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CONSTRUCTIVE APPROXIMATION

Polynomial Approximation on Convex Subsets of $\mathbf{R}^n$

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Abstract. Let K be a closed bounded convex subset of $\mathbf{R}^n$; then by a result of the first author, which extends a classical theorem of Whitney there is a constant $w_m(K)$ so that for every continuous function f on K there is a polynomial φ of degree at most $m-1$ so that

$$|f(x)-\varphi(x)| \le w_m(K) \sup_{x,x+mh\in K} |\Delta_h^m(f;x)|.$$

The aim of this paper is to study the constant $w_m(K)$ in terms of the dimension n and the geometry of K. For example, we show that $w_2(K) \le \frac{1}{2}[\log_2 n] + \frac{5}{4}$ and that for suitable K this bound is almost attained. We place special emphasis on the case when K is symmetric and so can be identified as the unit ball of finite-dimensional Banach space; then there are connections between the behavior of $w_m(K)$ and the geometry (particularly the Rademacher type) of the underlying Banach space. It is shown, for example, that if K is an ellipsoid then $w_2(K)$ is bounded, independent of dimension, and $w_3(K) \sim \log n$. We also give estimates for w_2 and w_3 for the unit ball of the spaces ℓ_p^n where $1 \le p \le \infty$.

1. Introduction

Basic Definitions. Let K be a closed subset of $\mathbf{R}^n$ and let $\mathcal{P}_m$ denote the space of polynomials of total degree at most m. If f is a continuous function on K we set

$$E_m(f;K) := \inf_{\varphi\in\mathcal{P}_{m-1}} \max_{x\in K} |f(x)-\varphi(x)|$$

and

$$\omega_m(f) = \omega_m(f;K) := \sup_{x,x+h,\dots,x+mh\in K} |\Delta_h^m(f;x)|,$$

where

$$\Delta_h^m(f;x) := \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} f(x+jh).$$

We then define the *Whitney constant* $w_m(K)$ by

$$w_m(K) := \sup\{E_m(f) : f\in C(K) \text{ and } \omega_m(f)\le 1\}. \tag{1.1}$$

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We will mainly be interested in the case when K belongs to the class $\mathcal{C}_b(\mathbf{R}^n)$ of *bounded convex subsets* of $\mathbf{R}^n$ or to the subclass $\mathcal{SC}_b(\mathbf{R}^n)$ of all centrally symmetric convex subsets of $\mathbf{R}^n$. In the latter case, K can be identified with the closed unit ball B_X of an n-dimensional Banach space X and it is natural to write $w_m(X)$ in place of $w_m(B_X)$. As we do not consider unbounded K except in the Introduction this notation does not lead to any ambiguity.

We also define the *global Whitney constant* by

$$w_m(n) := \sup\{w_m(K) : K \in \mathcal{C}_b(\mathbf{R}^n)\}. \tag{1.2}$$

In the spirit of the classical paper of Whitney [39] who considers the case of dimension one,[1] let us consider also the constants $w_m^*(n)$ and $w_m^{**}(n)$ defined by (1.1) with $K := \mathbf{R}^n_+ := \{x \in \mathbf{R}^n : x_i \geq 0\}$ and $K := \mathbf{R}^n$, respectively. Using the techniques of Beurling (see [39]) it is easy to prove the following estimates:

$$w_m^*(n) \leq 2, \qquad w_m^{**}(n) \leq \min_{1 \leq j \leq m} 1/\binom{m}{j}. \tag{1.3}$$

In contrast, the estimates for $w_m(n)$ are not independent of dimension, and in fact $\lim_{n\to\infty} w_m(n) = \infty$ if $m \geq 2$.

The main goal of this paper is to give "good" quantitative estimates for $w_m(n)$ and for $w_m(K)$ in terms of the geometry of the set K.

Remarks. (a) The inequalities (1.3) are relatively precise. For instance, $w_2^*(2) \geq 1$. Concerning the sharpness of the second inequality, even for $n = 1$, see [39]. In fact the Beurling method yields the more general inequality $w_m(K) \leq 2$ provided K satisfies the *unbounded cone condition*. This condition means that there is an unbounded cone C with vertex at the origin so that $K + C \subset K$.

(b) The asymptotic behavior of Whitney's constants does not change if the supremum in (1.2) is taken over *all* convex subsets of $\mathbf{R}^n$. Actually, let $\tilde{w}_m(n) := \sup w_m(K)$ where K runs over all unbounded convex subsets of $\mathbf{R}^n$. Then $w_m(n-1) \leq \tilde{w}_m(n)$ while compactness arguments show that $\tilde{w}_m(n) \leq w_m(n)$.

(c) If we let

$$w_m^{(s)}(n) := \sup_{\dim X = n} w_m(X), \tag{1.4}$$

then $w_m^{(s)}(n) \leq w_m(n)$. In the case $m = 2$ we have $w_2(n) \leq C w_2^{(s)}(n)$ for some universal constant C independent of dimension. However we do not know of a similar inequality when $m > 2$.

(d) In his paper [40] Whitney also proved the finiteness of similar constants in a more general situation in which $C[0,1]$ is replaced by the space $B[0,1]$ of bounded (not necessarily measurable) functions. He also posed the problem for the space $L_0[0,1]$ of measurable functions. Let us denote by $w_m(K; B)$ (respectively, $w_m(K; L_0)$) the corresponding constants defined by (1.1) allowing f to be bounded (respectively, measurable). One can then prove the inequality

$$w_m(K; B) \leq (2^{2m} - 1) w_m(K) + 2^m. \tag{1.5}$$

[1] In this case $w_m(1) = w_m([0,1])$.

A similar inequality holds for $w_m(K; L_0)$. Since we do not use this inequality we will omit its proof.

Prior Results: The One-Dimensional Case. In [39] Whitney proved that $w_m(1) < \infty$ for all m and gave numerical estimates for $w_m(1)$ when $m \le 5$. Using a different approach, the first-named author proved the analog of the Whitney inequality for translation-invariant Banach lattices and gave, in particular, an effective but rather rough estimate of $w_m(1)$ for all m. This estimate was subsequently improved by a research team (K. Ivanov, Binev, and Takev) headed by Sendov who finally showed that $w_m(1) \le 6$ for all m. The most recent result is due to Kryakin who proved that $w_m(1) \le 2$ for all m (see [20] for the references). The only known precise result is $w_2(1) = \frac{1}{2}$.

Prior Results: The Multidimensional Case. In 1970, the first-named author [2] established the multidimensional analog of Whitney's result for translation-invariant Banach lattices. From this it follows, in particular, that $w_m(n) < \infty$ for every m, n. Later in a lecture at Moscow State University he established an estimate $w_2(n) \le C \log(n+1)$. Following this lecture S. Konyagin suggested that $w_2(X) < \infty$ for every *infinite-dimensional* Banach space X belonging to the class $\mathcal{K}$ introduced by the second author (see [12]). In particular, this implies that $w_2(\ell_p^n)$ is bounded by a constant independent of dimension if $1 < p \le \infty$. This important observation led to the authors' collaboration on the current paper.

Discussion of the Main Results. Our main results concern the Whitney constants for $m = 2$ and $m = 3$ (see Section 5 for some results when $m > 3$). In Section 3, we give a fairly precise estimate for $w_2(n)$, i.e.,

$$\tfrac{1}{2} \log_2 \left(\left[\frac{n}{2} \right] + 1 \right) \le w_2(n) \le \tfrac{1}{2} [\log_2 n] + \tfrac{5}{4}.$$

Curiously enough $w_2(n)$ is almost attained not for the unit simplex S^n but for its Cartesian square. Meanwhile for S^n we prove in Theorem 3.6 the precise asymptotics are given by

$$\lim_{n\to\infty} \frac{w_2(S^n)}{\log_2 n} = \frac{1}{4}.$$

We also consider in this section the problem of estimating $w_2(\ell_p^n)$ for $1 \le p \le \infty$. In particular, we show in Theorem 3.9 that $w_2(\ell_1^n) \sim \log n$ while $\gamma(p) := \sup_n w_2(\ell_p^n)$ is finite for $1 < p \le \infty$. More precisely, $\gamma(p)$ is equivalent up to a logarithmic factor to $(p-1)^{-1}$ when $p \downarrow 1$; surprisingly, for $2 \le p \le \infty$, the constant $\gamma(p)$ is bounded by an absolute constant. This striking difference in asymptotic behavior is explained by Theorem 3.12 which gives an upper estimate of $w_2(X)$ in terms of the type p constant $T_p(X)$ of X.

In Section 4, we consider the problem of quadratic approximation on symmetric convex bodies. In particular we show in Theorem 4.1 that

$$c_1\sqrt{n} \le w_3^{(s)}(n) \le c_2\sqrt{n}\log(n+1)$$

for some absolute constants $0 < c_1, c_2 < \infty$. As in the linear case, however, better estimates are available for ℓ_p^n-spaces. Our results in this case are consequences of Theorem 4.5 giving upper and lower estimates of $w_3(X)$ by the type 2 constant of X, $T_2(X)$ and the cotype 2 constant of X^*, $C_2(X^*)$. Actually we show that

$$c_1 C_2(X^*)^{-8} \le \frac{w_3(X)}{\log(n+1)} \le c_2 T_2(X)^2$$

for some absolute positive constants c_1, c_2. As corollaries of this inequality we have, for example, that $w_3(\ell_2^n) \sim \log(n+1)$ while

$$c_1 \log(n+1) \le w_3(\ell_\infty^n) \le c_2(\log(n+1))^2$$

with absolute constants $c_1, c_2 > 0$. See Theorem 4.3.

In Section 5 we discuss a few estimates for $w_m^{(s)}(n)$ and $w_m(\ell_p^n)$ with $m \ge 4$. In particular, we prove in Corollary 5.5 that

$$w_m^{(s)}(X) \le cn^{m/2-1} \log(n+1), \tag{1.6}$$

where $n = \dim X$ and C is an absolute constant. Once again, for ℓ_p^n-spaces, we have better estimates. For instance,

$$w_m(\ell_p^n) \le Cn^{(m-3)/2} \log(n+1)$$

for $2 \le p \le \infty$ and $m \ge 3$ while $w_m(\ell_1^n) \sim \log(n+1)$. Particularly striking is the fact that there is a dimension-free upper bound for $w_m(\ell_p^n)$ for (fixed) arbitrary m if $0 < p < 1$ as in Theorem 5.8.

Our arguments depend, in part, on some deep results of the local theory of Banach spaces. Most of them are concentrated in the proofs of Theorems 3.6 and 4.5. We also need a refinement of the main result Theorem 1.1 of paper [14] and a version of Maurey's extension principle [24] using a dual cotype 2 assumption in place of the usual type 2 assumption. The proof of the first result is presented in Section 3 while the required ingredients of the proof are presented in Section 2. This section also contains the proof of the second result and those of two results related to the *homogeneous* versions of the Whitney constants.

Let us discuss our results in connection with the *curse of dimension*, which, roughly speaking, asserts that the computational complexity of a function of n variables grows exponentially in n. In situations where this can be precisely formulated and proved it is, in general, a statement of the complexity of a universal (e.g., linear) approximation method for functions in a given class. It may be anticipated that approximation methods for individual functions can be much more efficient. In these terms we can consider $w_m(K)$ as a measure of approximation of $f \in C(K)$ satisfying $\omega_m(f; K) \le 1$ by polynomials of degree $m-1$. We can then compare $w_m(K)$ with a linearized Whitney constant $w_m^l(K)$ which is defined by

$$w_m^l(K) = \inf_L \sup_{\omega_m(f:\, K) \le 1} \|f - Lf\|_{C(K)},$$

where L runs through all linear operators $L : C(K) \to \mathcal{P}_{m-1}$. In the case when $K = B_{\ell_2^n}$ this quantity has been estimated by Tsarkov [37], who proves

$$w_m^l(K) \sim n^{(m-1)/2}.$$

Our results show that $w_m(K) \le Cn^{(m-3)/2}\log(n+1)$ for $m \ge 3$. Thus we have a marked improvement over linear methods which is especially striking when $m = 3$ since $w_3^l(K) \sim n$ but $w_3(K) \sim \log(n+1)$.

Remarks on the Infinite-Dimensional Case. There is an obvious generalization of the Whitney constant $w_m(X)$ to the case when X is an infinite-dimensional Banach space (or even quasi-Banach space). In this case it is quite possible that $w_m(X) = \infty$. Let us consider first the case when $m = 2$. We recall (see [12] or [16]) that a Banach space X is called a $\mathcal{K}$-space if, whenever $f\colon X \to \mathbf{R}$ is a quasilinear map (see Section 2), then there is a linear functional $g\colon X \to \mathbf{R}$ with $\sup\{|f(x) - g(x)| : x \in B_X\} := \|f - g\|_{B_X} < \infty$. There is a clear connection between the above condition and $w_2(X) < \infty$. However, since the definition of a $\mathcal{K}$-space allows for discontinuous f (and g) it is not clear that these conditions are equivalent. They are equivalent if X has the bounded approximation property.

For the case $m \ge 3$ it is possible to show that $w_m(X) = \infty$ for most classical spaces. More precisely, $w_m(X) = \infty$ for $m \ge 3$ if X contains uniformly complemented ℓ_p^n's for some $1 \le p \le \infty$; this includes the case when X has nontrivial type. The same conclusion can also be reached if X^* has cotype 2 and this covers the case of the space P constructed by Pisier [29] as an example of a space which does not contain uniformly complemented finite-dimensional subspaces.

For infinite-dimensional quasi-Banach spaces, this situation is quite different. For example, $w_m(\ell_p) < \infty$ for any $m \in \mathbf{N}$ and $0 < p < 1$. The case of $L_p(0,1)$ is even more remarkable, since $w_m(L_p) < \infty$ for every $m \in \mathbf{N}$ and yet the only polynomials on L_p are constant (because L_p has trivial dual). Thus if $F\colon B_{L_p} \to \mathbf{R}$ is continuous, satisfies $F(0) = 0$ and $\omega_m(F) \le 1$, then $\|F\|_{B_{L_p}} \le C$ where $C = C(m, p)$.

It is worth perhaps remarking that although the paper does not explicitly use the theory of twisted sums of Banach and quasi-Banach spaces, this theory is implicit in many of the results, and there is a clear connection with ideas in [12], [15], [17], and [32].

The Stability of the Equation $\Delta_h^m f = 0$. There is an alternative viewpoint for the results presented in this paper. It is well known that a continuous function f defined on a convex set K is a polynomial of degree $m-1$ if and only if f satisfies the functional equation $\Delta_h^m f = 0$. So the Whitney constant $w_m(K)$ can be regarded as a measure of stability of this equation. Stability problems of this type go back to the work of Hyers and Ulam. We note in this connection the work of Casini and Papini [3] and a recent preprint of Dilworth, Howard, and Roberts [5] on stability of convexity conditions.

Conjectures. The work in this paper was motivated by certain conjectures, and it may be helpful to list them here:

(1) If $m \ge 2$, then

$$w_m(n) \sim w_m^{(s)}(n) \sim n^{m/2-1}\log(n+1)$$

as $n \to \infty$.

This conjecture is proved for $m = 2$ while the upper estimate for $w_m^{(s)}(n)$ is established for all $m \ge 2$. As the lower bound for $m \ge 3$ we have only the inequalities $w_m(n) \ge w_m^{(s)}(n) \ge c\sqrt{n}$.

(2) If $m \geq 3$ and $1 \leq p < \infty$, then

$$w_m(\ell_p^n) \sim \log(n+1)$$

as $n \to \infty$.

This result is established for $p = 1$ and for $m = 3$ and $2 \leq p < \infty$ while the lower bound is established for all $m \geq 3$. It is quite possible that this conjecture is way off the mark when $m \geq 4$.

(3) $w_2(\ell_\infty^n)$ is "small." We propose the conjecture that $w_2(\ell_\infty^n) \leq 2$ for all n. The only known results are $w_2(\ell_\infty^1) = \frac{1}{2}$ and $w_2(\ell_\infty^2) = 1$. Note that if our conjecture were to hold then for every convex function f on the n-cube Q^n we would have the inequality $E_2(f; Q^n) \leq \omega_2(f; Q^n)$.

(4) If X is an infinite-dimensional Banach space, then $w_3(X) = \infty$.

2. Preliminary Results

Homogeneous Whitney Constants. Suppose that X is an n-dimensional Banach space. We consider the homogeneous version of the Whitney problem. We say that a function $f : X \to \mathbf{R}$ is *m-homogeneous* if $f(ax) = a^m f(x)$ whenever $a \in \mathbf{R}$ and $x \in X$.

Definition 2.1. The homogeneous Whitney constant $v_m(X)$ for $m \geq 2$ is the least constant so that if f is an $(m-1)$-homogeneous continuous function on X there is an $(m-1)$-homogeneous polynomial φ so that for all $x \in X$,

$$|f(x) - \varphi(x)| \leq v_m(X)\|x\|^{m-1}\omega_m(f), \tag{2.1}$$

where $\omega_m(f) = \omega_m(f; B_X)$.

If f is continuous and homogeneous (i.e., 1-homogeneous), then

$$|f(x+y) - f(x) - f(y)| \leq \omega_2(f; B_X)\max(\|x\|, \|y\|).$$

Thus f is *quasilinear* in the sense of [12]. This connection was first noticed by S. Konyagin and the following result is essentially due to him (see remarks in the Introduction):

Proposition 2.2. *If X is a finite-dimensional normed space, then*

$$v_2(X) \leq w_2(X) \leq 4v_2(X) + \tfrac{3}{2}.$$

Proof. If $f\colon X \to \mathbf{R}$ is continuous and homogeneous, then an affine function of best approximation on the ball can be taken as a linear functional, x^* say, and then $|f(x) - x^*(x)| \leq w_2(X)\omega_2(f; B_X)$ so that $v_2(X) \leq w_2(X)$.

Conversely, suppose $f\colon B_X \to \mathbf{R}$ is continuous and that $\omega_2(f) \leq 1$. Let us note that any $x, y \in B_X$ we have

$$|f(tx + (1-t)y) - tf(x) - (1-t)f(y)| \leq 2E_1(f; [x, y]) \leq 1. \tag{2.2}$$

This follows from applying Whitney's one-dimensional result to the line-segment $[x, y]$, since $w_2(1) = \frac{1}{2}$.

We define g on X by $g(x) = \frac{1}{2}\|x\|(f(x/\|x\|) - f(-x/\|x\|))$ for $x \neq 0$ and $g(x) = 0$. Then g is continuous and homogeneous. We will show first that $\omega_2(g; B_X) \leq 4$.

Suppose $x, y \in X$ are not both zero. Let

$$\lambda = \frac{\|x\|}{\|x\| + \|y\|}, \qquad \mu = \frac{\|y\|}{\|x\| + \|y\|}, \qquad \nu = \frac{\|x+y\|}{\|x\| + \|y\|},$$

and choose $u, v, w \in B_X$ so that $\|u\| = \|v\| = \|w\| = 1$ and

$$\|x\|u = x, \qquad \|y\|v = y, \qquad \text{and} \qquad \|x+y\|w = x + y.$$

Then for $\varepsilon = \pm 1$,

$$I_\varepsilon := |f(\varepsilon(\lambda u + \mu v)) - \lambda f(\varepsilon u) - f(\varepsilon v)| \leq 1$$

by applying (2.2). Similarly

$$J_\varepsilon := |f(\varepsilon(\lambda u + \mu v)) - \nu f(\varepsilon w) - (1 - \nu) f(0)| \leq 1.$$

From the definition of g we have

$$\begin{aligned} |g(x) - 2g(\tfrac{1}{2}(x+y)) + g(y)| &= |g(x) - g(x+y) + g(y)| \\ &\leq \tfrac{1}{2}\|x+y\| \sum_{\varepsilon=\pm 1} (I_\varepsilon + J_\varepsilon) \\ &\leq 2\|x+y\| \leq 4. \end{aligned}$$

Hence $\omega_2(g; B_X) \leq 4$.

This implies that there exists $x^* \in X^*$ so that if $\|x\| \leq 1$,

$$|g(x) - x^*(x)| \leq 4v_2(X).$$

We will choose $\varphi(x) = x^*(x) + f(0)$ as an affine approximation to f. If $\|x\| = 1$, then

$$\begin{aligned} |f(x) - \varphi(x)| &\leq 4v_2(X) + |f(x) - f(0) - g(x)| \\ &\leq 4v_2(X) + \tfrac{1}{2}|f(x) + f(-x) - 2f(0)| \leq 4v_2(X) + \tfrac{1}{2}. \end{aligned}$$

Now suppose $\|y\| \leq 1$. We write $y = tx$ where $\|x\| = 1$ and $0 \leq t \leq 1$. By (2.2) we have

$$|f(y) - tf(x) - (1-t)f(0)| \leq 1$$

and hence

$$|f(y) - \varphi(y)| \leq 4v_2(X) + \tfrac{3}{2}.$$

This completes the proof. ■

The following lemma gives a uniform estimate on $w_m(X)$ for all X of dimension n (see [2]):

Lemma 2.3. *For any $m \geq 2$, and any n-dimensional Banach space X,*

$$w_m(X) \leq 2 + T_{m-1}(\sqrt{n})(2 + w_m(\ell_2^n)),$$

where $T_k(t) := \cos(k \arccos t)$ is the Chebyshev polynomial of degree k.

Proof. By a well-known result of John [10] there is a Euclidean norm $\|\cdot\|_E$ on X so that

$$n^{-1/2}\|x\|_E \leq \|x\|_X \leq \|x\|_E$$

for $x \in X$. Now suppose that $f\colon B_X \to \mathbf{R}$ is continuous and $\omega_m(f) \leq 1$. Restricting f to B_E we can find a polynomial $\varphi \in \mathcal{P}_{m-1}$ with $|f(x) - \varphi(x)| \leq w_m(\ell_2^n)$ for $x \in B_E$. Fix any $x \in B_X$. By the definition of the Whitney constant and Kryakin's theorem [20] there is a polynomial $\psi \in \mathcal{P}_{m-1}(\mathbf{R})$ so that

$$|f(tx) - \psi(t)| \leq w_m([0,1]) \leq 2$$

for $|t| \leq 1$. Hence for $|t| \leq n^{-1/2}$ we have

$$|\varphi(tx) - \psi(t)| \leq 2 + w_m(\ell_2^n).$$

According to the Chebyshev inequality (see, e.g., [33, p. 108]) it follows that for $|t| \leq 1$

$$|\varphi(tx) - \psi(t)| \leq T_{m-1}(\sqrt{n})(2 + w_m(\ell_2^n)).$$

The result now follows easily. ■

Let us also note at this point that essentially the same argument gives us the following elementary estimate:

Lemma 2.4. *Let X, Y be two n-dimensional normed spaces and let $d := d(X,Y)$ be the Banach–Mazur distance between them. Then*

$$w_m(Y) \leq 2 + T_{m-1}(d)(2 + w_m(X))$$

and

$$v_m(Y) \leq d^{m-1} w_m(X).$$

Proof. We may suppose that $\| \ \|_Y$ and $\| \ \|_X$ are two norms on $\mathbf{R}^n$ so that $d^{-1}\|x\|_X \leq \|x\|_Y \leq \|x\|_X$ for $x \in \mathbf{R}^n$. The first estimate is proved just as in Lemma 2.3. The second estimate follows easily from the definition of $v_m(X)$ using (2.1). ■

We now prove a much more general version of Proposition 2.2.

Proposition 2.5. *Suppose that $m \geq 2$. Then there is a constant $C = C(m)$ (independent of X) so that for every finite-dimensional Banach space X,*

$$C^{-1} \max_{2\leq k\leq m} v_k(X) \leq w_m(X) \leq C \max_{2\leq k\leq m} v_k(X).$$

Proof. First choose for each $0 \leq i \leq m-1$ real numbers $(c_{ij})_{j=1}^m$ so that for any polynomial φ in one variable of degree at most $m-1$ we have

$$\frac{\varphi^{(i)}(0)}{i!} = \sum_{j=1}^m c_{ij}\varphi\left(\frac{j}{m}\right). \tag{2.3}$$

In particular, we have

$$\sum_{j=1}^m c_{ij}\left(\frac{j}{m}\right)^k = \delta_{ik}. \tag{2.4}$$

for $0 \leq i, k \leq m$. Hence, if $\varphi \in \mathcal{P}_{m-1}$, then $\psi(x) := \sum_{j=1}^m c_{k-1,j}\varphi(jx/m)$ is a $(k-1)$-homogeneous polynomial.

Using this, let us first prove that

$$v_k(X) \leq C(m)w_m(X), \qquad 2 \leq k \leq m. \tag{2.5}$$

In fact, if $f : X \to \mathbf{R}$ is continuous and $(k-1)$-homogeneous with $\omega_k(f) = \omega_k(f; B_X) \leq 1$, then $\omega_m(f) \leq 2^{m-k}$ and so there exists a polynomial $\varphi \in \mathcal{P}_{m-1}$ with

$$|f(x) - \varphi(x)| \leq 2^{m-k}w_m(X).$$

Now $f(x) = \sum_{j=1}^m c_{k-1,j} f(jx/m)$ by the $(k-1)$-homogeneity of f and (2.4), and this inequality leads to the estimate

$$|f(x) - \psi(x)| \leq 2^{m-k}\left(\sum_{j=1}^m |c_{k-1,j}|\right) w_m(X)$$

for $x \in B_X$ where $\psi(x) := \sum_{j=1}^m c_{k-1,j}\varphi(jx/m)$ is a $(k-1)$-homogeneous polynomial. Hence (2.5) follows.

Conversely, let $V := \max_{2\leq k\leq m} v_k(X)$. Suppose $f \in C(B_X)$ with $\omega_m(f) \leq 1$. Then for each x with $\|x\| = 1$ and $1 \leq k \leq m-1$ we define $g_k(x) = \sum_{j=1}^m c_{kj} f(jx/m)$ and extend g_k to be k-homogeneous. It is easy to see that each g_k is continuous. We also let $g_0(x) = f(0)$ for all $x \in X$.

By the one-dimensional result [20] for each x with $\|x\| = 1$ there is a polynomial φ on [0, 1] of degree at most $m-1$ so that

$$|f(tx) - \varphi(t)| \leq 4$$

for $0 \leq t \leq 1$. Hence

$$\left|g_k(x) - \frac{\varphi^{(k)}(0)}{k!}\right| \leq 4\max\left(1, \sup_{1\leq l\leq m-1}\sum_{j=1}^m |c_{lj}|\right) \leq C_1,$$

where $C_1 = C_1(m)$. Then, for any $x \in B_X$ we have

$$\left| f(x) - \sum_{k=0}^{m-1} g_k(x) \right| \le 4 + mC_1 = C_2.$$

Using (2.4) for $1 \le k \le m-1$, we have the identity

$$\sum_{j=1}^{m} c_{kj} f\left(\frac{jx}{m}\right) - g_k(x) = \sum_{j=1}^{m} c_{kj} \left[f\left(\frac{jx}{m}\right) - \sum_{s=0}^{m-1} g_s\left(\frac{jx}{m}\right) \right]$$

and we can deduce

$$\left| g_k(x) - \sum_{j=1}^{m} c_{kj} f\left(\frac{jx}{m}\right) \right| \le C_2 \sum_{j=1}^{m} |c_{kj}| \le C_3(m).$$

Hence

$$\omega_m(g_k) \le 2^m C_3 + \sum_{j=1}^{m} |c_{kj}| = C_4 \tag{2.6}$$

for $1 \le k \le m-1$.

We now deduce from (2.6) that

$$\omega_{k+1}(g_k) \le C_5(m) \tag{2.7}$$

for $1 \le k \le m-1$. Indeed, let $x, x+(k+1)h \in B_X$ and let $F := \operatorname{span}\{x,h\}$ be the linear space generated by x, h. By Lemma 2.3 and the multivariate Whitney-type inequality (in dimension 2) [2] we can find a polynomial ψ_F of degree at most $m-1$ so that

$$|g_k(y) - \psi_F(y)| \le C_6 \omega_m(g_k)$$

for $y \in B_F$ where $C_6 = C_6(m)$. But, arguing as before, we can replace ψ_F by $\sum_{j=1}^{m} c_{kj} \psi_F(jx/m)$ and this allows us to assume that ψ_F is homogeneous of degree k (by similar arguments to those used above.) Hence

$$|\Delta_h^{k+1} g_k(x)| = |\Delta_h^{k+1}(g_k - \psi_F)(x)| \le 2^{k+1} C_6 \omega_m(g_k).$$

Combining with (2.6) we get (2.7). Then we can conclude that there is a k-homogeneous polynomial ψ_k on X so that

$$|g_k(x) - \psi_k(x)| \le C_7(m) V$$

for $x \in B_X$. Finally, if we set $\psi(x) = g_0(x) + \sum_{k=1}^{m-1} \psi_k(x)$, then

$$|f(x) - \psi(x)| \le (C_2 + mC_7) \le C_8(m) V$$

for $\|x\| \le 1$ and so $w_m(X) \le CV$ for a constant C depending only on m. ∎

Corollary 2.6. *If $2 \le l \le m$ there is a constant $C = C(l, m)$ so that*

$$w_l(X) \le C(l, m) w_m(X).$$

Remark. All the above results are clearly true (with constants also depending on r) for r-normed finite-dimensional spaces. Recall (see [16]) that $\|\cdot\|$ is an r-norm on X if we have:

(1) $\|x\| \ge 0$ with equality if and only if $x = 0$;
(2) $\|ax\| = |a|\|x\|$ for $a \in \mathbf{R}$ and $x \in X$; and
(3) $\|x_1 + x_2\|^r \le \|x_1\|^r + \|x_2\|^r$ for $x_1, x_2 \in X$.

We note only that in the proof of Lemma 2.3, John's theorem is replaced by its r-normed generalization due to Peck [27].

Indicators of Finite-Dimensional Banach Lattices. Let $X = \{\mathbf{R}^{n+1}, \|\cdot\|_X\}$ be an $(n+1)$-dimensional Banach lattice. In our setting this simply implies that if $x = (x_i)_{i=1}^{n+1}$ and $y = (y_i)_{i=1}^{n+1}$ with $|x| \le |y|$ (i.e., $|x_i| \le |y_i|$ for $i = 1, 2, \dots, n+1$), then $\|x\|_X \le \|y\|_X$.

Definition 2.7 ([14]). The indicator Φ_X of X is the function defined on the simplex $S^n := \{u \in \mathbf{R}^{n+1} : u \ge 0,\ \sum_{i=1}^{n+1} u_i = 1\}$ by

$$\Phi_X(u) := \sup_{\|x\|_X \le 1} \sum_{i=1}^{n+1} u_i \log_2 |x_i|. \tag{2.8}$$

Here we set $0 \log_2 0 = 0$. We remark first that we use logarithms base two in place of natural logarithms as in [14] for convenience. We also remark that in [26] the same function is called the *entropy function* of X.

We denote by Λ the functional $\Lambda(u) = \sum_{i=1}^{n+1} u_i \log_2 |u_i|$. Let us note the following straightforward properties of Φ_X:

Proposition 2.8.

(a) $\Phi_{\ell_1^{n+1}} = \Lambda$.
(b) *If $a_i > 0$ for $1 \le i \le n+1$, $1 \le p < \infty$, and $\ell_p^{n+1}(a)$ is defined by the norm*

$$\|x\|_{\ell_p^{n+1}(a)} := \left(\sum_{i=1}^{n+1} a_i^p |x_i|^p \right)^{1/p},$$

then

$$\Phi_{\ell_p^{n+1}(a)}(u) = \frac{1}{p} \left(\Lambda(u) - \sum_{i=1}^{n+1} u_i \log_2 a_i \right).$$

(c) *If* $\|\cdot\|_X$ *and* $\|\cdot\|_Y$ *are* C*-equivalent, i.e.,* $C^{-1}\|x\|_X \le \|x\|_Y \le C\|x\|_X$ *for all* $x \in \mathbf{R}^{n+1}$*, then*

$$|\Phi_X(u) - \Phi_Y(u)| \le \log_2 C$$

for $u \in S^n$.

Let us use $\langle x, y\rangle$ to denote the standard inner-product on $\mathbf{R}^{n+1}$. Then if X is a Banach lattice we define the dual space X^* by

$$\|x^*\|_{X^*} := \sup\{|\langle x^*, x\rangle| : \|x\|_X \le 1\}.$$

If X_0, X_1 are two $(n+1)$-dimensional Banach lattices we define the (Calderón) interpolation space $X_\theta = X_0^{1-\theta} X_1^\theta$ for $0 < \theta < 1$ by

$$\|x\|_{X_\theta} := \inf\{\|x_0\|_{X_0}^{1-\theta}\|x_1\|_{X_1}^{\theta}\}$$

where the infimum is taken over all $x_0, x_1 \in \mathbf{R}^{n+1}$ satisfying

$$|x| \le |x_0|^{1-\theta}|x_1|^\theta.$$

The following results are taken from [14]:

Theorem 2.9. (a) *For any Banach lattice* X *on* $\mathbf{R}^{n+1}$,

$$\Phi_X + \Phi_{X^*} = \Lambda.$$

(b) *If* X_0, X_1 *are two Banach lattices on* $\mathbf{R}^{n+1}$*, then*

$$\Phi_{X_0^{1-\theta}X_1^\theta} = (1-\theta)\Phi_{X_0} + \theta\Phi_{X_1}.$$

Note that (b) is a simple consequence of the definitions, while (a) follows from the deep duality theorem of Lozanovskii [23] (which is essentially equivalent to the statement that $X^{1/2}(X^*)^{1/2} = \ell_2^{n+1}$ for any Banach lattice X. It is not hard to see that Φ_X is a convex function satisfying $\delta_2(\Phi_X) \le 1$ where $\delta_2\colon C(S^n) \to \mathbf{R}$ is defined by

$$\delta_2(f) := \sup\{|f(\alpha u + (1-\alpha)v) - \alpha f(u) - (1-\alpha)f(v)|\},$$

where the supremum is taken over all $0 \le \alpha \le 1$ and $u, v \in S^n$.

The main result of [14] gives, in our setting, a form of converse to this statement.

Theorem 2.10. *For each* $0 < \varepsilon < \frac{1}{2}$ *there is a constant* $C = C(\varepsilon)$ *so that whenever* $n \in \mathbf{N}$*, and* $f \in C(S^n)$ *satisfies* $\delta_2(f) \le 1 - \varepsilon$ *there is a Banach lattice* X *so that*

$$|f(u) - (\Phi_X(u) - \Phi_{X^*}(u))| \le C$$

for all $u \in S^n$.

One of our goals is to refine this result to give a very general representation for functions on S^n in terms of the parameter $\omega_2(f)$. This will be achieved in Theorem 3.7 below.

Extension Theorems of Maurey Type. We recall that if X is a Banach space and $1 < p \leq 2$, then X is said to have *type* p if there is a constant C so that for any $x_1, \dots, x_n \in X$ we have

$$\left(\mathrm{Ave}_{\varepsilon_i=\pm 1}\left\|\sum_{i=1}^n \varepsilon_i x_i\right\|^p\right)^{1/p} \leq C\left(\sum_{i=1}^n \|x_i\|^p\right)^{1/p}.$$

The best constant C is called the type p constant of X and denoted by $T_p(X)$.

X is said to have *cotype* q where $2 \leq q < \infty$ if there is a constant C so that for any $x_1, \dots, x_n \in X$ we have

$$\left(\sum_{i=1}^n \|x_i\|^q\right)^{1/q} \leq C\left(\mathrm{Ave}_{\varepsilon_i=\pm 1}\left\|\sum_{i=1}^n \varepsilon_i x_i\right\|^q\right)^{1/q}.$$

The best such constant is denoted by $C_q(X)$.

We remark that if $\dim X = n$ then we have $T_p(X) \leq n^{1/p}$ and $C_q(X) \leq n^{1/p}$ where $1/p + 1/q = 1$. We also have a duality relationship, namely $C_q(X^*) \leq T_p(X)$.

Let X and Y be finite-dimensional Banach spaces and suppose E is a linear subspace of X.

Definition 2.11. The extension constant $\mathcal{E}_X(E, Y)$ is infimum of all constants M so that every linear map $T : E \to Y$ has a linear extension $T_1 : X \to Y$ with $\|T_1\| \leq M\|T\|$.

The Maurey extension principle [24] gives the following estimate for $Y = \ell_2^m$:

$$\mathcal{E}_X(E, \ell_2^m) \leq T_2(X). \tag{2.9}$$

In order to extend this principle to non-Hilbertian Y we can use the abstract Grothendieck theorem of Pisier. This states [30, Theorem 4.1] that if $T : E \to Y$ is a linear map then there is a factorization $V : X \to \ell_2^m$ and $U : \ell_2^m \to Y$ so that $\|U\|\|V\| \leq (2C_2(E^*)C_2(Y))^{3/2}$. (In fact, we can do a little better, i.e., we can obtain $\|U\|\|V\| \leq CC_2(X^*)C_2(Y)(1 + \log C_2(X^*)C_2(Y))$.) Putting these estimates together we obtain

$$\mathcal{E}_X(E; Y) \leq (2C_2(E^*)C_2(Y))^{3/2} T_2(X). \tag{2.10}$$

We will need an analogous result with a cotype assumption on X^* in place of the type restriction on X. The following result may be known to specialists but we have not been able to find it in the literature:

Theorem 2.12. *There is an increasing function* $\psi : (1, \infty) \to (1, \infty)$ *so that*

$$\mathcal{E}_X(E, \ell_2^m) \leq \psi(T_2(X/E))C_2(X^*). \tag{2.11}$$

Proof. Suppose that $T_0 : E \to \ell_2^m$ with $\|T_0\| \le 1$. We need to find an extension $T : X \to \ell_2^m$ of T_0 with norm majorized by the right-hand side of (2.11). To do this we follow an extension technique of Kisliakov which is used heavily in [17]. Consider the space $Z = X \oplus_1 \ell_2^m$, i.e., $Z = X \times \ell_2^m$ algebraically with norm $\|(x, y)\|_Z = \|x\|_X + \|y\|_{\ell_2^m}$. Then $Z^* = X^* \oplus_\infty \ell_2^m$, i.e., $Z^* = X^* \times \ell_2^m$ with norm $\|(x^*, y^*)\|_{Z^*} = \max(\|x^*\|_{X^*}, \|y^*\|_{\ell_2^m})$. Since $C_2(\ell_2^m) = 1$ we have

$$C_2(Z^*) \le \sqrt{2} C_2(X^*). \tag{2.12}$$

Let $G := \{(x, -T_0 x) : x \in E\} \subset Z$. Let $Y := Z/G$ and let $Q : Z \to Y$ be the quotient map. Note that Q maps $\{0\} \times \ell_2^m$ isometrically onto a subspace H of Y and that by (2.9) there is a projection $P : Y \to H$ with $\|P\| \le T_2(Y)$. Let $S : X \to Z$ be defined by $S(x) := (x, 0)$. Then PQS can be regarded as an extension of T_0; more precisely, $T := \mathrm{Pr}_2(Q^{-1}PQS)$ extends T_0 where $\mathrm{Pr}_2(x, y) := y$ and Q^{-1} is the inverse of Q on $\{0\} \times \ell_2^m$. Then $\|T\| \le \|P\| \le T_2(Y)$. It therefore remains only to estimate $T_2(Y)$.

Fix $1 < p < 2$. Note that Y/H is isometric to X/E. Hence by arguments that go back to paper [6] (see [13] for details) we have the estimate $T_p(Y) \le \varphi(T_2(X/E))$ for a suitable increasing function $\varphi : (1, \infty) \to (1, \infty)$. Now as a direct consequence of Pisier's characterization of K-convex spaces [31] we also have that an estimate on the K-convexity constant of Y in terms of $T_p(Y)$. Hence we get an estimate of the form

$$T_2(Y) \le \varphi_p(T_p(Y)) C_2(Y^*)$$

for a suitable increasing $\varphi_p : (1, \infty) \to (1, \infty)$. Putting these estimates together we have

$$\|T\| \le \psi(T_2(X/E)) C_2(Y^*),$$

where $\psi := \varphi_p \circ \varphi$. It remains to observe that $C_2(Y^*) \le C_2(Z^*) \le \sqrt{2} C_2(X^*)$ and we are done. ■

Using this theorem and Pisier's result as in (2.10) we have

Corollary 2.13.

$$\mathcal{E}_X(E; Y) \le \psi(T_2(X/E)) C_2(X^*) C_2(E^*)^{3/2} C_2(Y)^{3/2}.$$

3. Linear Approximation on Convex Subsets of $\mathbf{R}^n$

We begin with the proof of the basic estimate for $w_2(n)$ when $n \ge 2$. We recall that $w_2(1) = \frac{1}{2}$.

Theorem 3.1. *We have the estimate*

$$\tfrac{1}{2} \log_2 \left(\left[\frac{n}{2} \right] + 1 \right) \le w_2(n) \le \tfrac{1}{2} [\log_2 n] + \tfrac{5}{4}.$$

In particular,

$$\lim_{n \to \infty} \frac{w_2(n)}{\log_2 n} = \frac{1}{2}.$$

Remark. See [4], [9], and [5] for results on the corresponding problem for convex functions.

In the following discussion K will denote a closed bounded convex subset of $\mathbf{R}^n$. Note however that our first proposition does not need convexity.

Proposition 3.2. *If $f \in C(K)$, then*

$$E_2(f;K) = \tfrac{1}{2}\max\left\{\sum_{i=1}^{l} a_i f(x_i) - \sum_{j=1}^{m} b_j f(x_j)\right\},$$

where the maximum is computed over all pairs of positive integers l, m with $l+m \le n+2$, all subsets $\{x_1,\dots,x_l\}$, $\{y_1,\dots,y_m\}$ of K and all nonnegative reals $a_1,\dots,a_l, b_1,\dots,b_m$ with

$$\sum_{i=1}^{l} a_i = \sum_{j=1}^{m} b_j = 1 \qquad \textit{and} \qquad \sum_{i=1}^{l} a_i x_i = \sum_{j=1}^{m} b_j y_j.$$

Proof. We may choose φ affine so that $E_2(f-\varphi;K) = \|f-\varphi\|_K$. Then clearly $E_2(f-\varphi;K)$ dominates the expression on the right of the equation. To prove the converse, we observe (see, e.g., [34, p. 36]) that there exist nonempty subsets Σ_+ and Σ_- of K so that $|\Sigma_+|+|\Sigma_-| \le n+2$ and $(\operatorname{co}\Sigma_+)\cap(\operatorname{co}\Sigma_-) \ne \emptyset$ and so that for $x \in \Sigma_\pm$ we have

$$f(x) - \varphi(x) = \pm E_2(f;K).$$

Let $\Sigma_+ = \{x_1,\dots,x_l\}$ and $\Sigma_- = \{y_1,\dots,y_m\}$ then $l+m \le n+2$ and we can find convex combinations so that $\sum_{i=1}^{l} a_i x_i = \sum_{j=1}^{m} b_j y_j$. Then

$$\begin{aligned} E_2(f;K) &= \frac{1}{2}\left\{\sum_{i=1}^{l} a_i(f(x_i)-\varphi(x_i)) - \sum_{j=1}^{m} b_j(f(y_j)-\varphi(y_j))\right\} \\ &= \frac{1}{2}\left\{\sum_{i=1}^{l} a_i f(x_i) - \sum_{j=1}^{m} b_j f(y_j)\right\}. \end{aligned}$$ ∎

Let us define $\delta_m : C(K) \to \mathbf{R}$ (extending the definition of δ_2) by

$$\delta_m(f) := \sup\left| f\left(\sum_{k=1}^{m} a_k x_k\right) - \sum_{k=1}^{m} a_k f(x_k)\right|, \tag{3.1}$$

where the supremum is taken over all $x_1,\dots,x_m \in K$ and $a_1,\dots,a_m \in \mathbf{R}_+$ such that $\sum_{k=1}^{m} a_k = 1$. Let

$$\alpha_m(K) = \sup\{\delta_{m+1}(f) : f \in C(K),\ \omega_2(f;K) \le 1\}.$$

We then have:

Corollary 3.3.

$$E_2(f;K) \leq \tfrac{1}{2}\max\{\alpha_l(K)+\alpha_m(K)\colon\ l,m\geq 0,\ l+m=n\}.$$

Observe we have a trivial inequality $\alpha_m(K) \leq \alpha_m(S^m) =: \beta_m$ where S^m is, as usual, the m-dimensional simplex. Thus, combining with Corollary 3.3, we obtain the inequality

$$w_2(n) \leq \tfrac{1}{2}\max\{\beta_l+\beta_m\colon\ l+m=n\}. \tag{3.2}$$

Note that by Proposition 3.2, $\delta_{m+1}(f) \leq 2E_2(f;K)$ and so $\beta_m \leq 2w_2(m)$. In particular, $\beta_1 \leq 1$ by the results of Whitney [39]. To obtain an estimate for all m we need:

Lemma 3.4.

$$\beta_{2m} \leq \beta_m + \tfrac{1}{2}, \qquad m\in \mathbf{N}.$$

Proof. We set $S^m := \mathrm{co}\{e_1,\dots,e_{m+1}\}$ where $e_1,\dots,e_{m+1}$ is the canonical basis of $\mathbf{R}^{m+1}$. Replacing f by $f-\varphi$ where φ is an affine function satisfying $\varphi(e_k)=f(e_k)$ for $1\leq k\leq m+1$ we can obtain an alternate expression for β_m:

$$\beta_m = \sup\{|f(x)| : \omega_2(f)\leq 1 \text{ and } f(e_k)=0,\ 1\leq k\leq m+1\}. \tag{3.3}$$

Suppose then $f\in C(S^{2m})$ satisfies $\omega_2(f)\leq 1$ and $f(e_k)=0$ for $1\leq k\leq m+1$. Choose $x\in S^{2m}$. Let $x=\sum_{k=1}^{2m+1}\xi_k e_k$. Let $(r_k)_{k=1}^{2m+1}$ be a reordering of $\{1,2,\dots,2m+1\}$ so that ξ_{r_k} is increasing. Then we may choose signs $(\varepsilon_k)_{k=1}^m$ so that

$$0\leq a := \sum_{k=1}^m \varepsilon_k(\xi_{r_{2k-1}}-\xi_{r_{2k}}) \leq \max_{1\leq k\leq m}(\xi_{r_{2k}}-\xi_{r_{2k-1}}).$$

Then we can write $x=\frac{1}{2}(y+z)$ where

$$y = x+\sum_{k=1}^m \varepsilon_k(\xi_{r_{2k-1}}e_{r_{2k-1}}-\xi_{r_{2k}}e_{r_{2k}}) - ae_{r_{2m+1}}$$

and

$$z = x-\sum_{k=1}^m \varepsilon_k(\xi_{r_{2k-1}}e_{r_{2k-1}}-\xi_{r_{2k}}e_{r_{2k}}) + ae_{r_{2m+1}}.$$

Hence

$$|f(x)-\tfrac{1}{2}(f(y)+f(z))| \leq \tfrac{1}{2}.$$

If $y=\sum_{k=1}^{2m+1}\eta_k e_k$, then $\eta_k>0$ at most $m+1$ times and so $|f(y)-\sum_{k=1}^{2m+1}\eta_k f(e_k)|\leq \beta_m$. With a similar estimate for z we obtain

$$\left|f(x)-\sum_{k=1}^{2m+1}\xi_k f(e_k)\right| \leq \beta_m+\tfrac{1}{2}.$$

This leads immediately to the claimed estimate. ∎

Proof of the Upper Estimate in Theorem 3.1. Since $\beta_1 = 1$, Lemma 3.4 and induction gives us that $\beta_m \leq \frac{1}{2}\log_2 m + 1 = \frac{1}{2}k + 1$ when $m = 2^k$. Now suppose $2^k \leq n < 2^{k+1}$. Clearly if $l + m = n$ then at most one of them exceeds 2^k. Hence $\beta_l + \beta_m \leq \beta_{2^k} + \beta_{2^{k+1}} \leq k + \frac{5}{2}$. Applying inequality (3.2) we get the estimate $w_2(n) \leq \frac{1}{2}[\log_2 n] + \frac{5}{4}$.

For the lower estimate, we require the following general result:

Lemma 3.5. *Suppose K_1, K_2 are closed bounded convex subsets of $\mathbf{R}^{n_1}, \mathbf{R}^{n_2}$, respectively. Suppose $f_i \in C(K_i)$ for $i = 1, 2$ are convex and satisfy $\omega_2(f_i; K_i) \leq 1$. Then if $g : K_1 \times K_2 \to \mathbf{R}$ is defined by $g(x, y) = f_1(x) - f_2(y)$ we have*:

(a) $E_2(g; K_1 \times K_2) = E_2(f_1; K_1) + E_2(f_2; K_2)$; *and*
(b) $\omega_2(g; K_1 \times K_2) \leq 1$.

Proof. Suppose $h = (h_1, h_2) \in \mathbf{R}^{n_1} \times \mathbf{R}^{n_2}$. Then

$$\Delta_h^2 g(x, y) = \Delta_{h_1}^2 f_1(x) - \Delta_{h_2}^2 f_2(y).$$

Since f_1, f_2 are convex we obtain $\omega_2(g; K_1 \times K_2) \leq 1$. This proves (b). To prove (a) it suffices to apply Proposition 3.2 (see Theorem 6.2.5 in [34]). ■

Proof of the Lower Estimate in Theorem 3.1. Let $S^n = \mathrm{co}\{e_1, \ldots, e_{n+1}\}$ as before. Define the function

$$f_n(x) := \tfrac{1}{2}\Lambda(x) = \frac{1}{2}\sum_{k=1}^{n+1} x_k \log_2 x_k. \tag{3.4}$$

Since the function $\psi(t) := t\log_2 t,\ 0 \leq t \leq 1$, satisfies $0 \leq \Delta_h^2 f(t) \leq \Delta_{|h|}^2 f(0) = 2|h|$, the function f_n is convex and

$$0 \leq \Delta_h^2 f_n(x) = \frac{1}{2}\sum_{k=1}^{n+1} \Delta_{h_k}^2 \psi(x_k) \leq \sum_{k=1}^{n+1} |h_k|.$$

Now $h = \frac{1}{2}((x + 2h) - x)$ so that $\sum_{k=1}^{n+1} |h_k| \leq 1$. Thus $\omega_2(f_n) \leq 1$. Let $u := [1/(n+1)]\sum_{k=1}^{n+1} e_k$. Then by Proposition 3.2,

$$E_2(f_n) \geq \frac{1}{n+1}\sum_{k=1}^{n+1} f_n(e_k) - f_n(u) = \tfrac{1}{4}\log_2(n+1).$$

We remark that this function was essentially first considered in this context (in an equivalent formulation) by Ribe [32].

We can now apply Lemma 3.5. If $n = 2m$, putting $K_1 = K_2 = S^m$ and using f_n for both f_1 and f_2 of the lemma, we obtain the existence of g on $S^n \times S^n$ with $\omega_2(g) \leq 1$ and $E_2(g) \geq \frac{1}{2}\log_2(m+1)$. Hence $\omega_2(n) \geq \frac{1}{2}\log_2(n/2 + 1)$. If $n = 2m + 1$ we put $K_1 = S^m$, and $K_2 = S^{m+1}$ and use f_m, f_{m+1} to deduce that

$$\omega_2(n) \geq \tfrac{1}{4}(\log_2(m+1) + \log_2(m+2)) \geq \tfrac{1}{2}\log_2\left(\left[\frac{n}{2}\right] + 1\right).$$

The proof of Theorem 3.1 is now complete. ■

Remarks. (a) For small values of n we can use (3.2) directly to obtain better upper bounds for $w_2(n)$. Thus $\beta_2 \le \frac{3}{2}$, $\beta_3, \beta_4 \le 2$ and hence $w_2(2) \le 1$, $w_2(3) \le \frac{5}{4}$, and $w_2(4) \le \frac{3}{2}$.

On the other hand, if we use the piecewise linear function $f_\varepsilon(t) = \max((1-\varepsilon)(1-\varepsilon^{-1}t), 0)$ on $[0, 1]$, then f_ε is convex and satisfies $\omega_2(f_\varepsilon) = 1$ and $E_2(f_\varepsilon) = \frac{1}{2}(1-\varepsilon)$. Then using Lemma 3.5 and the functions $g_\varepsilon(x, y) = f_\varepsilon(x) - f_\varepsilon(y)$ we obtain that $w_2([0, 1]^2) \ge 1 - \varepsilon$. Combined with the upper estimates above we obtain

$$w_2(2) = w_2([0, 1]^2) = 1. \tag{3.5}$$

Notice that $w_2([0, 1]^2) = w_2(\ell_\infty^2) = w_2(\ell_1^2)$.

(b) The corresponding examples considered in [8] show that $\beta_2 = \frac{5}{3}$ and $\beta_3 = 2$.

We now show that for the case of the simplex the lower bound $\frac{1}{4}\log_2(n+1)$ is asymptotically sharp. More precisely:

Theorem 3.6.

$$\lim_{n\to\infty} \frac{w_2(S^n)}{\log_2 n} = \frac{1}{4}.$$

We remark first that the functions f_n constructed in (3.4) show that $w_2(S^n) \ge \frac{1}{4}\log_2(n+1)$ so that

$$\liminf_{n\to\infty} \frac{w_2(S^n)}{\log_2 n} \ge \frac{1}{4}. \tag{3.6}$$

The proof of Theorem 3.6 will follow from the following Theorem 3.7.

Theorem 3.7. *For any $0 < \varepsilon < \frac{1}{2}$ there is a constant $C = C(\varepsilon)$ such that whenever $n \in \mathbf{N}$, and $f \in C(S^n)$ satisfies $\omega_2(f) \le 1 - \varepsilon$ there is a Banach lattice X so that*

$$|f(u) - \tfrac{1}{2}(\Phi_X(u) - \Phi_{X^*}(u))| \le C$$

for all $u \in S^n$.

Before proving 3.7 let us complete the proof of Theorem 3.6 assuming Theorem 3.7:

Proof of Theorem 3.6. Fix $\varepsilon > 0$. If $f \in C(S^n)$ satisfies $\omega_2(f) \le 1-\varepsilon$, we determine X so that Theorem 3.7 holds. Let $\|\cdot\|_E$ be the Hilbertian norm determined by the John ellipsoid for B_X [10]. Then in the terminology of Proposition 2.8 we must have $E = \ell_2(a)$ for a suitable positive sequence $a = (a_1, \dots, a_{n+1})$. Then by Proposition 2.8 we have that $\Phi_E - \Phi_{E^*}$ is linear: in fact, $\Phi_E(u) - \Phi_{E^*}(u) = -2\langle u, \log a\rangle$.

From the properties of the John ellipsoid we have $B_E \subset B_X \subset (n+1)^{1/2} B_E$ so that $\Phi_E(u) \le \Phi_X(u) \le \Phi_E(u) + \frac{1}{2}\log_2(n+1)$. From Theorem 2.9(a) we get

$$\Phi_{E^*}(u) - \tfrac{1}{2}\log_2(n+1) \le \Phi_{X^*}(u) \le \Phi_{E^*}(u)$$

and so

$$|\tfrac{1}{2}(\Phi_X(u) - \Phi_{X^*}(u)) + \langle u, \log a\rangle| \le \tfrac{1}{4}\log_2(n+1).$$

It follows that

$$E_2(f) \leq C(\varepsilon) + \tfrac{1}{4}\log_2(n+1).$$

This implies that

$$w_2(S^n) \leq (1-\varepsilon)^{-1}(C(\varepsilon) + \tfrac{1}{4}\log_2(n+1)),$$

which in turn gives the required upper estimate

$$\limsup_{n\to\infty} \frac{w_2(S^n)}{\log_2 n} \leq \frac{1}{4}.$$

This completes the proof of Theorem 3.6. ■

We now turn to the proof of Theorem 3.7.

Proof. Let $f\colon S^n \to \mathbf{R}$ be a bounded function satisfying the condition $\omega_2(f) \leq 1-\varepsilon$ where $0 < \varepsilon < \frac{1}{2}$ is fixed. By Whitney's theorem applied to each line segment we have $\delta_2(f) \leq \omega_2(f) \leq 1-\varepsilon$. Let $\alpha := 1 - \frac{1}{2}\varepsilon$ and apply Theorem 2.10 to the function $\alpha^{-1} f$. Thus there is an $(n+1)$-dimensional Banach lattice Y with

$$\|f - \alpha(\Phi_Y - \Phi_{Y^*})\|_{S^n} \leq C(\varepsilon). \tag{3.7}$$

To complete the proof we will find a lattice X for which

$$\|\alpha(\Phi_Y - \Phi_{Y^*}) - \tfrac{1}{2}(\Phi_X - \Phi_{X^*})\|_{S^n} \leq C'(\varepsilon). \tag{3.8}$$

In order to do this we will show the existence of a Banach lattice X such that if we put $\theta := 1 - (2\alpha)^{-1}$, then the spaces Y and $X^{1-\theta}(\ell_2^{n+1})^\theta$ have equivalent norms with the constant of equivalence depending only on ε. Assuming this fact, let us show how the proof is completed. In this case, by Theorem 2.9 and Proposition 2.8, we have

$$\left\|\Phi_Y - \left((1-\theta)\Phi_X + \frac{\theta}{2}\Lambda\right)\right\|_{S^n} \leq C_1(\varepsilon).$$

Using the duality result Theorem 2.9(a) this implies that

$$\|(\Phi_Y - \Phi_{Y^*}) - (1-\theta)(\Phi_X - \Phi_{X^*})\|_{S^n} \leq 2C_1(\varepsilon).$$

Since $1 - \theta = (2\alpha)^{-1}$ this establishes (3.8) and combined with (3.7) the theorem is proved. ■

Thus it remains to construct X. We will need the following lemma:

Lemma 3.8. *Suppose p is defined by*

$$p := \left(1 + \frac{1}{2\alpha}\right)^{-1} + \left(1 + \frac{1-\varepsilon}{2\alpha}\right)^{-1}.$$

Then there is a constant C depending only on ε so that for every disjoint family of vectors $\{y_i\}_{i=1}^m \subset \mathbf{R}^{n+1}$, we have

$$\left\| \sum_{i=1}^m y_i \right\|_Y \le C \left(\sum_{i=1}^m \|y_i\|_Y^p \right)^{1/p} \tag{3.9}$$

and

$$\left\| \sum_{i=1}^m y_i \right\|_{Y^*} \le C \left(\sum_{i=1}^m \|y_i\|_{Y^*}^p \right)^{1/p}. \tag{3.10}$$

Before proving the lemma, let us show how to complete the construction of X assuming this lemma. We set

$$\frac{1}{r} := \frac{1}{2} + \frac{1}{4\alpha} = 1 - \frac{\theta}{2}. \tag{3.11}$$

Then $p > r$. By the lemma both Y and Y^* satisfy upper p-estimates with constants depending only on ε. According to a well-known theorem of Maurey and Pisier (see, e.g., [21]) this implies that Y and Y^* are both r-convex with constants depending only ε. This means that for any $y_1, \dots, y_m \in Y$ we have

$$\left\| \left(\sum_{i=1}^m |y_i|^r \right)^{1/r} \right\|_Y \le C \left(\sum_{i=1}^m \|y_i\|_Y^r \right)^{1/r},$$

where C depends only on ε, and a similar inequality holds in Y^*. Now by Propositions 1.d.4 and 1.d.8 of [21] there is a lattice Y_0 so that Y_0, Y_0^* are r-convex with constant one and the Y_0-norm is C-equivalent to the Y-norm with C depending only on ε. Finally we use the Pisier extrapolation theorem [28] to deduce that there is a Banach lattice X so that $Y_0 = X^{1-\theta}(\ell_2^{n+1})^\theta$. ■

We now turn to the proof of the lemma.

Proof. Let $g := \Phi_Y - \Phi_{Y^*}$. Using (3.7) we first estimate $\delta_m(g) \le \alpha^{-1}(C + \delta_m(f))$ where C depends only on ε. Since $\omega_2(f) \le 1 - \varepsilon$, Lemma 3.4 gives that $\delta_m(f) \le (1-\varepsilon)(\frac{1}{2}\log_2 m + 1)$. Hence

$$\delta_m(g) \le C_1 + \frac{1-\varepsilon}{2-\varepsilon} \log_2 m, \tag{3.12}$$

where $C_1 = C_1(\varepsilon)$.

Now suppose $u_1, u_2, \ldots, u_m \in S^n$ have disjoint supports and that $u = \sum_{i=1}^m a_i u_i \in S^n$ is a convex combination. Then

$$g(u) \geq \sum_{i=1}^m a_i g(u_i) - C_1 - \frac{1-\varepsilon}{2-\varepsilon} \log_2 m.$$

By duality (Proposition 2.9) $\Phi_Y = \frac{1}{2}(g + \Lambda)$ and direct calculation gives us that

$$\begin{aligned} \Lambda(u) &= \sum_{i=1}^m a_i \Lambda(u_i) + \sum_{i=1}^m a_i \log_2 a_i \\ &\geq \sum_{i=1}^m a_i \Lambda(u_i) - \log_2 m. \end{aligned}$$

Combining these estimates we have

$$\Phi_Y(u) \geq \sum_{i=1}^m a_i \Phi_Y(u_i) - \tfrac{1}{2} C_1 - \frac{1}{p_1} \log_2 m, \tag{3.13}$$

where

$$\frac{1}{p_1} := \frac{1}{2} + \frac{1-\varepsilon}{2(2-\varepsilon)}. \tag{3.14}$$

Note that we have a precisely similar estimate to (3.13) for Φ_{Y^*} in place of Φ_Y using instead the equation $\Phi_{Y^*} = \frac{1}{2}(\Lambda - g)$.

Now suppose $y_1, \ldots, y_m \in Y$ have disjoint supports. For any $u \in S^n$ we can write $u = \sum_{i=1}^m a_i u_i$ as a convex combination where $\operatorname{supp} u_i \supset \operatorname{supp} y_i$ and the (u_i) have disjoint supports. Let us write

$$\langle v, \log_2 |x| \rangle := \sum_{i=1}^{n+1} v_i \log_2 |x_i|$$

for $v \in S^n$ and $x \in R^{n+1}$ (with $-\infty$ as a possible value!). Then by (3.13)

$$\begin{aligned} \Phi_Y(u) &\geq \sum_{i=1}^m a_i \langle u_i, \log_2 |y_i| \rangle + \log_2(2^{-C_2} m^{-1/p_1}) \\ &= \langle u, \log_2(2^{-C_2} m^{-1/p_1} |y_1 + \cdots + y_m|) \rangle, \end{aligned}$$

where $C_2 := \frac{1}{2} C_1$. Now it is a consequence of Theorem 4.4 of [14] (which is much simpler in our finite-dimensional setting) that this implies

$$\|y_1 + \cdots + y_m\|_Y \leq 2^{C_2} m^{1/p_1}. \tag{3.15}$$

Again the same inequality holds in Y^*.

Now suppose $\{y_1, \ldots, y_m\}$ are any disjoint vectors with $\sum_{i=1}^m \|y_i\|_Y^p = 1$. Let $A_k := \{i : 2^{-k} < \|y_i\|_Y \leq 2^{1-k}\}$. If $|A_k|$ denotes the cardinality of A_k, then $|A_k| \leq 2^{kp}$ and by (3.13) we have

$$\left\| \sum_{i=1}^m y_i \right\|_Y \leq \sum_{k=1}^\infty \left\| \sum_{i \in A_k} y_i \right\|_Y \leq \sum_{k=1}^\infty 2^{C_2+1} |A_k|^{1/p_1} 2^{-k} \leq 2^{C_2+1} \sum_{k=1}^\infty 2^{k(p/p_1 - 1)}.$$

Since

$$\frac{p}{p_1} = \frac{1}{2}\left(\frac{3-2\varepsilon}{3-\varepsilon} + 1\right) < 1$$

this implies an estimate

$$\left\| \sum_{i=1}^{m} y_i \right\| \le C(\varepsilon) < \infty$$

and, combined with the similar estimate for Y^*, this establishes the lemma. ∎

We now turn our attention to the case when $K = B_X$ is the unit ball of a finite-dimensional Banach space. Our main result concerns the case when $X = \ell_p^n$ for $1 \le p \le \infty$.

Theorem 3.9. (a) *There exist constants* $c_1, c_2 > 0$ *so that*

$$c_1 \log_2(n+1) \le w_2(\ell_1^n) \le c_2 \log_2(n+1).$$

In addition

$$\limsup_{n\to\infty} \frac{w_2(\ell_1^n)}{\log_2 n} \le \frac{1}{4}. \tag{3.16}$$

(b) *If* $1 < p \le 2$, *then* $\gamma(p) := \sup_{n\in\mathbf{N}} w_2(\ell_p^n) < \infty$ *and further there exist constants* $d_1, d_2 > 0$ *so that*

$$\frac{d_1}{p-1} \le \gamma(p) \le \frac{d_2}{p-1}|\log(p-1)|.$$

(c) *If* $2 \le p \le \infty$, *then* $\gamma(p) := \sup_{n\in\mathbf{N}} w_2(\ell_p^n) < \infty$, *and further*

$$\gamma := \sup_{2\le p\le\infty} \gamma(p) \le 1602 < \infty.$$

Proof of (a). The upper estimate is an immediate consequence of Theorem 3.1. To prove the lower estimate, let $\tilde{f}_n : B_{\ell_1^n} \to \mathbf{R}$ be defined by

$$\tilde{f}_n(x) := \frac{1}{2}\sum_{i=1}^{n} x_i \log_2 |x_i| = \frac{1}{2}\sum_{i=1} \psi(t)$$

where $\psi(t) := t \log_2 |t|$ for $-1 \le t \le 1$. Since $|\Delta_h^2 \psi(t)| \le 2\log_2(1+\sqrt{2})|h|$, we obtain

$$|\Delta_h^2 \tilde{f}_n(x)| \le \log_2(1+\sqrt{2}) \sum_{i=1} |h_i|$$

so that $\omega_2(\tilde{f}_n) \le \log_2(1+\sqrt{2})$. Since $\tilde{f}_n|_{S^{n-1}} = f_{n-1}$ as defined in (3.4) we have

$$E_2(\tilde{f}_n; B_{\ell_1}) \ge E_2(f_{n-1}; S^{n-1}) \ge \tfrac{1}{4}\log_2 n.$$

This implies that

$$w_2(\ell_1^n) \geq \frac{1}{4\log_2(1+\sqrt{2})}\log_2 n.$$

It remains to prove (3.16). For this we need:

Lemma 3.10. $w_2(\ell_1^n) \leq w_2(S^{n-1}) + \frac{3}{2}$.

Proof. Suppose first f is a bounded continuous function on $B_{\ell_1^n}$ with $\omega_2(f) \leq 1$. Then there is an affine function g defined on S^{n-1} with $|\frac{1}{2}(f(x) - f(-x)) - g(x)| \leq w_2(S^{n-1})$. We can extend g to a linear functional on ℓ_1^n. We also have $|\frac{1}{2}(f(x) + f(-x)) - f(0)| \leq \frac{1}{2}$ for $x \in S^n$. Hence $|f(x) - g(x) - f(0)| \leq w_2(S^{n-1}) + \frac{1}{2}$ if $x \in \pm S^{n-1}$. Now if $x \in B_{\ell_1^n}$ we can find $u, v \in S^{n-1}$ and $0 \leq t \leq 1$ so that $x = tu - (1-t)v$. Hence $|f(x) - tf(u) - (1-t)f(-v)| \leq 1$ by the one-dimensional Whitney result which is essentially the fact that $\delta_2(f) \leq \beta_1 = 1$. Since g is linear, $|f(x) - g(x) - f(0)| \leq w_2(S^{n-1}) + \frac{3}{2}$. It follows that $w_2(\ell_1^n) \leq w_2(S^{n-1}) + \frac{3}{2}$. ∎

Now the inequality (3.16) follows from Theorem 3.6 and the proof of (a) is complete. ∎

We postpone the proof of (b) until after (c).

Proof of (c). We will need the following lemma (see (2.1) for the definition of $v_2(X)$):

Lemma 3.11. (a) *Let E be a subspace of a finite-dimensional Banach space X. Then $v_2(X/E) \leq 2v_2(X)$.*

(b) *Suppose X, Y are two n-dimensional Banach spaces. Then $v_2(Y) \leq d(X,Y)v_2(X)$ where $d(X,Y)$ is the Banach–Mazur distance between X and Y.*

Proof. Let $Q : X \to X/E$ be the quotient map. If $f : X/E \to \mathbf{R}$ is a continuous homogeneous function then there is a linear functional x^* on X so that

$$|f(Qx) - x^*(x)| \leq v_2(X)\omega_2(f; B_{X/E})\|x\|$$

for $x \in X$. For $x \in E$ we have

$$|x^*(x)| \leq v_2(X)\omega_2(f)\|x\|$$

and so by the Hahn–Banach theorem we can find a linear functional u^* with $u^*(e) = x^*(e)$ for $e \in E$ and $\|u^*\| \leq v_2(X)\omega_2(f)$. Then there exists $z^* \in (X/E)^*$ with $x^* - u^* = z^* \circ Q$ and we have

$$|f(Qx) - z^*(Qx)| \leq |f(Qx) - x^*(x)| + |u^*(x)| \leq 2v_2(X)\omega_2(f)\|x\|.$$

Part (a) now follows.

For part (b) suppose $T\colon X \to Y$ satisfies $\|T\| = 1$ and $\|T^{-1}\| = d(X, Y)$. Then if $f\colon Y \to \mathbf{R}$ is a continuous homogeneous function then $\omega_2(f \circ T; B_X) \le \omega_2(f; B_Y)$. Now there exists $x^* \in X^*$ so that $|f(Tx) - x^*(x)| \le v_2(X)\omega_2(f; B_Y)\|x\|$. Let $y^* = x^* \circ T^{-1}$. Then $|f(y) - y^*(y)| \le v_2(X)\omega_2(f; B_Y)d(X, Y)\|y\|$ and the lemma follows. ■

Now suppose $2 \le p \le \infty$. Then for any $n \in \mathbf{N}$ and $\varepsilon > 0$ there exists N so that ℓ_p^n is $(1+\varepsilon)$-isomorphic to a quotient of ℓ_∞^N. Hence $v_2(\ell_p^n) \le 2(1+\varepsilon)v_2(\ell_\infty^N)$.

However, the estimate $v_2(\ell_\infty^N) \le 200$ is proved in [18] (a factor 2 was omitted from the argument as pointed out in [22]). Hence $v_2(\ell_p^n) \le 400$ for all n. Now by Proposition 2.1 we have

$$w_2(\ell_p^n) \le 4v_2(\ell_p^n) + \tfrac{3}{2} \le 1602.$$

(Note that for $p = \infty$ we can eliminate a factor of 2 and get an estimate of 802.) ■

We now proceed to the proof of (b). Let us comment first that there is a striking difference between the cases $p < 2$ and $p > 2$ and this reflects the differing behavior of these spaces with respect to (Rademacher) type (see Section 2 for the definitions.)

We start by establishing the lower bound. For this we note that $d(\ell_1^n, \ell_p^n) \le n^{1/q}$ where $1/p + 1/q = 1$. Hence by Lemma 3.11 and part (a) we have $v_2(\ell_p^n) \ge c_1 n^{-1/q}\log(n+1)$. If we choose $n = [e^q]$ we obtain an estimate $\gamma(p) \ge d_1 q \ge d_1(p-1)^{-1}$ where $d_1 > 0$.

We will derive the upper bound from a general result about the relationship between the Whitney constants and the Rademacher-type p constant.

Theorem 3.12. *There is an absolute constant C so that if X is a finite-dimensional Banach space and $1 < p \le 2$, then*

$$w_2(X) \le \frac{C}{p-1}(1 + |\log(p-1)| + \log T_p(X)). \tag{3.17}$$

Proof. For this theorem we need the following elementary lemma:

Lemma 3.13. *Suppose Y is a Banach space of type p where $1 < p \le 2$ with type p constant $T_p(Y)$. Suppose $y_1, \dots, y_n \in B_Y$ and that $k \in \mathbf{N}$. Then there is a subset σ of $\{1, 2, \dots, n\}$ with $|\sigma| \le 2^{-k}n$ and so that*

$$\left\| \frac{1}{n}\sum_{i=1}^n y_i - \frac{2^k}{n}\sum_{i\in\sigma} y_i \right\| \le T_p(Y)n^{-1/q}\frac{2^{k/q}-1}{2^{1/q}-1},$$

where, as usual, $1/p + 1/q = 1$.

Proof. We prove this by induction on k, with $k = 0$ as the trivial starting point. Suppose σ_k is the subset satisfying the conclusions of the lemma for k. Then by the definition of the type p constant there is a choice of signs $\varepsilon_i = \pm 1$ with

$$\left\| \sum_{i\in\sigma_k} \varepsilon_i y_i \right\| \le T_p(Y)|\sigma_k|^{1/p} \le T_p(Y)2^{-k/p}n^{1/p}.$$

Without loss of generality we can assume $\sum_{i\in\sigma_k}\varepsilon_i \le 0$. Let $\sigma_{k+1} := \{i \in \sigma : \varepsilon_i = 1\}$. Then

$$\left\| \frac{2^k}{n}\sum_{i\in\sigma_k} y_i - \frac{2^{k+1}}{n}\sum_{i\in\sigma_{k+1}} y_i \right\| = \frac{2^k}{n}\left\|\sum_{i\in\sigma_k}\varepsilon_i y_i\right\| \le T_p(Y)2^{k/q}n^{-1/q}.$$

The induction step now follows easily ■

Returning to the proof of Theorem 3.12, we will estimate $\upsilon_2 := \upsilon_2(X)$. Suppose that f is any continuous homogeneous function on X with $\omega_2(f; B_X) \le 1$. We may pick $x^* \in X^*$ so that if $g := f - x^*$, then

$$E_2(f; B_X) = E_2(g\colon B_X) = \sup\{|g(x)|\colon \|x\| \le 1\} \le \upsilon_2. \tag{3.18}$$

By Proposition 3.2,

$$E_2(g; B_X) \le \sup\{\delta_m(g; B_X) : m \in \mathbf{N}\},$$

where $\delta_m(f; B_X)$ is defined in (3.1). Since g is continuous the right-hand side is equal to $\sup_{n\in\mathbf{N}} b_n$ where

$$b_n := \sup\left\{\left| g\left(\frac{1}{m}\sum_{i=1}^m x_i\right) - \frac{1}{m}\sum_{i=1}^m g(x_i)\right| : x_1,\dots,x_m \in B_X,\ m \le n\right\}.$$

We will show that

$$b_n \le 3 + 40q + 2q\log T_p + 2q\log\upsilon_2 \tag{3.19}$$

where $T_p := T_p(X)$.

To establish (3.19) choose an integer $N := [(T_p\upsilon_2)^q]$. By Theorem 3.1, $b_n \le 2w_2(n) \le 3 + \log_2 n$ and this shows that

$$b_N \le 3 + q\log_2 T_p + q\log_2 \upsilon_2 \le 3 + 2q\log T_p + 2q\log\upsilon_2.$$

In particular, (3.19) holds for all $n \le N$.

Suppose now $n > N$ and choose $k \in \mathbf{N}$ so that $2^{k-1}N < n \le 2^k N$. We consider the space $Y := X \oplus_\infty \mathbf{R}$; then $T_p(Y) \le 2T_p(X) = 2T_p$. If $x_1,\dots,x_n \in B_X$ we define elements of B_Y by $y_i := (x_i, \upsilon_2^{-1}g(x_i))$. By Lemma 3.13 there is a subset σ of $\{1,2,\dots,n\}$ with $|\sigma| \le 2^{-k}n$ so that

$$\left\|\frac{1}{n}\sum_{i=1}^n y_i - \frac{2^k}{n}\sum_{i\in\sigma} y_i\right\|_Y \le q(\log 2)^{-1}T_p(Y)2^{k/q}n^{-1/q} \le 8qT_pN^{-1/q}. \tag{3.20}$$

In particular, we have if $u := (1/n)\sum_{i=1}^n x_i$ and $w := (2^k/n)\sum_{i\in\sigma} x_i$

$$\|u - w\| \le 8qT_pN^{-1/q}. \tag{3.21}$$

Since $u, w \in B_X$ and g is homogeneous, we have $|g(u-w) - g(u) + g(w)| \le \omega_2(f; B_X) \le 1$. Hence, and by (3.18),

$$|g(u) - g(w)| \le 1 + |g(u-w)| \le 1 + \upsilon_2\|u - w\| \le 20q \tag{3.22}$$

by the choice of N. We also have from (3.20)

$$\left| \frac{1}{n} \sum_{i=1}^{n} g(x_i) - \frac{2^k}{n} \sum_{i \in \sigma} g(x_i) \right| \le 8q v_2 T_p N^{-1/q} \le 20q. \tag{3.23}$$

Finally we note that, since $|\sigma| \le 2^{-k} n \le N$,

$$\left| g(w) - \frac{2^k}{n} \sum_{i \in \sigma} g(x_i) \right| \le b_N \le 3 + 2q \log T_p + 2q \log v_2. \tag{3.24}$$

Combining (3.22), (3.23), and (3.24) gives us

$$\left| g(u) - \frac{1}{n} \sum_{i=1}^{n} g(x_i) \right| \le 3 + 40q + 2q \log T_p + 2q \log v_2$$

and so (3.19) holds.

Now (3.19) gives an estimate independent of n and so implies that

$$E_2(f; B_X) = E_2(g; B_X) \le 3 + 40q + 2q \log T_p + 2q \log v_2.$$

Since this estimate holds for all such f, we obtain

$$v_2 \le 3 + 40q + 2q \log T_p + 2q \log v_2.$$

Since $q \log v_2 \le \frac{1}{4} v_2 + q \log q + q \log 4$ this gives the required upper estimate in (b). ■

The proof of Theorem 3.12 is now also complete. ■

Corollary 3.14. *There is a universal constant C so that if X is an n-dimensional Banach space and* $2 < q < \infty$,

$$w_2(X) \le Cq(\log q + \log C_q(X^*) + \log(1 + \log n)).$$

Proof. If $1/p + 1/q = 1$, then $T_p(X) \le C(\log n + 1)C_q(X^*)$ (see [36]). It remains to apply the inequality (3.17). ■

Note that for the case of ℓ_∞^n this is weaker than the conclusion of Theorem 3.9(c). We conjecture that there is an estimate of the form $w_2(X) \le \varphi(q, C_q(X^*))$ for a suitable function φ. It is possible that the estimate $w_2(X) \le Cq(1 + \log C_q(X^*))$ holds, which would imply $w_2(X) \le C(p-1)^{-1}(1 + \log T_p(X))$ and $w_2(\ell_p^n) \le C(p-1)^{-1}$ giving a sharp estimate for $w_2(\ell_p^n)$.

4. Quadratic Approximation on Symmetric Convex Bodies

We now consider the problem of estimating $w_3(X)$ when X is a finite-dimensional Banach space. Our first result gives quite a sharp estimate of $w_3^{(s)}(n) := \sup_{\dim X = n} w_3(X)$.

Theorem 4.1. *There are absolute constants $0 < c, C < \infty$ so that for every $n \geq 1$*

$$c\sqrt{n} \leq w_3^{(s)}(n) \leq C\sqrt{n}\log(n+1).$$

Proof. The upper estimate is a special case of Theorem 5.2 (or Corollary 5.6), which we therefore postpone to the next section. For the lower estimate, we use the fact that the space ℓ_1^n contains a subspace V so that every linear projection $P : \ell_1^n \to V$ satisfies

$$\|P\| \geq c\sqrt{n}, \tag{4.1}$$

where $c > 0$ is an absolute constant. This follows from a well-known result of Kashin [19] that we may pick V with $\dim V = [n/2]$ and $d(V, \ell_2^{\dim V}) \leq C$ where C is independent of n. For convenience let Y be the space $\mathbf{R}^n$ with the norm, 2-equivalent to the ℓ_1-norm,

$$\|x\|_Y := \|x\|_{\ell_1^{2n}} + \|x\|_{\ell_2^{2n}}.$$

Then (4.1) holds for every linear projection $P : Y \to V$, with perhaps a different constant. Since Y is strictly convex, for every $x \in \mathbf{R}^n$ there is a unique $\Omega(x) \in V$ so that

$$\|x - \Omega(x)\|_Y = d_Y(x, V) := \inf_{v\in V} \|x - v\|_Y.$$

The map Ω is called the *metric projection* of Y onto V and the following properties are well known (see, e.g., [34, Sec. 5.1]):

Lemma 4.2. (a) *Ω is homogeneous and continuous.*
(b) *Ω is a (nonlinear) projection, $\|\Omega(x)\|_Y \leq 2\|x\|_Y$ for $x \in Y$ and $\Omega(x+v) = \Omega(x) + v$ if $x \in Y$, $v \in V$.*
(c) *For $x, y \in Y$,*

$$\|\Omega(x+y) - \Omega(x) - \Omega(y)\| \leq 2(d_Y(x, V) + d_Y(y, V)).$$

Now let $\langle\ ,\ \rangle$ be the standard inner-product on $\mathbf{R}^n$. Let π be the orthogonal projection onto V and let $\pi^\perp$ be the complementary projection onto $V^\perp$. Let $\|x\|_{Y^*} := \sup\{\langle x, y\rangle : \|y\|_Y \leq 1\}$ be the dual norm on $\mathbf{R}^n$.

We now define a norm $\|\ \|_X$ on $\mathbf{R}^n$ by the formula

$$\|x\|_X := d_{Y^*}(\pi x, V^\perp) + d_Y(\pi^\perp x, V), \tag{4.2}$$

where

$$d_{Y^*}(x, V^\perp) = \inf\{\|x - v^\perp\|_{Y^*}\colon v^\perp \in V^\perp\}.$$

Finally, let us define the continuous homogeneous function

$$F(x) := \langle \pi x, \Omega(\pi^\perp x)\rangle. \tag{4.3}$$

Now suppose $x, x+3h \in B_X$. Let $x = x_1 + x_2$ and $h = h_1 + h_2$ where $x_1, h_1 \in V$ and $x_2, h_2 \in V^\perp$. Then

$$\Delta_h^3 F(x) = \langle x_1, \Delta_{h_2}^3 \Omega(x_2)\rangle + 3\langle h_1, \Delta_{h_2}^2 \Omega(x_2 + h_2)\rangle. \tag{4.4}$$

Now we have

$$\begin{aligned}\|\Delta^3_{h_2}\Omega(x_2)\|_Y &\leq \|\Delta^2_{h_2}\Omega(x_2)\|_Y + \|\Delta^2_{h_2}\Omega(x_2+h_2)\|_Y \\ &\leq 2(d_Y(x_2, V) + d_Y(x_2+2h_2, V) + d_Y(x_2+h_2, V) + d_Y(x_2+3h_2, V)) \\ &\leq 8\end{aligned}$$

by Lemma 4.2. Similarly

$$\|\Delta^2_{h_2}\Omega(x_2+h_2)\| \leq 4.$$

Hence by (4.4) have

$$|\Delta^3_h F(x)| \leq 8 d_{Y^*}(x_1, V^\perp) + 12 d_{Y^*}(h_1, V^\perp) \leq 16 \tag{4.5}$$

since $d_{Y^*}(x_1, V^\perp) \leq 1$ and $d_{Y^*}(h_1, V^\perp) \leq \frac{2}{3}$. Thus (4.5) implies

$$\omega_3(F; B_X) \leq 16. \tag{4.6}$$

Let $\upsilon_3 := \upsilon_3(X)$. Then there is a quadratic form $Q(x)$ such that

$$|F(x) - Q(x)| \leq 16\upsilon_3\|x\|_X^2$$

for $x \in \mathbf{R}^n$. We can write $Q(x) = \langle x, Ax\rangle$ where A is a symmetric $n \times n$ matrix or equivalently a symmetric linear operator on $\mathbf{R}^n$.

Note for every $x \in \mathbf{R}^n$ we have $F(\pi x) = F(\pi^\perp x) = 0$. Hence

$$|\langle \pi x, A\pi x\rangle| \leq 16\upsilon_3\|\pi x\|_X^2 \leq 16\upsilon_3\|x\|_X^2$$

and

$$|\langle \pi^\perp x, A\pi^\perp x\rangle| \leq 16\upsilon_3\|\pi^\perp x\|_X^2 \leq 16\upsilon_3\|x\|_X^2.$$

It follows that

$$|F(x) - 2\langle \pi x, A\pi^\perp x\rangle| \leq 48\upsilon_3\|x\|_X^2. \tag{4.7}$$

We now define $P := \pi + 2\pi A\pi^\perp$. The linear operator P is a projection onto V; we will use (4.1) and so we estimate $\|P\|_Y$. Assume $\|y\|_Y = 1$ is chosen so that $\|Py\|_Y = \|P\|$. Then we may pick $x_1 \in V$ with $d_{Y^*}(x_1, V^\perp) \leq 1$ and

$$\langle x_1, Py\rangle = \|P\|_Y.$$

Now $x = x_1 + \pi^\perp(y) \in B_X$. Note that

$$F(x) = \langle x_1, \Omega(\pi^\perp(y))\rangle = \langle x_1, \Omega(y)\rangle - \langle x_1, \pi y\rangle.$$

Hence

$$|F(x) + \langle x_1, \pi y\rangle| \leq 2 d_{Y^*}(x_1, V^\perp)\|y\|_Y \leq 2$$

by Lemma 4.2. By (4.7) we obtain

$$|\langle x_1, 2\pi A\pi^\perp y + \pi y\rangle| \leq 2 + 48\upsilon_3$$

which implies $\|P\| \leq 2 + 48\upsilon_3$ and hence gives the estimate $\upsilon_3(X) \geq c\sqrt{n}$ for suitable $c > 0$. ∎

Our second main result of this section gives a rather sharp estimate of $w_3(\ell_p^n)$ when $p = 1$ or $2 \leq p \leq \infty$. It is a consequence of more general results which will proved later.

Theorem 4.3. *There are absolute constants $0 < c < C < \infty$ so that for every $n \geq 1$:*

(a) $c \log(n+1) \leq w_3(\ell_p^n) \leq Cp \log(n+1)$ *if $p = 1$ or $2 \leq p < \infty$; and*
(b) $c \log(n+1) \leq w_3(\ell_\infty^n) \leq C(\log(n+1))^2$.

Remark. We emphasize that c and C are independent of n and p. We do not have any really good upper estimate for $w_3(\ell_p^n)$ when $1 < p < 2$, but Theorem 4.3 gives a lower bound in that case.

Corollary 4.4. *There is a universal constant $c > 0$ so that for $1 < p < 2$,*

$$w_3(\ell_p^n) \geq c(p-1)\log(n+1).$$

Proof. We use the following fact proved in an equivalent form in [25, p. 21]. There is a universal constant C and for each n a subspace Y_n of ℓ_p^n, $1 < p < 2$, with $\dim Y = [n^{2/q}]$ (where $1/p + 1/q = 1$) so that:

(a) the Banach–Mazur distance $d(Y_n, \ell_2^{\dim Y_n}) \leq C$; and
(b) there is a projection $P\colon\ \ell_p^n \to Y_n$ with $\|P\| \leq C$.

Applying Lemma 4.7 and Lemma 2.4 to Y_n we can find a continuous 2-homogeneous function $f_0\colon\ Y_n \to \mathbf{R}$ with $\omega_3(f_0; B_{Y_n}) \leq 1$ and $E_3(f_0) \geq c(p-1)\log(n+1)$ where $c > 0$ is a universal constant. Defining $f := f_0 \circ P$ we easily have $\omega_3(f) \leq C$ but $E_3(f) \geq c(p-1)\log(n+1)$ and this proves the result. ∎

Except for the case $p = 1$, the estimates in Theorem 4.3 will follow from the following very general estimate:

Theorem 4.5. *There are absolute constants $0 < c < C < \infty$ so that for every n-dimensional Banach space we have*

$$\frac{c\log(n+1)}{C_2(X^*)^8} \leq w_3(X) \leq CT_2(X)^2\log(n+1).$$

Proof (The Upper Estimate). By Theorem 3.12 we have $w_2(X) \leq C(1 + \log T_2(X))$ and by Proposition 2.5 we have $w_3(X) \leq C\max(w_2(X), v_3(X))$. So it will suffice to show a similar estimate for $v_3(X)$. We obtain the result by a linearization technique. We can regard X as $\mathbf{R}^n$ with an appropriate norm. Now if P is an $n \times n$ positive-definite matrix, we can define an $\mathbf{R}^n$-valued Gaussian random variable ξ_P with covariance matrix P. Let Γ be the cone of positive-definite matrices.

Suppose now that f is a 2-homogeneous continuous function on X with $\omega_3(f; B_X) \leq 1$. We define a function $\hat{f}$ on Γ by putting

$$\hat{f}(P) := \mathbf{E}(f(\xi_P)).$$

Then $\hat{f}$ is 1-homogeneous on the cone Γ. Let Γ_0 be the convex hull of the set of matrices $\{x \otimes x : x \in B_X\}$ where $x \otimes x$ denotes the rank 1 matrix $(x_i x_j)_{1 \le i,j \le n}$. We need the estimate:

Lemma 4.6. *There is a universal constant so that for any $x_1, x_2 \in X$ we have*

$$|\tfrac{1}{2}(f(x_1 + x_2) + f(x_1 - x_2)) - f(x_1) - f(x_2)| \le C(\|x_1\|^2 + \|x_2\|^2). \tag{4.8}$$

Proof. By the main result of [2] there is a constant C_0 so that $w_3(Y) \le C_0$ for all two-dimensional subspaces. Let $Y := \operatorname{span}\{x_1, x_2\}$. By Proposition 2.5 there is a quadratic form h on Y so that $|f(y) - h(y)| \le C\|y\|^2$ for all $y \in Y$ (where again C is a universal constant). This immediately yields the lemma. ∎

Returning to the proof of the theorem we note that if ξ_P and ξ_Q are independent then $\xi_P + \xi_Q$ has the same distribution as ξ_{P+Q}. Hence

$$\begin{aligned} |\hat{f}(P+Q) - \hat{f}(P) - \hat{f}(Q)| &= |\mathbf{E}(f(\xi_P + \xi_Q)) - \mathbf{E}(f(\xi_P) + f(\xi_Q))| \\ &= |\mathbf{E}\tfrac{1}{2}(f(\xi_P + \xi_Q) + f(\xi_P - \xi_Q)) \\ &\quad - \mathbf{E}(f(\xi_P) + f(\xi_Q))| \\ &\le C\mathbf{E}(\|\xi_P\|^2 + \|\xi_Q\|^2). \end{aligned}$$

Now suppose that $P, Q \in \Gamma_0$. Then we can write $P = \sum_{i=1}^m a_i x_i \otimes x_i$ where $\|x_i\| \le 1$ for $1 \le i \le m$ and $a_i \ge 0$ with $\sum_{i=1}^m a_i = 1$. Then ξ_P has the same distribution as $\sum_{i=1}^m a_i^{1/2} g_i x_i$ where $g_1, \dots, g_m$ are independent normalized Gaussian random variables. Hence as is well known (see, e.g., [30, p. 25])

$$\mathbf{E}(\|\xi_P\|^2) = \mathbf{E}\left(\left\|\sum_{i=1}^m a_i^{1/2} g_i x_i\right\|^2\right) \le T_2(X)^2.$$

Using the similar inequality for Q, we obtain

$$|\hat{f}(P+Q) - \hat{f}(P) - \hat{f}(Q)| \le CT_2(X)^2$$

for a universal constant C. Hence $\omega_2(\hat{f}, \Gamma_0) \le CT_2(X)^2$. Since $\dim \Gamma_0 = \frac{1}{2}n(n-1) \le n^2$ we can apply Theorem 3.1 to Γ_0 to deduce the existence of an affine function φ on Γ_0 so that

$$|\hat{f}(P) - \varphi(P)| \le CT_2(X)^2 \log(n+1), \tag{4.9}$$

where C is again a universal constant. In particular, $|\varphi(0)|$ is dominated by $CT_2(X)^2 \log(n+1)$ so we can assume that φ is linear on the linear span of Γ_0. Let $h(x) = \varphi(x \otimes x)$. Then h is a quadratic form. Since $\hat{f}(x \otimes x) = \mathbf{E}(f(gx)) = f(x)\mathbf{E}(g) = f(x)$ where g is a normalized Gaussian, we have from (4.9)

$$|f(x) - h(x)| \le CT_2(X)^2 \log(n+1)$$

for all $x \in B_X$. This gives the desired estimate of $v_3(X)$ and completes the proof of the upper estimate.

(The Lower Estimate). We establish a lower estimate for $v_3(X)$; we first achieve this for the case of $X = \ell_2^n$.

Lemma 4.7. *There is an absolute constant $c > 0$ so that for all $n \geq 1$*

$$v_3(\ell_2^n) \geq c \log n. \tag{4.10}$$

Proof. Let $\varphi(t) := t^2 \log|t|$ for $-1 \leq t \leq 1$. Then, by the Mean Value Theorem

$$\Delta_h^3 \varphi(t) = 3h \Delta_{\theta h}^2 \varphi'(t + \theta h)$$

for some $0 < \theta < 1$. Hence

$$|\Delta_h^3 \varphi(t)| \leq 6 \log(1 + \sqrt{2})|h|^2.$$

Now define for $x \in B_{\ell_2^n}$,

$$f(x) = \sum_{i=1}^n \varphi(x_i).$$

Then for $x, x + 3h \in B_{\ell_2^n}$,

$$|\Delta_h^3 f(x)| \leq 6 \log(1 + \sqrt{2}) \sum_{i=1}^n h_i^2 < \tfrac{8}{3} \log(1 + \sqrt{2}).$$

Hence $\omega_3(f; B_{\ell_2^n}) < 6$.

Since f is even and $f(0) = 0$ we can find a quadratic form h on ℓ_2^n so that

$$\sup_{\|x\| \leq 1} |f(x) - h(x)| \leq 2E_3(f; B_{\ell_2^n}).$$

As the points $n^{-1/2} \sum_{i=1}^n \varepsilon_i e_i \in B_{\ell_2^n}$ for $\varepsilon_i = \pm 1$ the left-hand side is at least

$$\operatorname*{Ave}_{\varepsilon_i = \pm 1} \left| f\left(\frac{1}{\sqrt{n}} \sum_{i=1}^n \varepsilon_i e_i \right) - h\left(\frac{1}{\sqrt{n}} \sum_{i=1}^n \varepsilon_i e_i \right) \right| = \left| \tfrac{1}{2} \log n + \frac{1}{n} \sum_{i=1}^n h(e_i) \right|.$$

As $f(e_i) = 0$ for $1 \leq i \leq n$ we have

$$\frac{1}{n} \left| \sum_{i=1}^n h(e_i) \right| \leq 2E_3(f; B_{\ell_2}^n).$$

Putting these inequalities together gives $E_3(f; B_{\ell_2^n}) \geq \frac{1}{8} \log n$. ■

Next we need a lemma using the extension constants from Definition 2.11.

Lemma 4.8. *Let X be an n-dimensional Banach space and let E be a linear subspace of X. Let $\mathcal{E}_X(E, E^\perp) = M_1$ and $\mathcal{E}_X(E, X^*) = M_2$. Then*

$$v_3(X/E) \leq (M_1 + 1)(M_2 + 1) v_3(X). \tag{4.11}$$

Proof. It will be convenient to regard X as $\mathbf{R}^n$ with an appropriate norm and let $\langle\ ,\ \rangle$ be the usual inner-product on $\mathbf{R}^n$. Suppose f is a 2-homogeneous continuous function on X/E with $\omega_3(f; B_{X/E}) \le 1$. Let $Q : X \to X/E$ be the quotient map. Then $f \circ Q$ is continuous and 2-homogeneous on X and $\omega_3(f \circ Q; B_X) \le 1$. Hence there is a quadratic form $h : X \to \mathbf{R}$ such that

$$|f(Qx) - h(x)| \le v_3(X)\|x\|_X^2$$

for $x \in \mathbf{R}^n$. We can assume $h(x) = \langle x, Ax\rangle$ where A is a symmetric matrix.

Since $\langle x, Ay\rangle = \frac{1}{4}(h(x+y) - h(x-y))$ we have

$$|\langle x, Ay\rangle - \tfrac{1}{4}(f(Qx+Qy) - f(Qx-Qy))| \le \tfrac{1}{2}v_3(X)(\|x\|_X^2 + \|y\|_X^2).$$

Assume $y \in E$. Then $Qy = 0$ and so

$$|\langle x, Ay\rangle| \le \tfrac{1}{2}v_3(X)(\|x\|_X^2 + \|y\|_X^2).$$

Replacing x by αx and y by $\alpha^{-1}y$ and minimizing the right-hand side gives

$$|\langle x, Ay\rangle| \le v_3(X)\|x\|_X\|y\|_X.$$

This implies that

$$\|Ay\|_{X^*} \le v_3(X)\|y\|_X$$

when $y \in E$. From the definition of the extension constant there exists an $n \times n$ matrix A_1 so that $A_1 y = Ay$ for $y \in E$ and A_1 has norm at most $M_2 v_3(X)$ as an operator from X into X^*. Then $A - A_1$ maps E to 0 and hence the transpose $A - A_1^t$ maps $\mathbf{R}^n$ to $E^\perp$. Now $\|A_1^t\|_{X\to X^*} = \|A_1\|_{X\to X^*} \le M_2 v_3(X)$ and so $\|A - A_1^t\|_{E\to X^*} \le (M_2+1)v_3(X)$. Using the extension constant again we can find an $n \times n$ matrix A_2 which maps $\mathbf{R}^n$ into $E^\perp$ and such that $\|A_2\|_{X\to X^*} \le M_1(M_2+1)v_3(X)$.

Let $S = A - A_1^t - A_2$. Then S maps E to $\{0\}$ and $\mathbf{R}^n$ into $E^\perp$. It follows that $S = TQ$ where T is a linear operator from X/E to $E^\perp$ and we can define a quadratic form ψ on X/E by $\psi(Qx) = \langle x, Sx\rangle$.

Then

$$\begin{aligned} |f(Qx) - \psi(Qx)| &\le |f(Qx) - h(x)| + |\langle x, A_1^t x\rangle| + |\langle x, A_2 x\rangle| \\ &\le (1 + M_2 + M_1(M_2+1))v_3(X)\|x\|_X^2 \\ &= (M_1+1)(M_2+1)v_3(X)\|x\|_X^2. \end{aligned}$$

Now for given $u \in X/E$ we can choose $x \in X$ with $Qx = u$ and $\|x\|_X = \|u\|_{X/E}$. This implies $v_3(X/E) \le (M_1+1)(M_2+1)v_3(X)$. ∎

We can now complete the proof of the lower estimate in Theorem 4.5. Suppose X is a Banach space of dimension n. We use the following powerful form of the Dvoretzky theorem due to Figiel, Lindenstrauss, and Milman [7] (see [25, Theorem 9.6], where the theorem is formulated in the form required here). There is a subspace F of X^* which is 2-isomorphic to ℓ_2^m with

(4.12) $$m = \dim F \ge cC_2(X^*)^{-2}n$$

We note that the lower estimate in Theorem 4.5 is trivial for spaces such that $C_2(X^*) \geq \sqrt{c}n^{1/4}$. We will therefore consider only those spaces X for which $C_2(X^*) \leq \sqrt{c}n^{1/4}$. Then (4.12) gives $m \geq \sqrt{n}$.

Let us put $E := F^{\perp}$. Since E^* is isometric to X^*/F and $d(F, \ell_2^m) \leq 2$ we can apply Theorem 6.9 of [30] to obtain

$$C_2(E^*) \leq CC_2(X^*), \tag{4.13}$$

where, as usual, C is an absolute constant.

Now we use Corollary 2.13 to estimate the constants M_1, M_2 of Lemma 4.8 as follows:

$$\begin{aligned} M_1 &\leq \psi(T_2(X/E))C_2(X^*)(C_2(E^*)C_2(E^{\perp}))^{3/2}, \\ M_2 &\leq \psi(T_2(X/E))C_2(X^*)(C_2(E^*)C_2(X^*))^{3/2}, \end{aligned}$$

where $\psi : [1, \infty) \to [1, \infty)$ is a suitable increasing function. Since X/E is isometric to F^* we have $d(X/E, \ell_2^m) \leq 2$ and so $T_2(X/E) \leq 2$. Together with (4.13) this yields

$$M_1, M_2 \leq CC_2(X^*)^4.$$

Combining this with (4.11) and Lemma 2.4 we have

$$\tfrac{1}{4}\upsilon_3(\ell_2^m) \leq \upsilon_3(X/E) \leq CC_2(X^*)^8\upsilon_3(X).$$

Applying now (4.10) and the inequality $m \geq \sqrt{n}$ we have

$$\upsilon_3(X) \geq C^{-1}\frac{\log m}{C_2(X^*)^8} \geq c\frac{\log(n+1)}{C_2(X^*)^8}$$

for an absolute constant $c > 0$. The proof of Theorem 4.6 is now complete. ∎

Proof of Theorem 4.3. For the case $p = 1$ we postpone the proof to the next section (see Corollary 5.7 below). For $2 \leq p \leq \infty$ it suffices to apply Theorem 4.6 to $X = \ell_p^n$ noting that in this case $C_2(X^*)$ is uniformly bounded independent of n and p while $T_2(\ell_p^n) \leq C\sqrt{p}$ for $2 \leq p < \infty$ and $T_2(\ell_\infty^n) \leq C(\log(n+1))^{1/2}$; see, for example, [36]. ∎

5. Higher-Order Estimates

We now consider upper estimates for $w_n(X)$ when X is a finite-dimensional Banach space and $n \geq 3$ is arbitrary. In the proof we will use heavily the notion and characteristic properties of m-quasilinear functions, which we introduce next.

Definition 5.1. A map $F\colon X^m \to \mathbf{R}$ is said to be *m-quasilinear* if F is homogeneous in each variable separately and there is a constant $\lambda \geq 0$ so that for any $1 \leq j \leq m$ and any $(x_i)_{i\neq j}$ the map $g_j(x) := F(x_1, \ldots, x_{j-1}, x, x_{j+1}, \ldots, x_m)$ satisfies

$$\omega_2(g_j; B_X) \leq \lambda \prod_{i\neq j} \|x_i\|. \tag{5.1}$$

We then set $\tilde{\Delta}_m(F)$ to be the infimum of all λ so that (5.1) holds.

To formulate our main result, we recall that the *projection constant* $\lambda(Y)$ of a finite-dimensional Banach space Y is the smallest $\lambda \geq 1$ so that if Y is embedded isometrically in a Banach space Z, then there is a linear projection $P : Z \to Y$ with $\|P\| \leq \lambda$. See, for example, [41].

Theorem 5.2. *For any integers $m \geq k \geq 2$ there is a constant $C = C(m)$ so that*

$$w_m(X) \leq C\lambda(X^*)^{m-k} w_k(X). \tag{5.2}$$

Before proving this estimate we will establish some basic lemmas on m-quasilinear forms. We let C denote a constant which depends only on m.

Lemma 5.3. *Suppose $F : X^m \to \mathbf{R}$ is a symmetric m-quasilinear form and that $f : X \to \mathbf{R}$ is defined by $f(x) = F(x, \ldots, x)$. Then*

$$\left| f(x_1 + x_2) - \sum_{k=0}^{m} \binom{m}{k} F_k(x_1, x_2) \right| \leq C\tilde{\Delta}_m(F) \max(\|x_1\|^m, \|x_2\|^m), \tag{5.3}$$

where $F_k(x_1, x_2) = F(x_1, \ldots, x_1, x_2, \ldots, x_2)$ with x_1 repeated k times and x_2 repeated $n - k$ times.

More generally, there is a constant $C = C(m)$ so that if $x_1, \ldots, x_m \in X$

$$\left| f\left(\sum_{i=1}^{m} x_i\right) - \sum_{|\alpha|=m} \binom{m}{\alpha} F_\alpha(x_1, \ldots, x_m) \right| \leq C\tilde{\Delta}_m(F) \max(\|x_1\|^m, \ldots, \|x_m\|^m), \tag{5.4}$$

where we adopt the notation for $\alpha \in \mathbf{Z}_+^m$ of $|\alpha| := \sum_{i=1}^m \alpha_i$ and

$$F_\alpha(x_1, \ldots, x_m) := F(x_1, \ldots, x_1, x_2, \ldots, x_2, \ldots, x_m, \ldots, x_m)$$

with each x_k repeated α_k times.

Proof. This is established by expanding in each variable separately and collecting terms. We omit the details. ■

Suppose now that $f : X \to \mathbf{R}$ is a continuous m-homogeneous function. We associate with f the separately homogeneous function F: $X^m \to \mathbf{R}$ defined for $\|x_1\| = \|x_2\| = \cdots = \|x_n\| = 1$ by

$$F(x_1, \ldots, x_m) := \frac{1}{2^m m!} \sum_{\varepsilon_i = \pm 1} \varepsilon_1 \ldots \varepsilon_m f\left(\sum_{i=1}^{m} \varepsilon_i x_i\right). \tag{5.5}$$

and extended by homogeneity.

Lemma 5.4. *If $f : X \to \mathbf{R}$ is continuous and m-homogeneous then F defined by* (5.5) *is symmetric and m-quasilinear with $\tilde{\Delta}_m(F) \leq C\omega_{m+1}(f; B_X)$.*

Conversely, if F is continuous and m-quasilinear, then $f(x) := F(x, \ldots, x)$ is continuous and m-homogeneous with $\omega_{m+1}(f; B_X) \leq C\tilde{\Delta}_m(F)$.

Proof. Suppose first that f is continuous and m-homogeneous and that F is defined by (5.5). Suppose $(x_i)_{i\neq j} \in B_X$ and $x, x+2h \in B_X$. Let $E = \operatorname{span}(\{x_i\}_{i\neq j}, x, h)$. Then $\dim E \le m+1$ and so by the Whitney-type result of [2] there is a constant $C = C(m)$ so that $w_{m+1}(E) \le C$. By Proposition 2.5 we also have $v_{m+1}(E) \le C$. Since $\omega_{m+1}(f; B_E) \le \omega_{m+1}(f; B_X)$ there is a homogeneous polynomial of degree m on E so that

$$|f(u) - g(u)| \le C\|u\|^m \omega_{m+1}(f; B_X)$$

for $u \in E$. We can express g in the form $g(u) = G(u, \dots, u)$ where G is a symmetric m-linear form. Using the polarization formula from multilinear algebra, we have

$$|F(x_1, \dots, x_{j-1}, u, x_{j+1}, \dots, x_m) - G(x_1, \dots, x_{j-1}, u, x_{j+1}, \dots, x_m)| \le C\omega_{m+1}(f; B_X)$$

whenever $\|u\| \le 1$ and $u \in E$. Let $\varphi(u) = F(x_1, \dots, x_{j-1}, u, x_{j+1}, \dots, x_m)$. It now follows that

$$|\Delta_h^2 \varphi(x)| \le C\omega_{m+1}(f; B_X)$$

and so

$$\tilde{\Delta}_m(F) \le C\omega_{m+1}(f; B_X).$$

We now turn to the converse. Suppose that $x + jh \in B_X$ for $0 \le j \le m+1$. Using (5.3) we have

$$\left| f(x+jh) - \sum_{k=0}^{m} \binom{m}{k} j^k F_k(h, x) \right| \le C\tilde{\Delta}_m(F).$$

Hence

$$|\Delta_h^{m+1} f(x)| \le C\tilde{\Delta}_m(F)$$

as required. ∎

Our next result shows that symmetric m-quasilinear forms can be nicely approximated by m-linear forms.

Lemma 5.5. *Suppose $F\colon X^m \to \mathbf{R}$ is a continuous symmetric m-quasilinear form. Then there is a symmetric m-linear form $H\colon X^m \to \mathbf{R}$ so that*

$$|F(x_1, \dots, x_m) - H(x, \dots, x_m)| \le Cv_{m+1}(X)\tilde{\Delta}_m(F) \prod_{i=1}^{m} \|x_i\|.$$

Proof. Let $f(x) := F(x, \dots, x)$. By the previous lemma, $\omega_{m+1}(f; B_X) \le C\tilde{\Delta}_m(F)$. Hence there is a symmetric m-linear form H so that if $h(x) = H(x, \dots, x)$ then

$$|f(x) - h(x)| \le Cv_{m+1}(X)\tilde{\Delta}_m(F)\|x\|^m. \tag{5.6}$$

Now let us define F' using (5.5) to be separately homogeneous and for $\|x_1\| = \|x_2\| = \dots = \|x_n\| = 1$,

$$F'(x_1, \dots, x_n) := \frac{1}{2^m m!} \sum_{\varepsilon_i = \pm 1} \varepsilon_1 \dots \varepsilon_m f\left(\sum_{i=1}^{m} \varepsilon_i x_i\right). \tag{5.7}$$

Note that

$$\sum_{\varepsilon_i=\pm 1}\sum_{|\alpha|=m}\binom{m}{\alpha}\prod_{i=1}^{m}\varepsilon_i^{\alpha_i+1}=m!\,2^m \tag{5.8}$$

since $\sum_{\varepsilon_i=\pm 1}\prod_{i=1}^{m}\varepsilon_i^{\alpha_i+1}=0$ unless $\alpha_i=1$ for all i. Hence

$$\sum_{\varepsilon_i=\pm 1}\sum_{|\alpha|=m}\binom{m}{\alpha}\varepsilon_1\dots\varepsilon_m F_\alpha(\varepsilon_1x_1,\dots,\varepsilon_mx_m)=2^m m!\,F(x_1,\dots,x_m).$$

It follows, by Lemma 5.3, that for $\|x_1\|=\|x_2\|=\cdots=\|x_n\|=1$,

$$|F'(x_1,\dots,x_n)-F(x_1,\dots,x_n)|\le C\tilde{\Delta}_m(F).$$

We also have, again using Lemma 5.3,

$$|F'(x_1,\dots,x_n)-H(x_1,\dots,x_n)|\le Cv_{m+1}(X)\tilde{\Delta}_m(F)$$

and the lemma follows by homogeneity. ■

Proof of Theorem 5.2. We will prove by induction that

$$v_m(X)\le C\lambda(X^*)\max(v_{m-1}(X),v_2(X)) \tag{5.9}$$

when $m\ge 3$.

Let $f:X\to\mathbf{R}$ be a continuous m-homogeneous function with $\omega_{m+1}(f)\le 1$. We define $F:X^m\to\mathbf{R}$ using (5.5) so that $\tilde{\Delta}_m(F)\le C$. Now fixing $u\in X$ we define

$$g_u(x):=F(u,x,\dots,x)$$

so that g_u is $(m-1)$-homogeneous and $\omega_m(F)\le C\tilde{\Delta}_m(F)\|u\|\le C\|u\|$ by Lemma 5.4.

Now by Lemma 5.5 there is a symmetric $(m-1)$-linear form $H_u:X^{m-1}\to\mathbf{R}$ so that

$$|F(u,x_2,\dots,x_m)-H_u(x_2,\dots,x_m)|\le Cv_m(X)\|u\|\prod_{i=2}^{m}\|x_i\|.$$

We may clearly suppose that the map $u\to H_u$ is homogeneous. Then

$$|F(x_1,\dots,x_m)-H(x_1,\dots,x_m)|\le Cv_m(X)\prod_{i=1}^{m}\|x_i\|. \tag{5.10}$$

Now let Z be the space of all continuous homogenous functions on X with the norm $\|\varphi\|_Z=\sup_{\|x\|\le 1}|\varphi(x)|$. Then X^* is a linear subspace of Z and there is a projection $\pi:Z\to X^*$ with $\|\pi\|\le\lambda(X^*)$.

For $x_2,\dots,x_m\in X$ we define $H_{x_2,\dots,x_m}$ and $F_{x_2,\dots,x_m}\in Z$ by

$$H_{x_2,\dots,x_m}(x)=H(x,x_2,\dots,x_m)$$

and

$$F_{x_2,\dots,x_m}(x)=F(x,x_2,\dots,x_m).$$

Then

$$d(F_{x_2,\ldots,x_m}, X^*) \le v_2(X)\tilde{\Delta}_m(F)\prod_{i=2}^m \|x_i\|$$

and by (5.10)

$$\|F_{x_2,\ldots,x_m} - H_{x_2,\ldots,x_m}\| \le C v_m(X)\prod_{i=2}^m \|x_i\|.$$

Combining we obtain

$$d(H_{x_2,\ldots,x_m}, X^*) \le C(v_m(X) + v_2(X))\prod_{i=2}^m \|x_i\|.$$

Now set

$$G(x_1,\ldots,x_m) = \pi(h_{x_2,\ldots,x_m})(x_1)$$

so that G is m-linear. Then

$$|H(x_1,\ldots,x_m) - G(x_1,\ldots,x_m)| \le (1 + \|\pi\|)\prod_{i=1}^m \|x_i\|.$$

Hence appealing again to (5.10) we have

$$|F(x_1,\ldots,x_m) - G(x_1,\ldots,x_m)| \le C\lambda(X^*)\max(v_m(X), v_2(X))\prod_{i=1}^m \|x_i\|.$$

This implies (5.9).

Since $v_m(X) \le w_m(X) \le C\max(v_2(X),\ldots,v_m(X)$ by Proposition 2.5 the theorem is proved. ■

Corollary 5.6. *For any $m \ge 2$ there is a constant $C = C(m)$ so that*

$$w_m^{(s)}(n) \le Cn^{m/2-1}\log(n+1)$$

(*i.e., for any n-dimensional Banach space $w_m(X) \le Cn^{m/2-1}\log(n+1)$*).

Proof. Using Theorem 5.2 with $k = 2$ and the Kadets–Snobar inequality $\lambda(X^*) \le \sqrt{n}$ [11] and [41] we have $w_m(X) \le Cn^{m/2-1}w_2(X)$, but $w_2(X) \le C\log(n+1)$ by Theorem 3.1. ■

Corollary 5.7. *For any $m \in \mathbf{N}$ there exists a constant $C = C(m)$ so that*

$$C^{-1}\log(n+1) \le w_m(\ell_1^n) \le C\log(n+1).$$

Proof. Since $\lambda(\ell_\infty^n) = 1$ (see, e.g, [31]) by Theorem 5.2 with $k = 2$ we have $w_m(\ell_1^n) \le Cw_2(\ell_1^n) \le C\log(n+1)$. Conversely, by Corollary 2.6 and Theorem 3.9, we have $C^{-1}\log(n+1) \le w_2(\ell_1^n) \le Cw_m(\ell_1^n)$. ■

Corollary 5.8. *For any $m \geq 3$ and $2 \leq p < \infty$ there is a constant $C = C(m, p)$ so that*

$$w_m(\ell_p^n) \leq Cn^{(m-3)/2} \log(n+1).$$

Proof. Apply Theorem 5.2 with $k = 3$ and use Theorem 4.2. ■

There is a striking difference between the results for $p \geq 1$ and for $0 < p < 1$, when the sets $B_{\ell_p^n}$ are no longer convex. The following theorem is then true:

Theorem 5.9. *If $0 < p < 1$ and $m \geq 2$ there is a constant $C = C(p, m)$ so that $w_m(\ell_p^n) \leq C$ for all $n \geq 1$.*

Proof. It is easily checked that the proof of Theorem 5.2 goes through with trivial changes for r-normed spaces when $r < 1$ (see Remark after Corollary 2.6). Of course the constant C in its formulation depends now on r. Applying this result to ℓ_p^n with $r = p < 1$ we therefore have

$$w_m(\ell_p^n) \leq C(m, p)\lambda((\ell_p^n)^*)^{m-2} w_2(\ell_p^n).$$

But $(\ell_p^n)^* = \ell_\infty^n$ and it is essentially proved in [12] (in an equivalent formulation related to the notion of a $\mathcal{K}$-space) that $w_2(\ell_p^n) \leq C(1-p)^{-1}$ with C an absolute constant independent of n. This proves the theorem. ■

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