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LOCAL STRUCTURE THEORY FOR QUASI-NORMED SPACES (*)

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ABSTRACT. – We investigate the local structure of quasi-normed spaces. We show that some of the striking positive results in the local theory of Banach spaces are true also in the quasi-normed setting. In particular, we establish the quotient-subspace theorem and the cotype 2 theorem and extend the equivalence theorems for weak cotype 2 spaces. This shows that convexity is not essential for these results.

1. Introduction

A *quasi-norm* $\| \ \|$, defined on a real vector space X, is a map $X \to \mathbf{R}$ such that

(1) $\|x\| > 0$ for $x \neq 0$,

(2) $\|tx\| = |t| \, \|x\|$ for $t \in \mathbf{R}$, $x \in X$,

(3) $\|x+y\| \leqslant \kappa(\|x\| + \|y\|)$ for $x, y \in X$,

where $\kappa = \kappa(X)$ is a constant, the modulus of concavity of X. Here $\kappa = 1$ if, and only if, $\| \ \|$ is a norm.

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If $0<r\leqslant 1$ is chosen such that $1/r=\log_2(2\kappa)$ then the formula

$$\|x\|_r=\inf\{(\sum_{i=1}^n\|x_i\|^r)^{1/r};\ x_i\in X,\ i=1,2,\ldots n\}$$

defines another quasi-norm $\|\ \|_r$ which also satisfies the inequalities

(4) $\|x\|_r\leqslant\|x\|\leqslant 4^{1/r}\|x\|_r$ for $x\in X$

(5) $\|x+y\|_r^r\leqslant\|x\|_r^r+\|y\|_r^r$ for $x, y\in X$.

The space $(X, \|\ \|_r)$ is called an *r-normed space*. We refer to [8] for basic properties of quasi-norms and r-norms.

There is an extensive literature on local structure theory for normed spaces (*see* e.g. [14] and [18]), but relatively little has been done in the wider context of quasi-normed spaces. It is known, however, that Dvoretzky's theorem on almost spherical sections can be extended to the non-locally convex case [2], [4] and [5].

The aim of this note is to extend some other central results in the local theory to the quasi-normed case. All our results are formulated for r-normed spaces (and depend on r) but can be reformulated for general quasi-normed spaces to depend on the modulus of concavity κ.

We first consider Milman's subspace-quotient theorem. We show that this extends to r-normed spaces in the following form. There is a constant C depending only on r so that if X is an r-normed space of finite-dimension n and $0<\lambda<1$ then there is a subspace of a quotient space E of X so that $\dim E\geqslant\lambda n$ and $d_E\leqslant C(1-\lambda)^{-1}(1-\log(1-\lambda))^{(2/r)-1}$. Here d_E is the Banach-Mazur distance of E from a Hilbert space.

If X is of cotype 2 (or, more generally of weak cotype 2) then we can as in the convex case obtain a large Euclidean subspace of X. Thus there is a constant C depending only on r and the weak cotype 2 constant of X so that if $0<\lambda<1$ then X contains a subspace E such that $\dim E\geqslant\lambda n$ and $d_E\leqslant C((1-\lambda)^{-1}(1-\log(1-\lambda)))^{(2/r)-1}$. Some improvement on this result is possible if X is of cotype 2 and r is close to one. We note here that DILWORTH [2] first showed that cotype 2 spaces contain Euclidean subspaces of proportional dimension.

We also consider the standard characterization of weak cotype 2 spaces as spaces with bounded volume ratio. It follows from our results and those of DILWORTH [2] that a space with bounded volume ratios is of weak cotype 2; we prove the converse.

It is perhaps a little surprising that the convexity of the unit ball is not required for any of these results, even though convexity and duality are both exploited in the proofs (*see* [18]); thus the geometry of star-shaped bodies is similar to that of convex bodies. In fact, our proofs of the main results use the normed space results and an iteration argument. In each case we compare X with its Banach envelope $\hat{X}$. Here it is also perhaps worth pointing out that the geometrical properties of $\hat{X}$ can be quite different from those of X. For example, in the infinite-dimensional case, there is a subspace of l_p whose Banach envelope is isomorphic to $l_1 \oplus c_0$ (*see* [7]). We also note that the Banach envelope of a subspace is, in general, quite unrelated to the envelope of whole space.

2. The results

Let X be a finite-dimensional quasi-normed space equipped with an r-norm $\| \ \|$; let B_X denote the unit ball of X. Let $\hat{X}$ be the normed space whose unit ball is the convex hull co B_X. We define $\delta_X = d(X, \hat{X})$ and $\alpha_n = \sup \{ n^{-1} \| x_1 + \ldots + x_n \| ; x_i \in B_X \}$. The following lemma is essentially due to PECK [16].

LEMMA 1. — (1) $\delta_X = \|I\| \, \|I^{-1}\|$ *where* $I: X \to \hat{X}$ *is the identity map.*

(2) *If F is any normed space then* $\max \{ \delta_X, d(\hat{X}, F) \} \leqslant d(X, F)$.

(3) $\delta_X = \sup_n \alpha_n$.

Proof. — Suppose F is a normed space. Let $T: F \to X$ be an isomorphism (onto). Then $B_X \subset \|T^{-1}\| \, T(B_F) \subset \|T^{-1}\| \, \|T\| \, B_X$. Since $T(B_F)$ is convex it follows that $B_X \subset B_{\hat{X}} \subset \|T^{-1}\| \, T(B_F)$. Taking $F = \hat{X}$ and $T = I$ proves (1). We also immediately get (2). For (3) observe that if $x \in B_{\hat{X}}$ then given $\varepsilon > 0$ we can find n points $x_1, \ldots, x_n \in B_X$ so that $(1 - \varepsilon) x = 1/n (x_1 + \ldots + x_n)$. Thus $\|x\| \leqslant \sup \alpha_n$. It follows that $\delta_X = \sup \alpha_n$. ■

The definitions of type and cotype for Banach spaces were introduced and studied in [10] and [17]; the same definitions are appropriate for quasi-normed spaces. We will use the concept of equal-norms type. If X is an r-normed space and $1 \leqslant p \leqslant 2$ we define $\hat{T}_p(X)$ to be the least constant C so that we have the inequality

$$\mathrm{Ave}_{\varepsilon_i = \pm 1} \| \textstyle\sum_{i=1}^n \varepsilon_i x_i \| \leqslant C n^{1/p} \max_{1 \leqslant i \leqslant n} \| x_i \|.$$

For future reference we also note that the cotype 2 constant $C_2(X)$ of X is the least constant C such that one has the inequality

$$\sum_{i=1}^n \|x_i\|^2 \leqslant C^2 \operatorname{Ave}_{\varepsilon_i=\pm 1} \|\sum_{i=1}^n \varepsilon_i x_i\|^2.$$

If X is isomorphic to a Hilbert space H then we write $d_X = d(X, H)$.

LEMMA 2. – *Suppose* $0<r\leqslant 1<p\leqslant 2$. *Then there is a constant* $C=C(p, r)$ *so that if* X *is an* r*-normed space then* $\delta_X \leqslant C(\hat{T}_p(X))^{\phi}$ *where* $\phi=(1/r-1)/(1/r-1/p)$.

Proof. – Suppose $x_1, \ldots, x_n \in B_X$. Then we can choose signs $\varepsilon_i = \pm 1$ such that

$$\|\sum_{i=1}^n \varepsilon_i x_i\| \leqslant n^{1/p} \hat{T}_p(X).$$

If $A=\{i : \varepsilon_i = +1\}$ then we can assume $|A| \geqslant n/2$. Thus

$$\|\sum_{i \notin A} x_i\| \leqslant (n/2)\,\alpha_{[n/2]}.$$

Thus

$$\|\sum_{i=1}^n x_i\|^r = \|\sum_{i=1}^n \varepsilon_i x_i + 2\sum_{i \notin A} x_i\|^r \leqslant (\hat{T}_p n^{1/p})^r + n^r \alpha_{[n/2]}^r.$$

Hence

$$\alpha_n^r \leqslant (\hat{T}_p)^r n^{r(1/p-1)} + \alpha_{[n/2]}^r.$$

Thus, if $1 \leqslant k < n$ then

$$\alpha_{2^n}^r \leqslant \alpha_{2^k}^r + \hat{T}_p^r \sum_{j=k+1}^{\infty} 2^{rj(1/p-1)} \leqslant 2^{k(1/r-1)} + C^r \hat{T}_p^r 2^{kr(1/p-1)}$$

where $C=C(p, r)$. We thus obtain an estimate

$$\delta_X \leqslant C \min_k \max(2^{k(1/r-1)}, \hat{T}_p 2^{k(1/p-1)})$$

and the lemma follows by choosing k optimally. ■

An immediate consequence is:

LEMMA 3. – *Suppose* $0<r\leqslant 1$. *Then there is a constant* $C=C(r)$ *so that for any* r*-normed space* $\delta_X \leqslant C d_X^{\phi}$ *where* $\phi=(1/r-1)/(1/r-1/2)$.

Proof. – This follows from the estimate $\hat{T}_2(X) \leqslant d_X$. ■

Remark. – Note that $d_X \leqslant \delta_X d_{\hat{X}}$ whence we get the inequality $d_X \leqslant C d_{\hat{X}}^{1/(1-\phi)}$. This is a quantification of the result of [6] that if a quasi-Banach space has an Hilbertian Banach envelope then it is locally convex; the argument is also similar.

We now formulate the iteration lemma which we will use. This is a slight modification of Lemma 8.5 of [18].

LEMMA 4. – *Let f be a nonnegative bounded real function on* $(0, 1)$. *Suppose that g is a function on* $(0, 1)$ *such that for some* $\eta > 0$ *we have* $g(\lambda) \geqslant \eta$ *for* $0 < \lambda < 1$ *and such that there is a constant M with* $g(\lambda) \leqslant M g(\lambda^2)$ *for* $0 < \lambda < 1$ *and there exists* $0 < \theta < 1$ *such that*

$$f(\lambda^2) \leqslant g(\lambda) f(\lambda)^{\theta}.$$

Then for all $0 < \lambda < 1$ *we have*

$$f(\lambda) \leqslant M^{1/(1-\theta)^2} g(\lambda)^{1/(1-\theta)}.$$

We are now ready to prove the promised extension of Milman's subspace-quotient theorem [11].

THEOREM 5. – *Suppose* $0 < r \leqslant 1$. *Then there is a constant* $C = C(r)$ *so that if* $0 < \lambda < 1$ *and X is an r-normed space of dimension n then there is a subspace of a quotient space of X, E say, such that* $\dim E \geqslant \lambda n$ *and*

$$d_E \leqslant C\left((1-\lambda)^{-1} \log \frac{2}{1-\lambda}\right)^{(2/r)-1}.$$

Proof. – Suppose X is any r-normed space of dimension n. We apply Milman's theorem (*see* [3], [11], [15], [18], p. 129) to the normed space $\hat{X}$, to produce a subspace of a quotient space F with $d_F \leqslant C(1-\lambda)^{-1} \log(2(1-\lambda)^{-1})$ and $\dim F \geqslant \lambda n$. Let E be the same space but with the quotient quasi-norm inherited from X. Then clearly $d(E, F) \leqslant d(X, \hat{X}) = \delta_X$. Thus if $\phi = (1/r - 1)/(1/r - 1/2)$ then

$$d_E \leqslant \delta_X d_F \leqslant C d_X^{\phi} (1-\lambda)^{-1} \log \frac{2}{(1-\lambda)}.$$

We now use the iteration argument of MILMAN. For $0<\lambda<1$ define $f(\lambda)=\inf\{d_E : E\in\mathcal{Q}\mathcal{S}(X), \dim E\geqslant\lambda n\}$. Then we obviously have

$$f(\lambda^2)\leqslant C f(\lambda)^{\phi}(1-\lambda)^{-1}\log\frac{2}{(1-\lambda)}$$

and so by Lemma 4, we obtain an estimate

$$f(\lambda)\leqslant C\left((1-\lambda)^{-1}\log\frac{2}{(1-\lambda)}\right)^{(1-\phi)^{-1}}$$

which gives the theorem. ■

We now extend the definition of weak cotype 2 spaces, [13], to the quasi-normed case. Suppose X is a continuously quasi-normed space and that H is a finite-dimensional Hilbert space. If $u : H\to X$ is a linear operator we define

$$l(u)=(\mathbf{E}(\|\textstyle\sum_{k=1}^{n} g_i u(e_i)\|^2)^{1/2}$$

where $(g_1, \ldots, g_n)$ is a sequence of independent normalized gaussian random variables and $(e_1, \ldots, e_n)$ is an orthonormal basis of H. For $k\geqslant 1$ we define the k-th approximation number of u by $a_k(u)=\inf\{\|u-v\|; v : H\to X, \text{rank } v<k\}$. X is said to be weak cotype-2 with weak cotype-2 constant $w C_2(X)$ if $w C_2(X)$ is the least constant C such that the inequality $k^{1/2} a_k(u)\leqslant C l(u)$ holds for all such operators u.

It is easy to verify that the argument for normed spaces extends without alteration to the quasi-normed case to show that a cotype-2 space is also of weak cotype 2 and that $w C_2(X)\leqslant C_2(X)$.

The following theorem extends to quasi-normed spaces results of MILMAN and PISIER [13] (*see* also [18]); for similar results for cotype 2 spaces see MILMAN [11] and subsequent work in [3] and [15].

THEOREM 6. – *Suppose* $0<r<1$. *Then there is a constant* $C=C(r)$ *so that if* $0<\lambda<1$ *and* X *is an* r*-normed space of dimension* n *then* X *has a subspace* E *with* $\dim E\geqslant\lambda n$ *and*

$$d_E\leqslant C\left(w C_2(X)(1-\lambda)^{-1}\log\frac{2 w C_2(X)}{1-\lambda}\right)^{(2/r)-1}$$

Proof. — Suppose X is an r-normed space of dimension n. Then, using [9] or Theorem 3.1 of [18], there exists a Hilbert space H and a linear isomorphism $u: H \to \hat{X}$ such that $l(u) = l^*(u^{-1}) = n^{1/2}$. By Lemma 3.10 of [18] we have $l(u^{*-1}) \leqslant K(\hat{X}) n^{1/2}$ where $K(\hat{X})$ is the K-convexity constant of $\hat{X}$. Thus $K(\hat{X}) \leqslant C(1 + \log d_{\hat{X}})$ where C is a numerical constant (Theorem 2.5 of [18]). Now, by applying the results of [3], [12], [15], or [18] Theorem 5.8, p. 75, we can conclude that there is a subspace G_1 of H of codimension less than $1/2(1-\lambda)n$ such that

$$\| h \| \leqslant C(1-\lambda)^{-1/2} K(\hat{X}) \| uh \|_{\hat{X}}$$

for $h \in G_1$, where again C is a numerical constant.

Now consider $u: H \to X$. Then it is clear that $l_X(u)$ (*i.e.* $l(u)$ when u is considered with range X) can be estimated by $\delta_X n^{1/2}$. Thus there is subspace G_2 of H with codimension at most $1/2(1-\lambda)n$ such that

$$\| uh \|_X \leqslant w\, C_2(X)\, 2^{1/2} (1-\lambda)^{-1/2} \delta_X \| h \|$$

for $h \in G_2$.

If $\phi = (1/r - 1)/(1/r - 1/2)$ then we can use the estimates $\delta_X \leqslant d_X^{\phi}$ and $d_{\hat{X}} \leqslant d_X$. Let $E = u(G_1 \cap G_2)$. Then clearly $\dim E \geqslant \lambda n$ and

$$d_E \leqslant C(w\, C_2(X))(1-\lambda)^{-1} d_X^{\phi} (1 + \log d_X).$$

We conclude again by an iteration procedure. We define $f(\lambda)$ for $0 < \lambda < 1$ to be the infimum of d_E over all subspaces of X with $\dim E \geqslant \lambda n$. We have

$$f(\lambda^2) \leqslant C(w\, C_2(X))(1-\lambda)^{-1}(1 + \log f(\lambda)) f(\lambda)^{\phi}.$$

Pick any $\phi < \theta < 1$. Then we have an estimate

$$f(\lambda^2) \leqslant C(\theta)(w\, C_2(X))(1-\lambda)^{-1} f(\lambda)^{\theta}$$

and so by Lemma 4 we conclude that

$$f(\lambda) \leqslant C(w\, C_2(X))^{1/(1-\theta)} (1-\lambda)^{-1/(1-\theta)}.$$

where C depends only on θ. Returning to our original estimate we have

$$f(\lambda^2) \leqslant C(w\, C_2(X))(1-\lambda)^{-1} \left(1 + \log \frac{w\, C_2(X)}{1-\lambda}\right) f(\lambda)^{\phi}.$$

Now by Lemma 4 we get the estimate

$$f(\lambda)\leqslant C(w\,C_2(X))^{1/(1-\phi)}(1-\lambda)^{1/(1-\phi)}\left(1+\log\frac{w\,C_2(X)}{1-\lambda}\right)^{1/(1-\phi)}$$

for $0<\lambda<1$ and so the theorem is proved. ■

Remark. — If $4/5<r<1$ and X is of cotype 2 one can do slightly better. In this case one can estimate $C_2(\hat{X})\leqslant\delta_X C_2(X)$ and use the better estimates for cotype 2 spaces [3], [12], [15] to produce a subspace E of X with $\dim E\geqslant\lambda n$ so that

$$d_E\leqslant C\,\delta_X\,C_2(\hat{X})(1-\lambda)^{-1/2}\log\frac{2\,C_2(\hat{X})}{(1-\lambda)^{1/2}}.$$

One thus obtains

$$d_E\leqslant C\,d_X^{2\phi}\,C_2(X)(1-\lambda)^{-1/2}\left(1+\log\frac{C_2(X)\,d_X}{(1-\lambda)^{1/2}}\right)$$

and iteration will lead to a slightly improved order of growth as long as $\phi<1/3$, *i.e.*

$$d_E\leqslant C\left(C_2(X)(1-\lambda)^{-1/2}\left(\log\frac{2\,C_2(X)}{1-\lambda}\right)\right)^{1/(1-2\phi)}.$$

Now if X is any r-normed space and $0<\lambda<1$, we define $d_X(\lambda)$ to be the least constant $d\leqslant\infty$ such that if $E\subset X$ is a finite-dimensional subspace there is a subspace F of E with $\dim F\geqslant\lambda\dim E$ and so that $d_F\leqslant d$. The following theorem for normed spaces was proved by MILMAN and PISIER [13] (refined in [18]).

THEOREM 7. — *Let X be an r-normed space. Then the following conditions on E are equivalent*:

(1) *X is a weak cotype 2 space.*

(2) *There exists $0<\lambda<1$ such that $d_X(\lambda)<\infty$.*

(3) *For every $0<\lambda<1$ we have $d_X(\lambda)<\infty$.*

Furthermore there is a constant C depending only on r such that these conditions imply that

$$d_X(\lambda)\leqslant C\left(w\,C_2(X)(1-\lambda)^{-1}\log\frac{2\,w\,C_2(X)}{1-\lambda}\right)^{(2/r)-1}.$$

Proof. – (1) implies (3) by Theorem 6, and (3) clearly implies (2). The implication (2) implies (1) follows by mimicking the convex case (*see* [18], p. 153-155). Some adjustment must be made in Lemma 10.3 to handle the fact that X does not obey the triangle law, but otherwise the proof is unchanged. Finally the concluding estimate follows from Theorem 6. ■

Lastly we consider volume ratios. If X is an r-normed space of dimension n we define the volume ratio $\mathrm{vr}(X)$ by the formula

$$\mathrm{vr}(X) = \frac{(\mathrm{vol}\, B_X)^{1/n}}{(\mathrm{vol}\, D)^{1/n}}$$

where D is an ellipsoid of maximal volume contained in B_X. See Szarek [19], and also Dilworth [2] for the quasi-normed case. It was shown by Bourgain and Milman [1] that cotype 2 normed spaces have bounded volume ratios; that this property characterizes weak cotype 2 normed spaces was established by Milman and Pisier [13]. The reader is referred to [18] for more details.

Theorem 8. – *Let X be an infinite-dimensional r-normed space. Then X has weak cotype 2 if and only if there exists a constant $C = C(X)$ such that for every finite-dimensional subspace E of X we have* $\mathrm{vr}(E) \leqslant C$.

Proof. – If X has bounded volume ratios then by Theorem 16 of Dilworth [2] $d_X(\lambda) < \infty$ whenever $0 < \lambda < 1$ so that X is weak cotype 2.

We turn to the other direction. Notice first that we have an estimate $d_X(\lambda) \leqslant C(1-\lambda)^{-a}$ where C and a are positive constants depending on X.

Let E be an arbitrary N-dimensional subspace of X. Let D be an ellipsoid of maximal volume contained in B_E. We will identify E with $\mathbf{R}^N$ in such a way that D induces the standard inner-product norm $\| \ \|_2$; we calculate volumes with respect to usual product measure on $\mathbf{R}^N$. If we work in a subspace H we write vol_H for the volume of the intersection of a set with the subspace H (with respect to that subspace). In the special case when $H = \{0\}$, we write $\mathrm{vol}_H(A) = 1$ if $0 \in A$ and $\mathrm{vol}_H(A) = 0$ otherwise.

For $k = 2, 3, \ldots, E$ contains a subspace F_k with $\dim F_k \geqslant N(1-2^{-k})$ and such that $d_{F_k} \leqslant d_X(1-2^{-k}) \leqslant C\, 2^{ak}$. If $k > \log_2 N$ then $F_k = E$ of course. For each k there is an ellipsoid D_k with $D_k \subset B_E \cap F_k \subset d_{F_k} D_k$.

We claim that there is a further subspace G_k of F_k with $\dim G_k \geqslant \dim F_k - 2^{-k} N$ so that $G_k \cap D_k \subset 2^{(k+3)/r}(G_k \cap D)$. To see this, suppose not. Then by standard theory of quadratic forms there is a subspace V of F_k so that $\dim V > 2^{-k} N$ and so that

$$V \cap D_k \supset 2^{(k+3)/r}(V \cap D).$$

Form the orthogonal complement H of V in E. Then if

$$x \in 2^{-k/r}(V \cap D_k) \qquad \text{and} \qquad y \in (1-2^{-k})^{1/r}(H \cap D),$$

by the r-subadditivity of the norm, we have $\|x+y\| \leqslant 1$. Now let Δ be the ellipsoid defined by $x+y \in \Delta$ if $x \in V$, $y \in H$ and

$$8^{-2/r}\|x\|_2^2 + (1-2^{-k})^{-2/r}\|y\|_2^2 \leqslant 1.$$

Then $\operatorname{vol} \Delta \leqslant \operatorname{vol} D$. Hence if $v = \dim V$ then

$$8^{v/r}(1-2^{-k})^{(N-v)/r} \leqslant 1.$$

We conclude that $8 \leqslant (1-2^{-k})^{1-N/v}$. However $N/v \leqslant 2^k$ and so the right-hand side is smaller than 4. This contradiction establishes our claim.

Next we set $W_k = \bigcap_{j \geqslant k} G_j$. Then $\dim W_k \geqslant (1-4.2^{-k})N$. Further we have $B_E \cap W_k \subset 2^{(k+3)/r} d_{F_k} D$.

Before continuing the argument, we note a consequence of the Brunn-Minkowski theorem ([17], p. 3). Let $\gamma = 2^{1/r-1}$. Suppose A is a closed subset of E satisfying $A = -A$ and $A + A \subset 2^{1/r} A = 2\gamma A$. Then for any subspace H of E and any $e \in E$, we have the inequality $\operatorname{vol}_H(e+A) \leqslant \gamma^h \operatorname{vol}_H(A)$. To see this note that $\operatorname{vol}_H(e+A) = \operatorname{vol}_H(-e+A)$ and $((e+A) \cap H) + ((-e+A) \cap H) \subset (A+A) \cap H \subset 2\gamma A \cap H$. The inequality then follows from the Brunn-Minkowski inequality.

For $k \geqslant 3$, we define

$$\beta_k = \sup_{e \in E} \operatorname{vol}_{W_k}(e + B_E).$$

It then follows immediately from the argument above that

$$\beta_k \leqslant \gamma^{\dim W_k} \operatorname{vol}_{W_k}(B_E).$$

In particular, since $\dim W_3 \geqslant N/2$, we have $\beta_3 \leqslant C^N \operatorname{vol}_{W_3}(D)$ for a constant C depending only on X.

For $k \geqslant 3$, we suppose that Y_k is the orthogonal complement of W_k in W_{k+1}. For convenience we set $Y_2 = W_3$. Then $\dim Y_k \leqslant 4.2^{-k} N$. Let t be the first index for which $W_t = E$.

Let P_k be the orthogonal projection of E onto Y_k. Let $h=\dim Y_k$. For $e \in E$, using the volume argument described above,

$$(\mathrm{vol}_{Y_k}(P_k((e+B_E) \cap W_{k+1})) \leqslant \gamma^h (\mathrm{vol}_{Y_k}(P_k(B_E \cap W_{k+1}))).$$

However

$$P_k(B_E \cap W_{k+1}) \subset 2^{(k+4)/r} d_{F_{k+1}} D \cap Y_k.$$

We also have $d_{F_{k+1}} \leqslant C\, 2^{a(k+1)}$. Using these estimates we obtain:

$$\log_2(\mathrm{vol}_{Y_k}(P_k((e+B_E) \cap W_{k+1})) \leqslant Ck \dim Y_k + \log_2 \mathrm{vol}_{Y_k}(D)$$

where C depends only on X. (This has an appropriate interpretation if Y_k reduces to $\{0\}$ as explained earlier.)

Now (*cf.* [18], p. 132)

$$\begin{aligned} \mathrm{vol}_{W_{k+1}}(e+B_E) &= \int_{Y_k} \mathrm{vol}_{W_k}(e-x+B_E)\, dx \\ &= \int_{P_k((e+B_E) \cap W_{k+1})} \mathrm{vol}_{W_k}(e-x+B_E)\, dx \\ &\leqslant \beta_k \mathrm{vol}_{Y_k}(P_k((e+B_E) \cap W_{k+1})). \end{aligned}$$

Thus

$$\log_2 \beta_{k+1} \leqslant \log_2 \beta_k + Ck \dim Y_k + \log_2 \mathrm{vol}_{Y_k}(D).$$

Since $\dim Y_k \leqslant 4.2^{-k} N$ we obtain

$$\log_2 \beta_t - \log_2 \beta_3 \leqslant CN(1+\textstyle\sum_{k=3}^{\infty} k\, 2^{-k}) + \sum_{k=3}^{t-1} \log_2 \mathrm{vol}_{Y_k}(D)$$

for a suitable constant $C=C(X)$.

At this point, we introduce the subspaces

$$Z_2 = E \qquad \text{and} \qquad Z_k = Y_k + Y_{k+1} + \dots Y_{t-1} \qquad \text{for} \quad k \geqslant 3.$$

Thus Z_k is the direct sum of Z_{k+1} and Y_k. It follows from the explicit formula for the volume of the Euclidean ball (*see* [18], p. 11) that

$$\mathrm{vol}_{Y_k}(D)\, \mathrm{vol}_{Z_{k+1}}(D) \leqslant 2^{\dim Z_k} \mathrm{vol}_{Z_k}(D).$$

Hence

$$\sum_{k=2}^{t-1} \log_2 \mathrm{vol}_{Y_k}(D) \leqslant \sum_{k=2}^{t-1} \dim Z_k + \log_2 \mathrm{vol}(D)$$

Since $\dim Z_k \leqslant 8 . 2^{-k} N$, we conclude that

$$\log_2 \beta_t - \log_2 \mathrm{vol}(D) \leqslant CN + \log_2 \beta_3 - \log_2 \mathrm{vol}_{W_3}(D) \leqslant C' N$$

where C, C' depend only on X.

This implies $\mathrm{vr}(E) \leqslant C$ where $C = C(X)$. ■

We conclude by remarking that these results suggest the question whether every finite-dimensional r-normed space contains a large (proportional dimensional) locally convex subspace. More precisely one can define for an arbitrary, possibly infinite-dimensional, quasi-Banach space X, $\delta_X(\lambda)$ to be the infimum of all δ such that if E is a finite-dimensional subspace of X then E contains a subspace F with $\dim F \geqslant \lambda \dim E$ and $\delta_F \leqslant \delta$. The problem then is whether for some (or for every) $0 < \lambda < 1$ and every X we have $\delta_X(\lambda) < \infty$. Of course this is true for spaces such as L_p for $p < 1$ as they are of cotype 2.

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