows of A belong to E^β and it is easy to check that $A : (E, \tau(E, E^\beta)) \to F$ has closed graph; the continuity of A then follows from the closed graph theorem for barrelled spaces [**21**, p. 116].

(ii) $\Rightarrow$ (iii). This follows at once by taking A to be the identity matrix in (ii).

(iii) $\Rightarrow$ (iv). If (iii) holds and F is an FK-space containing E, then the inclusion mapping

$$(E, \sigma(E, E^\beta)) \to (F, \sigma(F, F'))$$

must be continuous [**21**, p. 39]; (iv) then follows since $(E, \sigma(E, E^\beta))$ is an AK-space.

(iv) $\Rightarrow$ (i). We first show that if (iv) holds, then every $\sigma(E^\beta, E)$ - bounded and closed subset of E^β is $\sigma(E^\beta, E)$ - sequentially compact. To see this, let K be $\sigma(E^\beta, E)$ - bounded and closed and let $\{a^{(i)}\}_{i=1}^{\infty}$ be a sequence of elements of K. By selecting a subsequence, if necessary, we may assume that

$$(*) \qquad \lim_{i\to\infty} a_j^{(i)} = a_j \text{ exists for } j = 1, 2, \ldots .$$

Putting

$$a_j^{(i)} = a_{ij} \quad (i, j = 1, 2, \ldots),$$

we have $E \subseteq m_A$, and so, for $x \in E$, (iv) gives

$$P_n(x) \to x \text{ weakly in } m_A.$$

Now $A : m_A \to m$ is continuous [**29**, Theorem 4.4(c)] and therefore weakly continuous [**21**, p. 39] so that

$$AP_n(x) \to Ax \text{ weakly in } c.$$

Consequently

$$\lim_{n\to\infty} \lim_{i\to\infty} (AP_n(x))_i = \lim_{i\to\infty} \lim_{n\to\infty} (AP_n(x))_i$$

so that $\sum_{j=1}^{\infty} a_j x_j$ exists and

$$\sum_{j=1}^{\infty} a_j x_j = \lim_{i\to\infty} \sum_{j=1}^{\infty} a_j^{(i)} x_j.$$

$x \in E$ being arbitrary, it follows that $a \in E^\beta$ and that

$$a^{(i)} \to a \ \sigma(E^\beta, E).$$

Since K is $\sigma(E^\beta, E)$ - closed, $a \in K$, and K is $\sigma(E^\beta, E)$ - sequentially compact. By [**6**, Theorem 6], K must also be $\sigma(E^\beta, E)$ - compact; consequently $\tau(E, E^\beta) = \beta(E, E^\beta)$, the strong topology on E so that $(E, \tau(E, E^\beta))$ is barrelled by [**21**, p. 66].

(i) $\wedge$ (iii) $\Rightarrow$ (v). If (i) holds, it is any easy consequence of a general result on Schauder bases in barrelled spaces [**4**, p. 505] that $(E, \tau(E, E^\beta))$ is an AK-space, but then (v) follows at once from (iii).

(v) $\Rightarrow$ (iv) This is obvious.

Theorem 4 applies to the class of all FK-AK-spaces, but the results obtained are all trivial in this case. Our next task is to augment this class and to show that Theorem 4 does have some interesting applications.

Garling [**6**; **7**] introduced the notion of B-invariance, where B denotes the unit ball of the BK-space

$$bv = \left\{x \in \omega : \sum_{j=1}^{\infty} |x_j - x_{j+1}| < \infty\right\}$$

under the norm

$$||x||_{bv} = \sum_{j=1}^{\infty} |x_j - x_{j+1}| + \lim_{j\to\infty} |x_j|.$$

For a subset E of ω we define

$$B(E) = \{\lambda x = \{\lambda_j x_j\}_{j=1}^{\infty} : \lambda \in B, x \in E\}.$$

Of course, $B(E)$ need not be a sequence space – even when E is – so we denote by $B^*(E)$ the linear span of $B(E)$. If $B(E) = E$, we say that E is *B-invariant.* Similarly we define the concept of B_0-invariance, where B_0 denotes the unit ball of the BK-space $bv_0 = bv \cap c_0$ under the norm $|| \cdot ||_{bv}$. As a corollary to Theorem 4 we obtain the following result due to Garling [**7**, Theorem 11].

COROLLARY. *If E is a B_0-invariant sequence space containing ϕ, then $(E, \tau(E, E^\beta))$ is a barrelled AK-space and E^β is $\sigma(E^\beta, E)$-sequentially complete.*

Proof. Let F be an FK-space containing E, and consider a fixed element x of E. Since $B_0(E) = E$, it follows that x has a representation $\lambda \cdot y$ where $\lambda \in B_0$ and $y \in E$. The mapping

$$\lambda \to \lambda \cdot y,$$

from bv_0 into F, is continuous by the closed graph theorem, so that, since bv_0 is an AK-space,

$$P_n(x) = P_n(\lambda \cdot y) = P_n(\lambda) \cdot y \to \lambda \cdot y = x$$

in F. It follows that $E \subseteq S_F$ and then, from Theorem 4, (v) $\Rightarrow$ (i), that $(E, \tau(E, E^\beta))$ is barrelled. That $(E, \tau(E, E^\beta))$ is an AK-space follows as in the proof of Theorem 4, (i) $\wedge$ (iii) $\Rightarrow$ (v). The last part is a consequence of a general result, viz: the dual of a barrelled space is weakly sequentially complete.

Our next result characterizes sequence spaces E for which E^β is $\sigma(E^\beta, E)$-sequentially complete. We recall that any solid or monotone sequence space has this property (see, for example, [**3**, Proposition 3 and Lemma 3]).

THEOREM 5. *The following conditions are equivalent for any sequence space E containing ϕ:*

(i) *E^β is $\sigma(E^\beta, E)$-sequentially complete;*

(ii) *if F is a separable FK-space and A is a matrix mapping E into F, then $A : (E, \tau(E, E^\beta)) \to F$ is continuous;*

(iii) *if F is a separable FK-space containing E, then the inclusion mapping $(E, \tau(E, E^\beta)) \to F$ is continuous;*

(iv) *if F is a separable FK-space containing E, then $E \subseteq W_F$.*

Proof. (i) $\Rightarrow$ (ii). If A is a matrix mapping E into F it is easy to see that $A : (E, \tau(E, E^\beta)) \to F$ has closed graph. Since E^β is $\sigma(E^\beta, E)$-sequentially complete, the continuity of A follows from Theorem 2.4 of [**11**].

(ii) $\Rightarrow$ (iii). This follows immediately by taking A to be the identity matrix in (ii).

(iii) $\Rightarrow$ (iv). See the proof of Theorem 4, (iii) $\Rightarrow$ (iv).

(iv) $\Rightarrow$ (i). Let $\{a^{(i)}\}_{i=1}^\infty$ be a $\sigma(E^\beta, E)$-Cauchy sequence in E^β and define the matrix A by

$$a_{ij} = a_j^{(i)} \quad (i, j = 1, 2, \ldots).$$

Clearly $E \subseteq c_A$ and, since c_A is a separable FK-space [**18**, Theorem 1.4.1] and [**2**, Corollary 1 to Theorem 5], (iv) gives, for each $x \in E$,

$$P_n(x) \to x \ \sigma(c_A, c_A').$$

Now the linear functional $\lim_A$ is continuous on c_A [**29**, Theorem 4.4(c)], where

$$\lim_A x = \lim_{i\to\infty} \sum_{j=1}^\infty a_{ij} x_j.$$

Consequently,

$$\lim_A x = \sum_{j=1}^{\infty} x_j \lim_A e^j = \sum_{j=1}^{\infty} a_j x_j,$$

where $a_j = \lim_{i\to\infty} a_{ij}$ denotes the jth column limit of A. (Since $\phi \subseteq E$, this limit exists.) Hence $a \in E^\beta$ and

$$\lim_{i\to\infty} \sum_{j=1}^{\infty} a_{ij} x_j = \sum_{j=1}^{\infty} a_j x_j \quad (x \in E),$$

so that $a^{(i)} \to a\ \sigma(E^\beta, E)$.

COROLLARY. *If E is a sequence space containing ϕ and E^β is $\sigma(E^\beta, E)$-sequentially complete, then $\sigma(E^\beta, E)$ and $\sigma(E^\beta, B^*(E))$ define the same convergent sequences and E^β is $\sigma(E^\beta, B^*(E))$-sequentially complete.*

Proof. Suppose $\{a^{(i)}\}_{i=1}^\infty$ is a sequence of elements of E^β with

$$a^{(i)} \to 0 \quad \sigma(E^\beta, E).$$

If A denotes the matrix given by

$$a_{ij} = a_j^{(i)} \quad (i, j = 1, 2, \ldots),$$

then $E \subseteq (c_0)_A$. If $x \in E$, Theorem 5, (i) $\Rightarrow$ (iv), shows that the set $\{P_n(x) : n = 1, 2, \ldots\}$ is bounded in $(c_0)_A$ and so for $\lambda \in B_0$,

$$\lambda x = \sum_{n=1}^{\infty} (\lambda_n - \lambda_{n+1}) P_n(x) \in (c_0)_A.$$

Consequently, $B_0(x) \subseteq (c_0)_A$ so that $B(x) \subseteq (c_0)_A$ giving

$$a^{(i)} \to 0\ \sigma(E^\beta, B^*(E)).$$

Thus $\sigma(E^\beta, E)$ and $\sigma(E^\beta, B^*(E))$ define the same convergent sequences. The last part follows from a general result of Webb [**28**, Corollary 1.5].

Theorems 4 and 5 naturally suggest the problem of characterizing those sequence spaces E for which $E \subseteq S_F$ whenever F is a *separable* FK-space containing E. To tackle this problem, the following structure theorem for spaces with weakly sequentially complete duals will be found useful.

PROPOSITION 2. *Let E be a sequence space containing ϕ and suppose that E^β is $\sigma(E^\beta, E)$-sequentially complete. Then $\tau(E, E^\beta)$ is the projective limit topology determined by the family of all separable FK-spaces F containing E and the associated inclusion mappings $E \to F$.*

Proof. Let τ be the projective limit topology mentioned above. Since E^β is $\sigma(E^\beta, E)$-sequentially complete, Theorem 5, (i) $\Rightarrow$ (iii), shows that $\tau \leqq \tau(E, E^\beta)$.

To establish the converse inclusion we consider an absolutely convex $\sigma(E^\beta, E)$-compact subset K of E^β. Now $\sigma(E^\beta, E)$ coincides on K with the coarser Hausdorff topology $\sigma(E^\beta, \phi)$ which is metrizable; consequently K is

$\sigma(E^\beta, E)$-separable. Thus we may choose a countable $\sigma(E^\beta, E)$-dense subset $\{a^{(i)}\}_{i=1}^\infty$ of K. Letting A denote the matrix given by

$$a_{ij} = a_j^{(i)} \quad (i, j = 1, 2, \ldots),$$

it is clear that $E \subseteq m_A$ and that the inclusion mapping $(E, \tau(E, E^\beta)) \to m_A$ is continuous (K is $\tau(E, E^\beta)$-equicontinuous). Since $(E, \tau(E, E^\beta))$ is separable (ϕ is $\sigma(E, E^\beta)$-dense and therefore $\tau(E, E^\beta)$-dense in E [**21**, p. 34]), it follows that E is separable in m_A. Hence F, the closure of E in m_A, is a separable FK-space. Considering the inclusion $E \to F$, we see that the seminorm

$$x \to \sup_{a \in K} \left| \sum_{j=1}^\infty a_j x_j \right| = \sup_i \left| \sum_{j=1}^\infty a_{ij} x_j \right|$$

is continuous on E in the projective limit topology τ. Thus $\tau(E, E^\beta) \leqq \tau$ and the result is established.

THEOREM 6. *The following conditions are equivalent for any sequence space E containing ϕ:*

(i) *$(E, \tau(E, E^\beta))$ is an AK-space and E^β is $\sigma(E^\beta, E)$-sequentially complete;*

(ii) *if F is a separable FK-space containing E then $E \subseteq S_F$.*

Proof. (i) $\Rightarrow$ (ii). This follows at once from the proof of Theorem 5, (i) $\Rightarrow$ (iii).

(ii) $\Rightarrow$ (i). If (ii) holds, then E^β is $\sigma(E^\beta, E)$-sequentially complete by the proof of Theorem 5, (iv) $\Rightarrow$ (i). That $(E, \tau(E, E^\beta))$ is an AK-space follows from Proposition 2 and the elementary properties of projective limits. (See for example, [**21**, Chapter V].)

It follows, for example, from Propositions 2 and 3 of [**3**] that Theorem 6 applies to any monotone sequence space E. The space m, however, has the additional property that every $\tau(m, l)$-bounded set is $\tau(m, l)$-relatively compact, i.e., $(m, \tau(m, l))$ is a *semi-Montel space*. Our next result characterizes these spaces in terms of inclusion mappings.

THEOREM 7. *The following conditions are equivalent for any sequence space E containing ϕ:*

(i) *$(E, \tau(E, E^\beta))$ is a semi-Montel space;*

(ii) *if F is a separable FK-space containing E, then the inclusion mapping $E \to F$ is compact.*

Proof. (i) $\Rightarrow$ (ii). We first show that (i) implies that E^β is $\sigma(E^\beta, E)$-sequentially complete. To do this, we consider the topology $\rho(E^\beta, E)$ on E^β of uniform convergence on the $\tau(E, E^\beta)$-compact subsets of E. Let f be a linear functional on E whose restrictions to $\tau(E, E^\beta)$-compact sets are $\sigma(E, E^\beta)$-continuous. If $x \in E$ then $\{P_n(x) : n = 1, 2, \ldots\}$ is $\sigma(E, E^\beta)$-bounded and so $\tau(E, E^\beta)$-relatively compact. It follows that $f(P_n(x)) \to f(x)$ so that $f \in E^\beta$ and then, by Grothendieck's completeness theorem (see, for example, [**5**, Proposition 1]), that E^β is $\rho(E^\beta, E)$-complete. If $a^{(n)} \to 0$ $\sigma(E^\beta, E)$, then K, the closed absolutely convex cover of $\{a^{(n)} : n = 1, 2, \ldots\}$ is $\sigma(E^\beta, E)$-compact [**20**, § 20, 9.(6)]. Furthermore, by Grothendieck's precompactness theorem [**5**, Proposition 2], every absolutely convex $\sigma(E^\beta, E)$-compact set is $\rho(E^\beta, E)$-compact. Consequently, $\rho(E^\beta, E)$ and $\sigma(E^\beta, E)$ have the same convergent sequences so that by [**28**, Corollary 1.5] E^β is $\sigma(E^\beta, E)$-sequentially complete. The implication (i) $\Rightarrow$ (ii) now follows from Theorem 5, (i) $\Rightarrow$ (iii).

(ii) $\Rightarrow$ (i). Suppose (ii) holds and that F is a separable FK-space containing E. If $x \in E$, then $\{P_n(x) : n = 1, 2, \ldots\}$ is relatively compact in F and it follows that $P_n(x) \to x$ in F. Thus $E \subseteq S_F$ and Theorem 6, (ii) $\Rightarrow$ (i), shows that E^β is $\sigma(E^\beta, E)$-sequentially complete. That $(E, \tau(E, E^\beta))$ is a semi-Montel space now follows from Proposition 2 and the familiar properties of projective limits (see [**21**, p. 85]).

We note here that the natural analogue of Theorem 7 fails to hold for *semi-reflexive spaces* (i.e., spaces in which every bounded set is weakly relatively compact). To see this we take E to be the space bv; then $E^\beta = cs$, the space of all convergent series, $(bv, \tau(bv, cs))$ is semi-reflexive (since $(cs, \beta(cs, bv))' = bv$) yet cs is *not* $\sigma(cs, bv)$-sequentially complete. However, we do have the following result for reflexive spaces, i.e., barrelled, semi-reflexive spaces. The proof is omitted.

THEOREM 8. *The following conditions are equivalent for any sequence space E containing ϕ:*

(i) *$(E, \tau(E, E^\beta))$ is reflexive (respectively Montel);*

(ii) *if F is an FK-space and A is a matrix mapping E into F, then $A : (E, \tau(E, E^\beta)) \to F$ is weakly compact (respectively compact);*

(iii) *if F is an FK-space containing E, then the inclusion mapping $E \to F$ is weakly compact (respectively compact).*

4. Matrix transformations of l^p spaces. In this short section, we study some inclusion properties of the summability domains $(l^p)_A$. Our first result improves theorems of Lorentz [**16**], Mehdi [**22**], Peyerimhoff [**20**] and Zeller [**30**].

THEOREM 9. *Let $1 \leqq p \leqq \infty$ and let N denote the set of positive integers. Then the following conditions are equivalent for any matrix A:*

(i) *A maps m into l^p;*

(ii) $\displaystyle \sup_{J \subseteq N} \sum_{i=1}^{\infty} \left| \sum_{j \in J} a_{ij} \right|^p < \infty$;

(iii) $\displaystyle \sum_{i=1}^{\infty} \left| \sum_{j \in J} a_{ij} \right|^p < \infty$, *for each $J \subseteq N$.*

Proof. (i) $\Rightarrow$ (ii). If $A : m \to l^p$ then A is continuous [**29**, Theorem 4.4(c)]; if S denotes the set of all sequences of zeros and ones then S is bounded in m so that $\sup_{x \in S} \|Ax\|_p < \infty$, giving (ii).

(ii) $\Rightarrow$ (iii). This is immediate.

(iii) $\Rightarrow$ (i). Condition (iii) is clearly equivalent to $m_0 \subseteq (l^p)_A$ so that, by the Corollary to Theorem 1, (i) follows from (iii). Alternatively, for a proof not involving the results of Seever, we may use Theorem 24 of [**5**] noting that $(l^p)_A$ is a separable FK-space by [**2**, Theorem 4].

We note here that the equivalence of (i) and (ii) for $p = 1$ was first observed by Lorentz [**16**, Lemma 1] and by Zeller (see [**30**, p. 344] and the literature cited there). Later, Peyerimhoff gave another proof [**20**, Theorem 1] which was subsequently generalized by Mehdi (see [**22**, p. 161]) to cover the case $1 \leqq p \leqq \infty$.

Peyerimhoff [**20**, Corollary (a)] and later Pelczynski and Szlenk [**19**, Corollary 1] showed that a necessary condition for a matrix A to map m into l is that

$$\sum_{j=1}^{\infty} |a_{jj}| < \infty.$$

This may be interpreted as saying that if A maps m into l then so does the associated diagonal matrix D. Using this observation we obtain rather quickly the following result of Tong [**27**, Theorem 2.1].

THEOREM 10. *If A is a matrix which maps the monotone sequence space E into the perfect space F, then so does the associated diagonal matrix D.*

Proof. Clearly

$$(*) \qquad E = \bigcup_{x \in E} m_0 \cdot x \quad \text{and} \quad F = \bigcap_{y \in F^\alpha} y^\alpha,$$

so that $yAx = \{y_i a_{ij} x_j\}_{i,j=1}^{\infty}$ maps m_0 into l for each $x \in E$ and each $y \in F^\alpha$. By the above remarks, yDx has the same property, so that D maps E into F by $(*)$.

An immediate consequence is the following extension of the result of Peyerimhoff and Pelczynski-Szlenk.

COROLLARY. *If $p > q \geqq 1$, then a necessary condition for the matrix A to map l^p into l^q is that*

$$\sum_{j=1}^{\infty} |a_{jj}|^{pq/p-q} < \infty.$$

THEOREM 11. *If A is a matrix which maps m into l^p and*

$$\max\{1, 2p/(p+2)\} \leqq q \leqq 2,$$

then

$$\sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} |a_{ij}|^q \right)^{2p/2q-2p+pq} < \infty.$$

Moreover, we have

$$\sum_{j=1}^{\infty}\left(\sum_{i=1}^{\infty}|a_{ij}|^q\right)^{2q/3q-2}<\infty, \quad \text{if } p=1;$$

$$\sum_{j=1}^{\infty}\left(\sum_{i=1}^{\infty}|a_{ij}|^p\right)^{2/p}<\infty, \quad \text{if } 1<p\leqq 2;$$

$$\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}|a_{ij}|^p<\infty, \quad \text{if } 2\leqq p<\infty.$$

Proof. By Theorem 4.4(c) of [**29**], the mapping $A: m\rightarrow l^p$ is continuous. Denoting by $\rho_A^{(i)}$ the ith row of A, we have $\rho_A^{(i)}\in l$, $i=1,2,\ldots$, and

$$\sup_{\|x\|_\infty\leqq 1}\sum_{i=1}^{\infty}|\langle\rho_A^{(i)},x\rangle|^p<\infty.$$

By a result of Kwapien [**13**, Theorem 1.1], the inclusion mapping of l into l^q is $(2q/3q-2, 1)$-absolutely summing (see [**13**] for definitions). It follows from Proposition 0.7 of [**13**] that the inclusion is also $(2pq/2q-2p+pq, p)$-absolutely summing; consequently

$$\sum_{i=1}^{\infty}\|\rho_A^{(i)}\|_q^{2pq/2q-2p+pq}<\infty,$$

i.e.,

$$\sum_{i=1}^{\infty}\left(\sum_{j=1}^{\infty}|a_{ij}|^q\right)^{2p/2q-2p+pq}<\infty.$$

If $p=1$, the transposed matrix maps m into l, and the second inequality follows from the first.

For the third inequality, we use a result of Lindenstrauss and Pelczynski [**14**, Theorem 4.3] which asserts that $A: m\rightarrow l^p (1\leqq p\leqq 2)$ is $(2,2)$-absolutely summing. Now we clearly have

$$\sup_{f\in m', \|f\|\leqq 1}\sum_{j=1}^{\infty}|f(e^j)|<\infty$$

so that

$$\sum_{j=1}^{\infty}\|Ae^j\|_p^2<\infty,$$

i.e.,

$$\sum_{j=1}^{\infty}\left(\sum_{i=1}^{\infty}|a_{ij}|^p\right)^{2/p}<\infty.$$

The last inequality follows similarly since every continuous linear mapping of m into l^p $(2\leqq p<\infty)$ is $(p,2)$-absolutely summing [**14**, Proposition 8.2].

Putting $q=4p/2+p$ in Theorem 11 gives the following generalization of a result of Littlewood [**15**, p. 165, where the case $p=1$ is given].

COROLLARY. *If A is a matrix which maps m into l^p and $1 \leqq p \leqq 2$, then*

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |a_{ij}|^{4p/2+p} < \infty.$$

Our final result is obtained from Theorem 11 by considering the transposed matrix.

THEOREM 12. *If A is a matrix which maps $l^p (p > 1)$ into l and*

$$\max\{1, 2p/(3p-2)\} \leqq q \leqq 2,$$

then

$$\sum_{j=1}^{\infty} \left(\sum_{i=1}^{\infty} |a_{ij}|^q \right)^{2p/3pq-2p-2q} < \infty.$$

Moreover, we have

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |a_{ij}|^{p/p-1} < \infty, \qquad \text{if } 1 < p \leqq 2,$$

$$\sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} |a_{ij}|^{p/p-1} \right)^{2(p-1)/p}, \quad \text{if } 2 \leqq p < \infty.$$

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