Bol. Soc. Mat. Mexicana (3) Vol. 3, 1997

A NOTE ON PAIRS OF PROJECTIONS

N.J. KALTON

Abstract

We give a brief proof of a recent result of Avron, Seiler and Simon.

In [1], it is proved that if P, Q are (not necessarily self-adjoint) projections on a Hilbert space and $(P-Q)^n$ is trace-class (i.e. nuclear) for some odd integer n then $\operatorname{tr}(P-Q)^n$ is an integer and in fact, if P and Q are self-adjoint, $\operatorname{tr}(P-Q)^n = \dim E_{10} - \dim E_{01}$ where $E_{ab} = \{x : Px = ax,\ Qx = bx\}$; (see also [2]). The proof given in [1] uses the structure of the spectrum of $P-Q$ and Lidskii's theorem; it is therefore not applicable to more general Banach spaces. The purpose of this note is to give a very brief proof of the same result which involves only simple algebraic identities and is valid in any Banach space with a well-defined trace (i.e. with the approximation property). We use $[A, B]$ to denote the commutator $AB - BA$.

The basic material about operators on Banach spaces which we use can be found in the book of Pietsch [3]. We summarize the two most important ingredients.

We will need the following basic result from Fredholm theory. Suppose X is a Banach space and $A: X \to X$ is an operator such that for some m, A^m is compact. Let $S = I - A$; then $F = \bigcup_{k\geq 1} S^{-k}(0)$ is finite-dimensional and if $Y = \bigcap_{k\geq 1} S^k(X)$ then Y is closed and X can be decomposed as a direct sum $X = F \oplus Y$. Furthermore F and Y are invariant for S and S is invertible on Y. We refer to [3] 3.2.9 (p. 141-142) for a slightly more general result.

We will also need the following properties of nuclear operators and the trace. If X is a Banach space then an operator $T: X \to X$ is called nuclear if it can written as a series $T = \sum_{n=1}^{\infty} A_n$ where each A_n has rank one and $\sum_{n=1}^{\infty} \|A_n\| < \infty$. The nuclear operators form an ideal in the space of bounded operators. When X has the approximation property, one can then define the trace of T *unambiguously* by $\operatorname{tr} T = \sum_{n=1}^{\infty} \operatorname{tr} A_n$ (where the trace of a rank one operator $A = x^* \otimes x$ is defined in the usual way by $\operatorname{tr} A = x^*(x)$.) The trace is then a linear functional on the ideal of nuclear operators and has the property that $\operatorname{tr}[A, T] = 0$ if A is bounded and T is nuclear. See Chapter 4 of [3] and particularly Theorem 4.7.2.

1991 *Mathematics Subject Classification*: 47A53.

Supported by NSF grant DMS-9500125

LEMMA. *Let X be a Banach space and suppose P and Q are two projections on X. Let $M = P - Q$, $U = (I - Q)(I - P) + QP$, $V = (I - P)(I - Q) + PQ$ and suppose T is any operator which commutes with both P and Q. Then*
(1) M^2 commutes with both P and Q.
(2) $[(I - 2Q)TM, PV] = TM(I - M^2)$.
(3) If $I - M^2$ is invertible $[(I - 2Q)TM(I - M^2)^{-1}, PV] = TM$.

Proof. (1) was first observed by Dixmier, Kadison and Mackey as remarked in [1]. For (2) observe that $QU = UP$ and $UV = VU = I - M^2$. Hence $M(I - M^2) = PUV - QUV = PVU - UPV = -[U, PV] = [I - U, PV] = [(I - 2Q)M, PV]$. If T commutes with P and Q then (2) follows. Note that (3) is immediate from (2), replacing T by $T(I - M^2)^{-1}$. ■

THEOREM. *Let X be a Banach space with the approximation property, and suppose n is an odd integer. If P, Q are two projections on X so that $(P - Q)^n$ is nuclear, then $\operatorname{tr}(P - Q)^n = \dim E_{10} - \dim \tilde{E}_{01} = \dim \tilde{E}_{10} - \dim E_{01}$, where $E_{ab} = \{x \in X : Px = ax,\ Qx = bx\}$ and $\tilde{E}_{ab} = \{x^* \in X^* : P^*x^* = ax^*,\ Q^*x^* = bx^*\}$.*

Remark. If X is a Hilbert space and P and Q are self-adjoint this is equivalent to the result of Avron, Seiler and Simon.

Proof. We use the notation of the lemma. If M^n is nuclear then some power of M^2 is compact. Let $S = I - M^2$ and let $F = \bigcup_{k\geq 1} S^{-k}(0)$ and $Y = \bigcap_{k\geq 1} S^k(X)$. Then as noted above we have that $\dim F < \infty$, $X = F \oplus Y$ and S is invertible on Y. Since P and Q commute with S both F and Y are invariant for P and Q. We denote the restriction of an operator T to F or Y by T_F or T_Y.

Now $(I - M_Y^2)$ is invertible on Y so that (3) of the lemma expresses M_Y^n as the commutator of a nuclear operator and a bounded operator. Hence $\operatorname{tr} M_Y^n = 0$.

On the other hand, by (2) of the Lemma, $M_F - M_F^n$ is a commutator on F which is finite-dimensional so that $\operatorname{tr} M_F^n = \operatorname{tr} M_F = \operatorname{tr} P_F - \operatorname{tr} Q_F \in \mathbb{Z}$.

It is easy to see from elementary computations that $\operatorname{tr} P_F - \operatorname{tr} Q_F = \dim F - \operatorname{rank}(I - P_F) - \operatorname{rank} Q_F = \dim F - \dim((I - P)F + Q(F)) - \dim E_{01}$. Now $\dim F - \dim((I - P)F + Q(F))$ is the dimension of the subspace of F^* of all f^* such that $P_F^* f^* = f^*$ and $Q_F^* f^* = 0$; if we identify F^* with $Y^{\perp}$ via the direct sum decomposition this space coincides with $\tilde{E}_{10}$. Now, it follows easily from the properties of the trace that $\operatorname{tr} M^n = \operatorname{tr} M_F^n + \operatorname{tr} M_Y^n$. This gives the second formula for $\operatorname{tr} M^n$. The other formula is similar. ■

UNIVERSITY OF MISSOURI
DEPARTMENT OF MATHEMATICS,
COLUMBIA, MO 65211, U.S.A.
nigel@math.missouri.edu

References

[1] J. Avron, R. Seiler and B. Simon, *The index of a pair of projections*, J. Funct. Anal. **120** (1994), 220–237.

[2] E. G. Effros, *Why the circle is connected*, Math. Intelligencer **11** (1989), 27–34.

[3] A. Pietsch, Eigenvalues and s-numbers, Cambridge Studies in Advanced Mathematics 13, Cambridge University Press, Cambridge, 1987.