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CONSTRUCTIVE APPROXIMATION


# The Thresholding Greedy Algorithm, Greedy Bases, and Duality

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**Abstract.** Some new conditions that arise naturally in the study of the Thresholding Greedy Algorithm are introduced for bases of Banach spaces. We relate these conditions to best $n$-term approximation and we study their duality theory. In particular, we obtain a complete duality theory for greedy bases.

## 1. Introduction

Let $X$ be a Banach space with a seminormalized basis $(e_n)$. An approximation algorithm $(F_n)_{n=1}^{\infty}$ is a sequence of maps $F_n : X \to X$ such that for each $x \in X$, $F_n(x)$ is a linear combination of at most $n$ of the basis elements $(e_j)$. The most natural algorithm is the *linear algorithm* $(S_n)_{n=1}^{\infty}$ given by the partial sum operators.

Recently, Konyagin and Temlyakov [6] introduced the *Thresholding Greedy Algorithm* (TGA) $(G_n)_{n=1}^{\infty}$, where $G_n(x)$ is obtained by taking the largest $n$ coefficients (precise definitions are given in Section 2). The TGA provides a theoretical model for the thresholding procedure that is used in image compression and other applications.

They defined the basis $(e_n)$ to be *greedy* if the TGA is optimal in the sense that $G_n(x)$ is essentially the best $n$-term approximation to $x$ using the basis vectors, i.e., there exists a constant $C$ such that, for all $x \in X$ and $n \in \mathbf{N}$, we have

$$\|x - G_n(x)\| \le C \inf \left\{ \left\| x - \sum_{j \in A} \alpha_j e_j \right\| : |A| = n,\ \alpha_j \in \mathbf{R},\ j \in A \right\}. \tag{1.1}$$

They then showed that greedy bases can be simply characterized as unconditional bases with the additional property of being *democratic*, i.e., for some $\Delta > 0$ we have $\|\sum_{j\in A} e_j\| \le \Delta \|\sum_{j \in B} e_j\|$ whenever $|A| \le |B|$.

They also defined a basis to be *quasi-greedy* if there exists a constant $C$ such that $\|G_m(x)\| \le C\|x\|$ for all $x \in X$ and $n \in \mathbf{N}$. Subsequently, Wojtaszczyk [11] proved that these are precisely the bases for which the TGA merely converges, i.e., $\lim_{n\to\infty} G_n(x) = x$ for $x \in X$.

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In this paper we introduce two natural intermediate conditions. Let us denote the biorthogonal sequence by $(e_n^*)$. We say $(e_n)$ is *almost greedy* if there is a constant $C$ such that

$$\|x - G_n(x)\| \le C \inf\left\{\left\|x - \sum_{j\in A} e_j^*(x)e_j\right\| : |A| = n\right\}, \qquad x \in X, \quad n \in \mathbf{N}. \tag{1.2}$$

Comparison with (1.1) shows that this is formally a weaker condition; in fact Wojtaszczyk's examples of conditional quasi-greedy bases of $\ell_2$ [11] are almost greedy but not greedy. We give two characterizations of almost greedy bases in Theorem 3.3. First, a basis is almost greedy if and only if it is quasi-greedy and democratic. Second, if $\lambda > 1$, then $(e_n)_{n=1}^\infty$ is almost greedy if and only if there exists a constant $C$ such that, for all $x \in X$ and $n \in \mathbf{N}$, we have

$$\|x - G_{[\lambda n]}(x)\| \le C \inf\left\{\left\|x - \sum_{j\in A} \alpha_j e_j\right\| : |A| = n,\ \alpha_j \in \mathbf{R},\ j \in A\right\}. \tag{1.3}$$

Equation (1.2) is a very natural weakening of (1.1).

We also introduce *partially greedy* bases. These are bases such that, for some $C$, we have

$$\|x - G_n(x)\| \le C \left\|\sum_{k=n+1}^{\infty} e_k^*(x)e_k\right\|, \qquad x \in X, \quad n \in \mathbf{N}. \tag{1.4}$$

We give a characterization in Theorem 3.4.

Next we study duality of these conditions. In Theorem 5.1 we show that if $(e_n)$ is a greedy basis of a Banach space $X$ with nontrivial Rademacher type, then $(e_n^*)$ is a greedy basis of $X^*$. However, examples at the end of the paper show that if $X$ has trivial type, then $(e_n^*)$ need not be a greedy basic sequence. Theorem 5.4 concerns duality for almost greedy sequences. It is proved that $(e_n)$ and $(e_n^*)$ are both almost greedy if and only if they are both partially greedy. It is also proved that if $(e_n)$ is almost greedy, then $(e_n^*)$ is almost greedy if and only if $(e_n)$ is *bidemocratic*, i.e., for some $C$ we have

$$\left\|\sum_{j\in A} e_j\right\| \left\|\sum_{j\in A} e_j^*\right\| \le Cn, \qquad |A| = n, \quad n \in \mathbf{N}.$$

Using this result we extend Theorem 5.1 by showing that if $X$ has nontrivial type and $(e_n)$ is almost greedy, then $(e_n^*)$ is an almost greedy basic sequence.

We use standard Banach space notation throughout (see, e.g., [8]). For clarity, however, we recall here the notation that is used most heavily. Let $X$ be a Banach space. The *dual space* of $X$, denoted $X^*$, is the Banach space of all continuous linear functionals $F$ equipped with the norm

$$\|F\| = \sup\{F(x) : \|x\| = 1\}.$$

The closed linear span of a set $A \subseteq X$ (resp., a sequence $(x_n)$) is denoted $[A]$ (resp., $[x_n]$). A *basis* for $X$ is a sequence of vectors $(e_n)$ such that every $x \in X$ has a unique expansion as a norm-convergent series

$$x = \sum_{k=1}^{\infty} e_n^*(x)e_n.$$

Here $(e_n^*)$ is the sequence of *biorthogonal functionals* in $X^*$ defined by $e_n^*(e_m) = \delta_{n,m}$. The basis is said to be *unconditional* if the series expansion converges unconditionally for every $x \in X$. It is said to be *monotone* if

$$\left\| \sum_{k=1}^{n} e_k^*(x)e_k \right\| \le \|x\|, \qquad (x \in X, n \ge 1).$$

Finally, more specialized notions from Banach space theory, such as *type* and *cotype*, will be introduced as needed.

## 2. Greedy Conditions for Bases

Let $(e_n)_{n\in\mathbf{N}}$ be a seminormalized basis of a Banach space $X$ (i.e., $1/C \le \|x_n\| \le C$ for some $C$); let $(e_n^*)_{n\in\mathbf{N}}$ be the biorthogonal sequence in $X^*$. Let us denote by $S_m$ the partial-sum operators

$$S_m(x) = \sum_{j=1}^{m} e_j^*(x)e_j.$$

We also define the remainder operators $R_m = I - S_m$. For any $x \in X$ we define the *greedy ordering for* $x$ as the map $\rho : \mathbf{N} \to \mathbf{N}$ such that $\rho(\mathbf{N}) \supset \{j : e_j^*(x) \ne 0\}$ and so that if $j < k$, then either $|e_{\rho(j)}^*(x)| > |e_{\rho(k)}^*(x)|$ or $|e_{\rho(j)}^*(x)| = |e_{\rho(k)}^*(x)|$ and $\rho(j) < \rho(k)$. The $m$th greedy approximation is given by

$$G_m(x) = \sum_{j=1}^{m} e_{\rho(j)}^*(x)e_{\rho(j)}.$$

We will also introduce the $m$th greedy remainder

$$H_m(x) = x - G_m(x).$$

The basis $(e_n)$ is called *quasi-greedy* if $G_m(x) \to x$ for all $x \in X$. This is equivalent (see [11]) to the condition that for some constant $C$ we have

$$\sup_m \|G_m(x)\| \le C\|x\|, \qquad x \in X. \tag{2.1}$$

It will be convenient to define the *quasi-greedy constant* $K$ to be the least constant such that

$$\|G_m(x)\| \le K\|x\| \qquad \text{and} \qquad \|H_m(x)\| \le K\|x\|, \qquad x \in X.$$

If $(e_n)$ is any basis we denote

$$\sigma_m(x) = \inf\left\{ \left\| x - \sum_{j\in A} \alpha_j e_j \right\| : |A| = m,\ \alpha_j \in \mathbf{R} \right\}.$$

A basis $(e_n)$ is called *greedy* [6] if there is a constant $C$ such that, for any $x \in X$ and $m \in \mathbf{N}$, we have

$$\|H_m(x)\| \le C\sigma_m(x). \tag{2.2}$$

It is natural to introduce two slightly weaker forms of greediness. For any basis $(e_n)$ let

$$\tilde{\sigma}_m(x) = \inf\left\{\left\|x - \sum_{k\in A} e_k^*(x)e_k\right\| : |A| \le m\right\}.$$

Note that

$$\sigma_m(x) \le \tilde{\sigma}_m(x) \le \|R_m(x)\| \to 0 \qquad \text{as} \quad m \to \infty.$$

Let us say that a basis $(e_n)$ is *almost greedy* if there is a constant $C$ so that

$$\|H_m x\| \le C\tilde{\sigma}_m(x). \tag{2.3}$$

We will say that a basis $(e_n)$ is *partially greedy* if there is a constant $C$ so that, for any $x \in X,\ m \in \mathbf{N}$,

$$\|H_m(x)\| \le C\|R_m x\|. \tag{2.4}$$

It is clear that for any basis we have the following implications:

greedy ⇒ almost greedy ⇒ partially greedy ⇒ quasi-greedy.

Next we prove two useful lemmas concerning quasi-greedy bases. These are both essentially due to Wojtaszczyk [11]. The first lemma says that every quasi-greedy basis is *unconditional for constant coefficients*.

**Lemma 2.1.** *Suppose* $(e_n)_{n\in\mathbf{N}}$ *has quasi-greedy constant* $K$. *Suppose* $A$ *is a finite subset of* $\mathbf{N}$. *Then, for every choice of signs* $\varepsilon_j = \pm 1$, *we have*

$$\frac{1}{2K}\left\|\sum_{j\in A} e_j\right\| \le \left\|\sum_{j\in A} \varepsilon_j e_j\right\| \le 2K\left\|\sum_{j\in A} e_j\right\|, \tag{2.5}$$

*and, hence, for any real numbers* $(a_j)_{j\in A}$,

$$\left\|\sum_{j\in A} a_j e_j\right\| \le 2K \max_{j\in A}|a_j| \left\|\sum_{j\in A} e_j\right\|. \tag{2.6}$$

**Proof.** First note that if $B \subset A$ and $\varepsilon > 0$, then

$$\left\|\sum_{j\in B}(1+\varepsilon)e_j\right\| \le K\left\|\sum_{j\in B}(1+\varepsilon)e_j + \sum_{j\in A\setminus B} e_j\right\|.$$

Letting $\varepsilon \to 0$, we obtain $\|\sum_{j\in B} e_j\| \le K\|\sum_{j\in A} e_j\|$ and, hence, for any choice of signs $\varepsilon_j = \pm 1$, we have

$$\left\|\sum_{j\in A} \varepsilon_j e_j\right\| \le 2K\left\|\sum_{j\in A} e_j\right\|.$$

This gives the right-hand inequality in (2.5) and the left-hand inequality is similar. By convexity, (2.6) follows immediately. ■

**Lemma 2.2.** *Suppose $(e_n)_{n\in\mathbf{N}}$ has quasi-greedy constant $K$. Suppose $x \in X$ has greedy ordering $\rho$. Then*

$$|e^*_{\rho(m)}(x)| \left\| \sum_{j=1}^{m} e_{\rho(j)} \right\| \le 4K^2 \|x\| \tag{2.7}$$

*and, hence, if $A$ is any subset of $\mathbf{N}$ and $(a_j)_{j\in A}$ are any real numbers,*

$$\min_{j\in A} |a_j| \left\| \sum_{j\in A} e_j \right\| \le 4K^2 \left\| \sum_{j\in A} a_j e_j \right\|. \tag{2.8}$$

**Proof.** We prove (2.7), and then (2.8) is immediate. Let $a_j = e^*_j(x)$. Let $\varepsilon_j = \operatorname{sgn} a_j$ and put $1/|a_\rho(0)| = 0$. Then

$$\begin{aligned} |a_{\rho(m)}| \left\| \sum_{j=1}^{m} \varepsilon_{\rho(j)} e_{\rho(j)} \right\| &= |a_{\rho(m)}| \left\| \sum_{j=1}^{m} \left( \frac{1}{|a_{\rho(j)}|} - \frac{1}{|a_{\rho(j-1)}|} \right) (H_{j-1}(x) - H_m(x)) \right\| \\ &\le 2K\|x\|. \end{aligned}$$

We then use (2.5). ■

We conclude this section by considering direct and inverse theorems for approximation with regard to almost greedy bases.

We define the *fundamental function* $\varphi(n)$ of a basis $(e_n)$ by

$$\varphi(n) = \sup_{|A|\le n} \left\| \sum_{k\in A} e_k \right\|.$$

For $x \in X$ with greedy ordering $\rho$, let

$$a_k(x) := |e^*_{\rho(k)}(x)|.$$

The following theorem was proved in [10]:

**Theorem 2.3.** *Let $1 < p < \infty$ and let $(e_n)$ be a greedy basis with $\varphi(n) \asymp n^{1/p}$. Then, for any $0 < r < \infty$ and $0 < q < \infty$, we have the following equivalence*:

$$\sum_n \sigma_n(x)^q n^{rq-1} < \infty \quad \Leftrightarrow \quad \sum_n a_n(x)^q n^{rq-1+q/p} < \infty.$$

We generalize this theorem as follows:

**Theorem 2.4.** *Let $1 < p < \infty$ and let $(e_n)$ be a democratic quasi-greedy basis with $\varphi(n) \asymp n^{1/p}$. Then, for any $0 < r < \infty$ and $0 < q < \infty$, we have the following equivalence*:

$$\sum_n \sigma_n(x)^q n^{rq-1} < \infty \quad \Leftrightarrow \quad \sum_n a_n(x)^q n^{rq-1+q/p} < \infty.$$

The proof of this theorem is similar to the proof of Theorem 2.3 and is based on the following lemmas which are analogous to the corresponding lemmas from [10]. See the Introduction (or the next section) for the definition of a democratic basis.

**Lemma 2.5.** *Let $(e_n)$ be a democratic quasi-greedy basis with $\varphi(n) \asymp n^{1/p}$. Then there exists a constant $C$ such that, for any two positive integers $N < M$ and any $x \in X$, we have*

$$a_M(x) \leq C\|H_N(x)\|(M-N)^{-1/p}.$$

**Proof.** This lemma follows from (2.8) of Lemma 2.2. ■

**Lemma 2.6.** *Let $(e_n)$ be a democratic quasi-greedy basis with $\varphi(n) \asymp n^{1/p}$. Then there exists a constant $C$ such that, for any sequence $m_0 < m_1 < \cdots$ of nonnegative integers, we have*

$$\|H_{m_s}(x)\| \leq C \sum_{l=s}^{\infty} a_{m_l}(x)(m_{l+1} - m_l)^{1/p}.$$

**Proof.** This lemma follows from (2.6) of Lemma 2.1. ■

By Theorem 3.3 below we get that a democratic quasi-greedy basis is almost greedy and also has the following property (setting $\lambda = 2$ in (3) of Theorem 3.3):

$$\sigma_{2n}(x) \leq \|H_{2n}(x)\| \leq C\sigma_n(x).$$

This inequality implies that

$$\sum_n \|H_n(x)\|^q n^{rq-1} < \infty \quad \Leftrightarrow \quad \sum_n \sigma_n(x)^q n^{rq-1} < \infty.$$

Therefore Theorem 2.3 holds with the assumption that $(e_n)$ is greedy replaced by the assumption that $(e_n)$ is almost greedy, which yields Theorem 2.4.

**Remark 2.7.** We note that the version of Theorem 2.4 with $\sigma_n(x)$ replaced by $\|H_n(x)\|$ was proved in [4] (it also follows from Lemmas 2.5 and 2.6). Our version of Theorem 2.4 is based on Theorem 3.3 below.

## 3. Democratic and Conservative Bases

We recall that a basis $(e_n)$ in a Banach space $X$ is called *democratic* if there is a constant $\Delta$ such that

$$\left\|\sum_{k\in A} e_k\right\| \leq \Delta \left\|\sum_{k\in B} e_k\right\| \qquad \text{if} \quad |A| \leq |B|. \tag{3.1}$$

This concept was introduced in [6].The following characterization of greedy bases was also proved in [6].

**Theorem 3.1.** *A basis $(e_n)$ is greedy if and only if it is unconditional and democratic*.

Recall that the fundamental function $\varphi(n)$ of $(e_n)$ is defined by

$$\varphi(n) = \sup_{|A|\leq n} \left\| \sum_{k\in A} e_k \right\|.$$

The *dual fundamental function* is given by

$$\varphi^*(n) = \sup_{|A|\leq n} \left\| \sum_{k\in A} e_k^* \right\|.$$

Note that $\varphi$ (and $\varphi^*$) is subadditive (i.e., $\varphi(m+n) \leq \varphi(m) + \varphi(n)$) and increasing. It may also be seen that $\varphi(n)/n$ (and $\varphi^*(n)/n$) is decreasing since, for any set $A$ with $|A| = n$, we have

$$\sum_{k\in A} e_k = \frac{1}{n-1} \sum_{k\in A} \sum_{j\neq k} e_j.$$

By convexity, for any set $A$ and any scalars $\{a_j : j \in A\}$, we have

$$\left\| \sum_{j\in A} a_j e_j \right\| \leq \max_{j\in A} |a_j| \max_{\pm} \left\| \sum_{A} \pm e_j \right\|.$$

Hence

$$\left\| \sum_{j\in A} a_j e_j \right\| \leq 2\varphi(|A|) \max_{j\in A} |a_j|. \tag{3.2}$$

It is clear that $(e_k)$ is democratic with constant $\Delta$ in (3.1) if and only if

$$\Delta^{-1}\varphi(|A|) \leq \left\| \sum_{k\in A} e_k \right\| \leq \varphi(|A|), \qquad |A| < \infty. \tag{3.3}$$

**Lemma 3.2.** *Let $(e_n)$ be a democratic quasi-greedy basis. Let $K$ be the quasi-greedy constant and $\Delta$ the democratic constant. Then, for $x \in X$, if $\rho$ is the quasi-greedy ordering*

$$|e^*_{\rho(m)}(x)| \leq \frac{4K^2\Delta}{\varphi(m)} \|x\|, \tag{3.4}$$

*and*

$$\sup_{k\in \mathbf{N}} |e_k^*(H_m x)| \leq \frac{4K^2\Delta}{\varphi(m+1)} \|x\|. \tag{3.5}$$

**Proof.** This follows directly from (3.3) and Lemma 2.2 (2.7). ■

Next we compare almost greedy bases with greedy bases. It follows from (3) of the theorem below that in an almost greedy basis the convergence of the TGA is "almost" optimal. It follows from (2) of the theorem below and [11] that any conditional quasi-greedy basis of a Hilbert space is actually almost greedy. See also [3] for a conditional almost greedy basis of $\ell_1$.

**Theorem 3.3.** *Suppose* $(e_n)$ *is a basis of a Banach space. The following are equivalent*:

(1) $(e_n)$ *is almost greedy.*
(2) $(e_n)$ *is quasi-greedy and democratic*.
(3) *For any* (*resp*., *every*) $\lambda > 1$ *there is a constant* $C = C_\lambda$ *such that*

$$\|H_{[\lambda m]}x\| \le C_\lambda \sigma_m(x).$$

**Proof.** We start by showing (1) implies (2). It is immediate that $(e_n)$ is quasi-greedy. Now suppose $|A| \le |B|$. Suppose $\delta > 0$ and define

$$x = \sum_{j\in A} e_j + \sum_{j\in B\setminus A} (1+\delta)e_j.$$

Then, if $r = |B\setminus A|$ we have $H_r(x) = \sum_{j\in A} e_j$. However,

$$\tilde{\sigma}_r(x) \le \left\| \sum_{j\in B} e_j^*(x)e_j \right\| \le \left\| \sum_{j\in B} e_j \right\| + \delta \left\| \sum_{j\in B\setminus A} e_j \right\|.$$

Letting $\delta \to 0$, it follows from (2.3) that $(e_n)$ is democratic.

Next we show that (2) implies (1) so that (1) and (2) are equivalent. Suppose $x \in X$ and $m \in \mathbf{N}$. Let

$$G_m(x) = \sum_{j\in A} e_j^*(x)e_j$$

where $|A| = m$. Suppose $|B| = r \le m$. Then

$$H_m(x) = \left( x - \sum_{j\in B} e_j^*(x)e_j \right) + \sum_{j\in B\setminus A} e_j^*(x)e_j - \sum_{j\in A\setminus B} e_j^*(x)e_j.$$

Then $|B\setminus A| \le s := |A\setminus B|$. Thus

$$\left\| \sum_{j\in B\setminus A} e_j^*(x)e_j \right\| \le 2K \left( \max_{j\in B\setminus A} |e_j^*(x)| \right) \varphi(s)$$

(by (2.6))

$$\le 2K \left( \min_{j\in A\setminus B} |e_j^*(x)| \right) \varphi(s)$$

$$\le 8K^3\Delta \left\| \sum_{j\in A\setminus B} e_j^*(x)e_j \right\|$$

(by (3.4))

$$= 8K^3\Delta \left\| G_s\left(x - \sum_{j\in B} e_j^*(x)e_j\right)\right\|$$

$$\le 8K^4\Delta \left\| \left(x - \sum_{j\in B} e_j^*(x)e_j\right)\right\|.$$

We also have

$$\left\| \sum_{j\in A\setminus B} e_j^*(x)e_j \right\| = \left\| G_s\left(x - \sum_{j\in B} e_j^*(x)e_j\right)\right\|.$$

Thus it follows that

$$\|H_m(x)\| \le (8K^4\Delta + K + 1)\left\| x - \sum_{j\in B} e_j^*(x)e_j\right\|$$

and so, optimizing over $B$ with $|B| \le m$,

$$\|H_m(x)\| \le (8K^4\Delta + K + 1)\tilde{\sigma}_m(x).$$

Let us prove that (2) implies (3) for every $\lambda > 1$. Assume $K$ is the quasi-greedy constant and $\Delta$ is the democratic constant. Assume $m, r \in \mathbf{N}$. For $x \in X$ and $A$ a finite subset of cardinality $m$, let $v = \sum_{j\notin A} e_j^*(x)e_j$. Now suppose $y$ is such that $e_j^*(y) \ne e_j^*(x)$ only if $j \in A$. Then

$$G_r(y) = \sum_{j\in B} e_j^*(y)e_j$$

where $|B| = r$. Let $|A \cap B| = s$ where $0 \le s \le \min(r, m)$. Then

$$H_r(y) - H_{r-s}(v) = H_s(y - v) = \sum_{j\in A\setminus B} e_j^*(y)e_j.$$

Now, by (3.5),

$$\max_{j\in A\setminus B} |e_j^*(y)| \le \frac{4K^2\Delta}{\varphi(r+1)}\|y\|.$$

Hence, by (2.6),

$$\|H_r(y) - H_{r-s}(v)\| \le \frac{8K^3\Delta\varphi(m)}{\varphi(r+1)}\|y\|. \tag{3.6}$$

For $\varepsilon > 0$ we can choose $y$ so that $\|y\| < \sigma_m(x) + \varepsilon$ and $\{j : e_j^*(y) \ne e_j^*(x)\}$ is contained in a set $A$ of cardinality $m$ as above. Note that

$$\tilde{\sigma}_{m+r}(x) \le \tilde{\sigma}_{m+r-s}(x) \le \|H_{r-s}(v)\|$$

and hence (3.6) and the triangle inequality yield

$$\begin{aligned}\tilde{\sigma}_{m+r}(x) &\leq \|H_{r-s}(v)\| \\ &\leq \|H_r(y)\| + \|H_r(y) - H_{r-s}(v)\| \\ &\leq K\|y\| + \frac{8K^3\Delta\varphi(m)}{\varphi(r+1)}\|y\|.\end{aligned}$$

Since $\|y\| \leq \sigma_m(x) + \varepsilon$ and $\varepsilon$ is arbitrary, we obtain

$$\tilde{\sigma}_{m+r}(x) \leq \left(\frac{8K^3\Delta\varphi(m)}{\varphi(r+1)} + K\right)\sigma_m(x).$$

Next suppose $\lambda > 1$ and $r = [\lambda m] - m$. Now $\varphi(m)/\varphi(r+1) \leq m/(r+1)$, so we have

$$\tilde{\sigma}_{[\lambda m]}(x) \leq \left(\frac{8K^3\Delta}{\lambda - 1} + K\right)\sigma_m(x).$$

This implies (3) with $C_\lambda \asymp (\lambda - 1)^{-1}$.

It remains to show (3) (for some fixed $\lambda > 1$) implies (2). That $(e_n)$ is quasi-greedy is immediate. Note that if $|D| = [\lambda m]$, then

$$\left\|\sum_{j\in D} e_j\right\| \leq \varphi(\lambda m) \leq \lambda\varphi(m).$$

So to prove that $(e_n)$ is democratic it is enough to show that

$$\left\|\sum_{j\in D} e_j\right\| \geq \varphi(m)/C_\lambda.$$

Suppose $|A| \leq m < \infty$. For any set $B$ of cardinality $[\lambda m]$ disjoint from $A$ we have (by a similar argument as in the case (1) implies (2))

$$\left\|\sum_{j\in A} e_j\right\| \leq C_\lambda \sigma_m\left(\sum_{j\in A\cup B} e_j\right) \leq C_\lambda \left\|\sum_{j\in D} e_j\right\|$$

whenever $D \subset A \cup B$ with $|D| \geq [\lambda m]$. Thus, maximizing over all $A$ with $|A| \leq m$,

$$\inf_{|D|=[\lambda m]} \left\|\sum_{j\in D} e_j\right\| \geq \varphi(m)/C_\lambda$$

and so $(e_j)$ is democratic. ∎

If $A$, $B$ are subsets of $\mathbf{N}$ we use the notation $A < B$ to mean that $m \in A, n \in B$ implies $m < n$. We write $n < A$ for $\{n\} < A$. Let us define a basis $(e_n)$ to be *conservative* if there is a constant $\Gamma$ such that

$$\left\|\sum_{k\in A} e_k\right\| \leq \Gamma \left\|\sum_{k\in B} e_k\right\| \qquad \text{if} |A| \leq |B| \text{ and } A < B. \tag{3.7}$$

The analogue of Theorems 3.1 and 3.3 is

**Theorem 3.4.** *A basis* $(e_n)$ *is partially greedy if and only if it is quasi-greedy and conservative.*

**Proof.** Clearly a partially greedy basis is also quasi-greedy. Suppose $(e_n)$ is partially greedy (with constant $C$ in (2.4)) and $A < B$ with $|A| = |B| = m$. Let $r = \max A$. Let $D = [1, r]\backslash A$ and then, for $\delta > 0$, let

$$x = \sum_{k\in A} e_k + (1+\delta)\sum_{k\in D\cup B} e_k.$$

Then

$$\|H_r(x)\| = \left\|\sum_{k\in A} e_k\right\|$$

and

$$\|R_r(x)\| = (1+\delta)\left\|\sum_{k\in B} e_k\right\|,$$

so that letting $\delta \to 0$ gives (3.7) with $\Gamma = C$.

Conversely, let us suppose $(e_n)$ is quasi-greedy with constant $K$ and conservative with constant $\Gamma$. Suppose $x \in X$ and $m \in \mathbf{N}$. Let $\rho$ be the greedy ordering for $x$. Then, let $D = \{\rho(j) : j \le m,\ \rho(j) \le m\}$ and $B = \{\rho(j) : j \le m,\ \rho(j) > m\}$. Let $A = [1, m]\backslash D$. Then $|A| = |B| = r$, say, and $A < B$. Now

$$\left\|\sum_{k\in B} e_k^*(x)e_k\right\| = \|G_r(R_m x)\| \le K\|R_m x\|.$$

Also

$$\begin{aligned}\left\|\sum_{k\in A} e_k^*(x)e_k\right\| &\le 2K\left(\max_{k\in A}|e_k^*(x)|\right)\left\|\sum_{k\in A} e_k\right\| \\ &\le 2K\Gamma\left(\min_{k\in B}|e_k^*(x)|\right)\left\|\sum_{k\in B} e_k\right\| \\ &\le 8K^3\Gamma\left\|\sum_{k\in B} e_k^*(x)e_k\right\|\end{aligned}$$

(by (2.8))

$$\le 8K^4\Gamma\|R_m x\|.$$

Combining gives us

$$\begin{aligned}\|H_m x\| &\le \|R_m x\| + \left\|\sum_{k\in A} e_k^*(x)e_k\right\| + \left\|\sum_{k\in B} e_k^*(x)e_k\right\| \\ &\le (8K^4\Gamma + K + 1)\|R_m x\|.\end{aligned}$$

■

## 4. Bidemocratic Bases

Suppose $(e_n)$ is a democratic basis. We shall say that $(e_n)$ has the *upper regularity property* (URP) if there exists an integer $r > 2$ so that

(4.1) $$\varphi(rn) \leq \tfrac{1}{2} r\varphi(n), \qquad n \in \mathbf{N}.$$

This of course implies $\varphi(r^k n) \leq 2^{-k} r^k \varphi(n)$ and is therefore easily equivalent to the existence of $0 < \beta < 1$ and a constant $C$ so that, if $m > n$,

(4.2) $$\varphi(m) \leq C\left(\frac{m}{n}\right)^{\beta} \varphi(n).$$

We say $(e_n)$ has the *lower regularity property* (LRP) if there exists $r > 1$ so that, for all $n \in \mathbf{N}$, we have

(4.3) $$\varphi(rn) \geq 2\varphi(n), \qquad n \in \mathbf{N}.$$

This is similarly equivalent to the existence of $0 < \alpha < 1$ and $c > 0$ so that, if $m > n$,

(4.4) $$\varphi(m) \geq c\left(\frac{m}{n}\right)^{\alpha} \varphi(n).$$

Let us recall that a Banach space $X$ has (Rademacher) type $1 < p \leq 2$ if there is a constant $C$ such that

$$\left(\operatorname*{Ave}_{\varepsilon_j=\pm 1} \left\| \sum_{j=1}^{n} \varepsilon_j x_j \right\|^p\right)^{1/p} \leq C \left(\sum_{j=1}^{n} \|x_j\|^p\right)^{1/p}, \qquad x_1, \dots, x_n \in X, \quad n \in \mathbf{N}.$$

The least such constant $C$ is called the type $p$-constant $T_p(X)$. We say that $X$ has nontrivial (resp., trivial) type if $X$ has (resp., does not have) type $p$ for some (resp., any) $p > 1$. Recall also that $X$ has (Rademacher) cotype $2 \leq q < \infty$ if there exists a constant $C$ such that

$$\left(\sum_{j=1}^{n} \|x_j\|^q\right)^{1/q} \leq C \left(\operatorname*{Ave}_{\varepsilon_j=\pm 1} \left\| \sum_{j=1}^{n} \varepsilon_j x_j \right\|^q\right)^{1/q}, \qquad x_1, \dots, x_n \in X, \quad n \in \mathbf{N}.$$

The least such constant $C$ is called the cotype $q$-constant $C_q(X)$. We say that $X$ has nontrivial (resp., trivial) cotype if $X$ has (resp., does not have) cotype $q$ for some (resp., any) $q < \infty$.

**Proposition 4.1.**

(1) *If $(e_n)$ is an almost greedy basis of a Banach space with nontrivial cotype, then $(e_n)$ has the LRP.*
(2) *If $(e_n)$ is an almost greedy basis of a Banach space with nontrivial type, then $(e_n)$ has the LRP and the URP.*

**Proof.** (1) Suppose $K$ is the quasi-greedy constant of $(e_n)$ and $\Delta$ is the democratic constant. Suppose $X$ has cotype $q < \infty$ with constant $C_q(X)$. Let $B_1, \ldots, B_m$ be disjoint sets with $|B_k| = n$ and let $A = \bigcup_{k=1}^m B_k$. Then, using Lemma 2.1, (2.5), and (3.3),

$$\begin{aligned} m^{1/q}\varphi(n) &\le \Delta \left( \sum_{k=1}^m \left\| \sum_{j\in B_k} e_j \right\|^q \right)^{1/q} \\ &\le 2K\Delta \left( \sum_{k=1}^m \operatorname*{Ave}_{\varepsilon_j=\pm 1} \left\| \sum_{j\in B_k} \varepsilon_j e_j \right\|^q \right)^{1/q} \\ &\le 2K\Delta C_q(X) \left( \operatorname*{Ave}_{\varepsilon_j=\pm 1} \left\| \sum_{j\in A} \varepsilon_j e_j \right\|^q \right)^{1/q} \\ &\le 4K\Delta C_q(X)\varphi(mn). \end{aligned}$$

It is clear this implies (4.4) for some suitable constant $c > 0$ and $\alpha = 1/q$.

(2) Since nontrivial type implies nontrivial cotype we obtain the LRP immediately. The proof of the URP (with $\beta = 1/p$ when $X$ has type $p$) is very similar. Using the same notation, and assuming $X$ has type $p > 1$ with constant $T_p(X)$, we have

$$\begin{aligned} \varphi(mn) &\le 2K\Delta \left( \operatorname*{Ave}_{\varepsilon_j=\pm 1} \left\| \sum_{j\in A} \varepsilon_j e_j \right\|^p \right)^{1/p} \\ &\le 2K\Delta T_p(X) \left( \sum_{k=1}^m \operatorname*{Ave}_{\varepsilon_j=\pm 1} \left\| \sum_{j\in B_k} \varepsilon_j e_j \right\|^p \right)^{1/p} \\ &\le 4K\Delta T_p(X) m^{1/p}\varphi(n). \end{aligned}$$

This implies (4.2) for suitable constants. ∎

We now say that a basis $(e_n)$ is *bidemocratic* if there is a constant $\Delta$ so that

$$\varphi(n)\varphi^*(n) \le \Delta n. \tag{4.5}$$

**Proposition 4.2.** *If* $(e_n)$ *is bidemocratic* (*with constant* $\Delta$), *then* $(e_n)$ *and* $(e_n^*)$ *are both democratic* (*with constant* $\Delta$) *and are both unconditional for constant coefficients.*

**Proof.** If $A$ is any finite set we have

$$|A| \le \left\| \sum_{j\in A} e_j^* \right\| \left\| \sum_{j\in A} e_j \right\| \le \varphi^*(|A|) \left\| \sum_{j\in A} e_j \right\|.$$

Hence

$$\Delta^{-1}\varphi(|A|) \le \left\| \sum_{j\in A} e_j \right\|$$

and so $(e_n)$ is democratic with constant $\Delta$. Let $(\varepsilon_j)_{j\in A}$ be any choice of signs $\pm 1$. Then

$$|A| \le \left\|\sum_{j\in A}\varepsilon_j e_j^*\right\| \left\|\sum_{j\in A}\varepsilon_j e_j\right\| \le 2\varphi^*(|A|)\left\|\sum_{j\in A}\varepsilon_j e_j\right\|.$$

Hence

$$\frac{1}{2\Delta}\varphi(|A|) \le \left\|\sum_{j\in A}\varepsilon_j e_j\right\| \le 2\varphi(|A|).$$

Hence $(e_n)$ is unconditional for constant coefficients. Similar calculations work for $(e_j^*)$ to obtain the theorem. ■

**Proposition 4.3.** *A basis $(e_n)$ is bidemocratic if and only if there is a constant $C$ so that, for any finite set $A \subset \mathbf{N}$,*

$$\left\|\sum_{k\in A} e_k\right\| \left\|\sum_{k\in A} e_k^*\right\| \le C|A|. \tag{4.6}$$

**Proof.** One direction is trivial. Assume (4.6) holds with $C \ge 1$. Suppose $n \in \mathbf{N}$. By passing to an equivalent norm on $X$, if necessary, we may assume that $(e_n)$ and $(e_n^*)$ are both monotone. There exist $A, B \subset \mathbf{N}$ with $|A| \le n$, $|B| \le n$, and

$$\left\|\sum_{j\in A} e_j\right\| \ge \tfrac{1}{2}\varphi(n), \qquad \left\|\sum_{j\in B} e_j^*\right\| \ge \tfrac{1}{2}\varphi^*(n).$$

By monotonicity of $(e_n)$ and $(e_n^*)$ we may assume that $|A| = |B| = n$. Let $D = A \cup B$, $E = D\backslash A$.

If $\|\sum_{j\in D} e_j\| \ge (1/8C)\varphi(n)$ and $\|\sum_{j\in D} e_j^*\| \ge (1/8C)\varphi^*(n)$ we obtain immediately that

$$\varphi(n)\varphi^*(n) \le 2^6 C^3 |D| \le 2^7 C^3 n.$$

Consider when one of these inequalities fails; we need only treat the case $\|\sum_{j\in D} e_j\| < (1/8C)\varphi(n)$. Then

$$\left\|\sum_{j\in E} e_j\right\| \ge \left\|\sum_{j\in A} e_j\right\| - \left\|\sum_{j\in D} e_j\right\| > \frac{\varphi(n)}{2} - \frac{\varphi(n)}{8C} > \frac{\varphi(n)}{4}$$

and thus, as $|E| \le n$, (4.6) gives

$$\left\|\sum_{j\in E} e_j^*\right\| \le 4Cn\varphi(n)^{-1}.$$

We also have from (4.6) that

$$\left\|\sum_{j\in A} e_j^*\right\| \le 2Cn\varphi(n)^{-1}.$$

Hence

$$\left\|\sum_{j\in D} e_j^*\right\| \le 6Cn\varphi(n)^{-1}$$

and so

$$n \le |D| \le \left\|\sum_{j\in D} e_j\right\| \left\|\sum_{j\in D} e_j^*\right\| \le \left(\frac{6Cn}{\varphi(n)}\right)\left(\frac{\varphi(n)}{8C}\right) = \frac{3n}{4}$$

which is a contradiction. ■

**Proposition 4.4.** *If $(e_n)$ is a democratic quasi-greedy basis with the URP, then $(e_n)$ is bidemocratic.*

**Proof.** We assume (4.2) holds, that $(e_n)$ is quasi-greedy with constant $K$, and democratic with constant $\Delta$. Suppose $A$ is a finite subset of $\mathbf{N}$. Pick $x \in X$ so that $\|x\| = 1$ and $\sum_{j\in A} e_j^*(x) > \frac{1}{2}\|\sum_{j\in A} e_j^*\|$. Let $\rho$ be the greedy ordering for $x$. Then by (3.5), if $|A| = n$,

$$\begin{aligned}\varphi(n)\left\|\sum_{j\in A} e_j^*\right\| &\le 2\varphi(n)\sum_{j\in A}|e_j^*(x)|\\ &\le 2\varphi(n)\sum_{k=1}^{n}|e_{\rho(k)}^*(x)|\\ &\le 8K^2\Delta\sum_{k=1}^{n}\frac{\varphi(n)}{\varphi(k)}\\ &\le 8K^2\Delta Cn^{\beta}\sum_{k=1}^{n}k^{-\beta}\\ &\le C_1 n\end{aligned}$$

for a suitable constant $C_1$. This implies $\varphi(n)\varphi^*(n) \le C_1 n$. ■

**Corollary 4.5.** *Let $(e_n)$ be a quasi-greedy basis for a Hilbert space. Then $(e_n)$ is bidemocratic.*

**Proof.** Wojtaszczyk [11] proved that $(e_n)$ is democratic and that $\varphi(n) \asymp \sqrt{n}$. So the result follows from Proposition 4.4. ■

**Remark 4.6.** Proposition 4.4 breaks down for bases that are not quasi-greedy. To see this, let $(e_n^p)$ be the unit vector basis of $\ell_p$. We define a normalized basis $(f_n)$ of $\ell_2 \oplus_2 \ell_p$ as follows:

$$f_{2n-1} = \frac{1}{\sqrt{2}}(e_n^2 + e_n^p), \qquad f_{2n} = \tfrac{1}{2}e_n^2 + \frac{\sqrt{3}}{2}e_n^p.$$

Suppose that $1 < p < 2$. It is easy to check that $(f_n)$ and $(f_n^*)$ are both democratic and unconditional for constant coefficients, that $\varphi(n) \asymp n^{1/p}$, and that $\varphi^*(n) \asymp \sqrt{n}$. So both $(f_n)$ and $(f_n^*)$ have the URP but $(f_n)$ is not bidemocratic.

## 5. Duality of Almost Greedy Bases

**Theorem 5.1.** *Let $(e_n)$ be a greedy basis with the URP. Then $(e_n^*)$ is a greedy basic sequence. In particular, if $(e_n)$ is a greedy basis of a Banach space $X$ with nontrivial type, then $(e_n^*)$ is a greedy basis of $X^*$.*

**Proof.** Since $(e_n^*)$ is automatically unconditional this follows from Proposition 4.4 and Theorem 3.1. The second part follows from Proposition 4.1; note that any space with nontrivial type and an unconditional basis is reflexive by James's theorem [5]. ■

**Remark 5.2.** The Haar system is a greedy basis of $H_1$. However, Oswald [9] proved that the Haar system is not a greedy basic sequence in BMO (i.e., $H_1^*$). This provides a natural illustration of the fact that the assumption of nontrivial type in Theorem 5.1 cannot be eliminated.

**Corollary 5.3.** *For $1 < p < \infty$ the Banach space $L_p[0,1]$ has a greedy basis not equivalent to a rearranged subsequence of the Haar system.*

**Proof.** For $p > 2$, Wojtaszczyk [11] constructed such a basis with $\varphi(n) \asymp n^{1/p}$, hence with the URP. The case $p < 2$ follows by duality using Theorem 5.1. ■

**Theorem 5.4.** *Let $(e_n)$ be a quasi-greedy basis of a Banach space $X$. Then the following are equivalent*:

(1) $(e_n)$ *is bidemocratic*.
(2) $(e_n)$ *and* $(e_n^*)$ *are both almost greedy.*
(3) $(e_n)$ *and* $(e_n^*)$ *are both partially greedy.*

**Proof.** We first prove (1) implies (2). Let $\Delta$ denote the bidemocratic constant. In fact, by Theorem 3.3 and Proposition 4.2, we only need show that $(e_n^*)$ is quasi-greedy. Let us denote by $G_m^*$ and $H_m^*$ the greedy operator and greedy remainder operators associated to the dual basic sequence $(e_n^*)$. Suppose $x^* \in X^*$ and $x \in X$.

First note that if $|A| = m$, then

$$\begin{aligned}\sum_{j\in A} |x^*(e_j)| &\leq \|x^*\| \sup_{\varepsilon_j=\pm1} \left\| \sum_{j\in A} \varepsilon_j e_j \right\| \\ &\leq 2\varphi(m)\|x^*\|.\end{aligned}$$

Hence

$$\sup_{j\in\mathbf{N}} |(H_m^* x^*)(e_j)| \leq 2\frac{\varphi(m+1)}{m+1}\|x^*\|. \tag{5.1}$$

On the other hand, (3.5) implies that

$$\sup_{j\in\mathbf{N}} |e_j^*(H_m(x))| \le \frac{4K^2\Delta}{\varphi(m+1)}\|x\|. \tag{5.2}$$

Suppose $G_m(x) = \sum_{j\in A} e_j^*(x)e_j$ and $G_m^*(x^*) = \sum_{j\in B} x^*(e_j)e_j^*$ where $|A| = |B| = m$. Then

$$\begin{aligned}
|(H_m^* x^*)(G_m(x))| &= \left|\left(\sum_{j\in A\setminus B} x^*(e_j)e_j^*\right)(x)\right| \\
&\le \left\|\sum_{j\in A\setminus B} x^*(e_j)e_j^*\right\| \|x\| \\
&\le 4\frac{\varphi(m+1)\varphi^*(m)}{m+1}\|x\|\|x^*\|
\end{aligned}$$

(by (3.2) and (5.1))

$$\le 4\Delta\|x\|\|x^*\|.$$

Also,

$$\begin{aligned}
|(G_m^* x^*)(H_m(x))| &= \left|x^*\left(\sum_{j\in B\setminus A} e_j^*(x)e_j\right)\right| \\
&\le \|x^*\|\frac{4K^2\Delta\|x\|}{\varphi(m+1)}(2\varphi(m))
\end{aligned}$$

(by (5.2))

$$\le 8K^2\Delta\|x\|\|x^*\|.$$

Now

$$G_m^* x^*(x) = x^*(G_m x) - (H_m^* x^*)(G_m x) + G_m^*(x^*)(H_m x).$$

Hence

$$|G_m^* x^*(x)| \le (K + 4\Delta + 8K^2\Delta)\|x\|\|x^*\|$$

so that

$$\|G_m^* x^*\| \le (K + 4\Delta + 8K^2\Delta)\|x^*\|.$$

This implies $(e_n^*)$ is a quasi-greedy basic sequence, and proves (1) implies (2).

Of course (2) implies (3), so it remains to prove (3) implies (1). By Theorem 3.4, (3) implies that both $(e_n)$ and $(e_n^*)$ are quasi-greedy and conservative. Let us assume that $K$ is a quasi-greedy constant for both $(e_n)$ and $(e_n^*)$, and that $\Gamma$ is a conservative constant for both $(e_n)$ and $(e_n^*)$.

Suppose $A$ is any finite subset of $\mathbf{N}$. For $x \in [e_j]_{j \notin A}$, let $y = \sum_{j \in A} e_j + x$. First suppose that $|e_j^*(x)| \neq 1$ for all $j$. Then

$$\begin{aligned}\left\|\sum_{j \in A} e_j\right\| &\leq \left\|\sum_{|e_j^*(y)| \leq 1} e_j^*(y) e_j\right\| + \left\|\sum_{|e_j^*(y)| < 1} e_j^*(y) e_j\right\| \\ &\leq 2K\|y\|.\end{aligned}$$

By continuity, $\|\sum_{j \in A} e_j\| \leq 2K\|y\|$ for all $x \in [e_j]_{j \notin A}$. Thus, by Nikol'skii's Duality Theorem (see, e.g., [7]), there exists $x^* \in [e_j^*]_{j \in A}$ with $\|x^*\| = 1$ and

$$\left|x^*\left(\sum_{j \in A} e_j\right)\right| \geq \frac{1}{2K}\left\|\sum_{j \in A} e_j\right\|. \tag{5.3}$$

Now suppose $m \in \mathbf{N}$. Choose $A_0, B_0$ with $|A_0|, |B_0| \leq m$ and

$$\left\|\sum_{j \in A_0} e_j\right\| \geq \tfrac{1}{2}\varphi(m), \qquad \left\|\sum_{j \in B_0} e_j^*\right\| \geq \tfrac{1}{2}\varphi^*(m).$$

Now let $A$ be any subset of $\mathbf{N}$ with $|A| = 2m$ and $A > \max(A_0, B_0)$.

Note that if $D \subset A$ and $|D| \geq m$, then since $(e_n)$ and $(e_n^*)$ are conservative with constant $\Gamma$,

$$\left\|\sum_{j \in D} e_j\right\| \geq \frac{1}{2\Gamma}\varphi(m), \qquad \left\|\sum_{j \in D} e_j^*\right\| \geq \frac{1}{2\Gamma}\varphi^*(m). \tag{5.4}$$

Let us choose $u^* \in [e_j^*]_{j \in A}$ such that $\sum_{j \in A} |u_j^*(e_j)|^2$ is minimized subject to $\|u^*\| \leq 1$ and

$$\sum_{j \in A} u^*(e_j) \geq \frac{\varphi(m)}{4\Gamma K}. \tag{5.5}$$

This is possible by (5.3) and (5.4).

Now let $G_m^*(u^*) = \sum_{j \in B} u^*(e_j) e_j^*$ where $B \subset A$ and $|B| = m$. Let $D = A \backslash B$. We observe that by (2.7) we have

$$\min_{j \in B} |u^*(e_j)| \left\|\sum_{j \in B} e_j^*\right\| \leq 4K^2$$

and hence, by (5.4),

$$\min_{j \in B} |u^*(e_j)| \leq \frac{8K^2\Gamma}{\varphi^*(m)}. \tag{5.6}$$

We then again use (5.3) to find $v^* \in [e_j^*]_{j \in D}$ with $\|v^*\| = 1$ and

$$\sum_{j \in D} v^*(e_j) \geq \frac{\varphi(m)}{4\Gamma K}.$$

It follows from the minimality assumption on $u^*$ that

$$\sum_{j\in A}((1-t)u^*(e_j)+tv^*(e_j))^2 \geq \sum_{j\in A}(u^*(e_j))^2$$

for $0 \leq t \leq 1$ and so, using (2.8) and (5.6),

$$\begin{aligned}\sum_{j\in A} u^*(e_j)^2 &\leq \sum_{j\in A} u^*(e_j)v^*(e_j)\\ &\leq \min_{j\in B}|u^*(e_j)|\sum_{j\in D}|v^*(e_j)|\\ &\leq \frac{8K^2\Gamma}{\varphi^*(m)}\max_{\varepsilon_j=\pm1}\left\|\sum_{j\in D}\varepsilon_j e_j\right\|\\ &\leq \frac{16K^2\Gamma\varphi(m)}{\varphi^*(m)}.\end{aligned}$$

Thus, from (5.5),

$$\begin{aligned}(\varphi(m))^2 &\leq 2^4\Gamma^2K^2\left(\sum_{j\in A}|u^*(e_j)|\right)^2\\ &\leq 2^4\Gamma^2K^2 m\sum_{j\in A}u^*(e_j)^2\\ &\leq \frac{2^8\Gamma^3K^4 m\varphi(m)}{\varphi^*(m)}\end{aligned}$$

which gives the estimate

$$\varphi(m)\varphi^*(m) \leq 2^8\Gamma^3K^4 m,$$

so that $(e_n)$ is bidemocratic. ■

**Corollary 5.5.** *Let $X$ be a Banach space with nontrivial type. If $(e_n)$ is an almost greedy basis of $X$, then $(e_n^*)$ is an almost greedy basic sequence in $X^*$.*

**Proof.** This follows directly from Theorem 5.4 and Proposition 4.4. ■

**Remark 5.6.** In [3] there is an example of an almost greedy basis $(e_n)$ of $\ell_1$ such that $(e_n^*)$ is not unconditional for constant coefficients, thus not quasi-greedy. The example localizes to give a quasi-greedy basis of the reflexive space $(\sum\oplus\ell_1^n)_2$ whose dual basis is not quasi-greedy. On the other hand, it follows from Corollary 4.5 and Theorem 5.4 above that in a Hilbert space the dual basis of a quasi-greedy basis is always quasi-greedy (in fact, both the basis and its dual are almost greedy).

**Corollary 5.7.** *Suppose that $(e_n)$ and $(e_n^*)$ are both partially greedy and that $\varphi(n) \asymp n$. Then $(e_n)$ is equivalent to the unit vector basis of $\ell_1$.*

**Proof.** By Theorem 5.4, $(e_n)$ is bidemocratic. Hence

$$\varphi^*(n) \asymp n/\varphi(n) \asymp 1.$$

But this implies that $(e_n^*)$ is equivalent to the unit vector basis of $c_0$, which gives the result. ■

**Example 5.8.** Let us conclude this section by showing that if $\varphi : \mathbf{N} \to (0, \infty)$ is an increasing function satisfying $\varphi(1) = 1$ and $\varphi(n)/n$ is decreasing, but failing (4.1), then it is possible to construct a Banach space with a greedy basis $(e_n)$ with a fundamental function equivalent to $\varphi(n)$ and such that the dual basic sequence $(e_n^*)$ is not greedy. This will show that the preceding theorem is, in some sense, sharp. In Example 5.10 we will show, under very mild additional conditions on $\varphi$, how to make a reflexive example.

Let us define the sequence space $X_\varphi$ to be the completion of $c_{00}$ for the norm

$$\|\xi\|_\varphi = \sup_n \sup_{\substack{|A|=n \\ n<A}} \frac{\varphi(n)}{n} \sum_{k\in A} |\xi_k|.$$

It is clear that the canonical basis is unconditional. It is also democratic since if $|A| = n$,

$$\tfrac{1}{2}\varphi(n) \le \left\| \sum_{k\in A} e_k \right\| \le \varphi(n).$$

Let us suppose the dual basic sequence $(e_n^*)$ is democratic with democratic constant $\Delta$. We note that if $A > n$, then

$$\left\| \sum_{k\in A} e_k^* \right\| \le n/\varphi(n).$$

It follows from the democratic assumption that

$$\left\| \sum_{k=1}^n e_k^* \right\| \le \Delta n/\varphi(n).$$

Now consider

$$\xi = \sum_{k=1}^n \frac{1}{\varphi(k)} e_k.$$

Clearly $\|\xi\| \le 1$ and so

$$\left\| \sum_{k=1}^n e_k^* \right\| \ge \sum_{k=1}^n \frac{1}{\varphi(k)}.$$

We deduce that

$$\sum_{k=1}^n \frac{1}{\varphi(k)} \le \Delta \frac{n}{\varphi(n)}.$$

Now, any $m, n$ with $m \geq 2$, we have

$$\begin{aligned} \frac{n}{\varphi(n)} \log m &\leq \frac{n}{\varphi(n)} \sum_{k=n}^{mn} \frac{1}{k} \\ &\leq \sum_{k=1}^{mn} \frac{1}{\varphi(k)} \\ &\leq \Delta \frac{mn}{\varphi(mn)}. \end{aligned}$$

Hence

$$\varphi(mn) \leq \frac{\Delta}{\log m} m\varphi(n).$$

For large $m$ this shows that (4.1) holds, contradicting our assumption.

**Remark 5.9.** The end of the proof of Example 5.8 actually establishes one direction of the following equivalence (the other direction is easier): $(\varphi(n))$ satisfies the URP if and only if $(1/\varphi(n))$ is *regular*, i.e., if and only if there exists $C > 0$ such that

$$\frac{1}{\varphi(n)} \geq \frac{C}{n} \sum_{j=1}^{n} \frac{1}{\varphi(j)}.$$

Regular weight sequences also arise in the theory of Lorentz spaces.

**Example 5.10.** Now let us suppose, in addition, that $\varphi(n)/n^\delta$ is increasing for some choice of $\delta > 0$. We show how to make the preceding example reflexive.

Let $\psi(n) = \varphi(n)^{1+\delta} n^{-\delta}$. Then $\psi(n)/n$ is decreasing and $\psi(n)$ is increasing. Define $X_\psi$ as in Example 5.8 for the function $\psi$. Let $\theta = (1+\delta)^{-1}$.

Let $T$ denote Tsirelson space (see [2]). For our purposes it is only necessary to know that this space is reflexive,

$$\tfrac{1}{2} n \leq \Big\| \sum_{j \in A} e_j \Big\|_T \leq n \qquad \text{if} \quad |A| = n$$

and

$$\Big\| \sum_{j \in A} e_j^* \Big\|_{T^*} \leq 2 \qquad \text{if} \quad |A| = n \text{ and } n < A.$$

Now let $Y = [T, X_\psi]_\theta$ be the space obtained by complex interpolation. Since $T$ is reflexive it follows from a result of Calderón [1] that $Y$ is reflexive. Note that $Y^* = [T^*, X_\psi^*]_\theta$.

Now suppose $A \subset \mathbf{N}$ and $|A| = n$. Then

$$\Big\| \sum_{j \in A} e_j \Big\|_Y \leq n^{1-\theta} \Big\| \sum_{j \in A} e_j \Big\|_{X_\psi}^{\theta} \leq \varphi(n).$$

On the other hand, if $n < A$ we have

$$\left\|\sum_{j\in A} e_j^*\right\|_{Y^*} \le \left\|\sum_{j\in A} e_j^*\right\|_{T^*}^{1-\theta} \left\|\sum_{j\in A} e_j^*\right\|_{X_{\psi^*}}^{\theta} \le 2\left(\frac{n}{\psi(n)}\right)^{\theta} = 2\frac{n}{\varphi(n)}.$$

Hence, for any $A$ with $|A| = 2n$, we have

$$\left\|\sum_{j\in A} e_j\right\|_Y \ge \frac{\varphi(n)}{2}.$$

Thus $(e_j)$ is democratic with fundamental function equivalent to $\varphi$. Now suppose $(e_j^*)$ is democratic with constant $\Delta$. Then

$$\left\|\sum_{j=1}^{n} e_j^*\right\|_Y \le 2\Delta\frac{n}{\varphi(n)}.$$

Now $Y^* = (T^*)^{1-\theta}(X_\psi^*)^\theta \subset Z := (\ell_\infty)^{1-\theta}(X_\psi^*)^\theta$ and so we have

$$\left\|\sum_{j=1}^{n} e_j^*\right\|_{Y^*} \ge \left\|\sum_{j=1}^{n} e_j^*\right\|_Z = \left\|\sum_{j=1}^{n} e_j^*\right\|_\infty^{1-\theta} \left\|\sum_{j=1}^{n} e_j^*\right\|_{X_\psi^*}^{\theta} = \left\|\sum_{j=1}^{n} e_j^*\right\|_{X_\psi^*}^{\theta}.$$

We deduce that

$$\left\|\sum_{j=1}^{n} e_j^*\right\|_{X_\psi^*} \le \left(2\Delta\frac{n}{\varphi(n)}\right)^{1/\theta} = (2\Delta)^{1/\theta}\frac{n}{\psi(n)}.$$

Hence, by the argument presented in Example 5.8, we have that

$$\psi(m) \le C_1\left(\frac{m}{n}\right)^{\beta}\psi(n), \qquad m > n,$$

for some $\beta < 1$ and $C_1$. Now

$$\varphi(m) \le C_1^{1/(1+\delta)}\left(\frac{m}{n}\right)^{(\beta+\delta)/(1+\delta)}\varphi(n), \qquad m > n.$$

This implies $\varphi$ satisfies (4.1), contradicting our assumption.

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