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$L^p(X)$ $(1 \leqslant p < \infty)$ HAS THE PROPERTY (u) WHENEVER X DOES

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ABSTRACT. — We show that if X is a Banach space having the property (u) then $L^p(X)(1 \leqslant p < \infty)$ has the same property. An application of the techniques used to prove this result is given concerning unconditionally converging operators on $C(K, X)$ spaces.

If X is Banach space, $(\Omega, \Sigma, \lambda)$ a probability space and $1 \leqslant p < \infty$, we denote by $L^p(X)$ the space of p-Bochner integrable functions from Ω to X equipped with the norm $\| f \|_p = \left(\int_\Omega \| f(\omega) \|^p \, d\lambda(w) \right)^{1/p}$. If X is the scalar field, then $L^p(X)$ will be denoted by L^p. In the sequel p will always be in the interval $[1, +\infty)$. For a series $\sum_n x_n$ in the Banach space X we say that $\sum_n x_n$ is a **weakly unconditionally Cauchy** (w. u. c) **series** in X if it satisfies one of the following equivalent statements

(a) $\sum_n |x^*(x_n)| < \infty$, *for every* $x^* \in X^*$;

(b) $\sup \{ \| \sum_{n \in \sigma} x_n \| ; \sigma$ finite subset of $\mathbf{N} \} < \infty$;

(c) $\sup_n \sup_{\varepsilon_i = \pm 1} \| \sum_{i=1}^n \varepsilon_i x_i \| < \infty$.

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In [11] spaces with property (u) were introduced, for this recall that a Banach space E has **property** (u) if for any weakly Cauchy sequence (e_n) in E there exists a weakly unconditionally Cauchy series $\sum_n x_n$ in E such that the sequence $(e_n - \sum_{i=1}^n x_i)$ converges weakly to zero in E. Any Banach space E with unconditional basis or more generally any space with unconditional reflexive decomposition has (u) and so is the case of any weakly sequentially complete Banach space and any order continuous Banach lattice [9]. In particular, any L^p, $1 \leqslant p < \infty$, has the property (u). Another class of spaces having property (u) are those spaces which are M-ideals in their biduals [5] and under certain conditions, spaces of compact operators on a Banach space X have the property (u) [6]. It is clear that a Banach space that has the property (u) is weakly sequentially complete if and only if X does not contain a copy of c_0. In [8] Kwapien showed that $L^p(X)$ contains a copy of c_0 if and only if X contains a copy of c_0. Talagrand [13] showed that if X is weakly sequentially complete then the same is true for $L^p(X)$. In this paper we show that if X is a Banach space having the property (u) then $L^p(X)$ has the same property. The techniques used to prove this result allow us to extend and strengthen a result in [12].

Let $\mathscr{F}$ be the set of finite subsets of the natural numbers $\mathbf{N}$. All notions and notation used and not defined can be found in [2], [3], [4] or [9].

Proposition 1. — *Let X be a Banach space with the property (u). There exists a constant $C>0$ so that for any weakly Cauchy sequence $(x_n)_{n \geqslant 1}$, there is a weakly unconditionally Cauchy series $\sum_{i=1}^{\infty} y_i$ such that*

1° $(x_n - \sum_{i=1}^n y_i)$ converges weakly to zero in X.

2° $\sup_{\sigma \in \mathscr{F}} \| \sum_{i \in \sigma} y_i \| \leqslant C \liminf_{n \to \infty} \| x_n \|$.

Proof. — Let

$$\beta_1(X) = \{ u \in X^{**};\ u \text{ is a weak* limit of a sequence in } X \}.$$

By [10], the space $\beta_1(X)$ is norm closed in X^{**}. Let

$$E = \{ (y_n)_{n \geqslant 1} \in X^{\mathbf{N}};\ \textstyle\sum_{i=1}^{\infty} y_i \text{ is w. u. c.} \}.$$

On E, we put the norm

$$||| ((y_n)_{n \geqslant 1}) ||| = \sup_{\sigma \in \mathscr{F}} \| \textstyle\sum_{i \in \sigma} y_i \|.$$

It is easy to check that E equipped with this norm is a Banach space. Let T: $E \to \beta_1(X)$ be defined by

$$T((y_n)_{n \geqslant 1}) = \text{weak*-lim}_{n \to \infty} \sum_{i=1}^n y_i.$$

It is clear that T is linear. Notice that if $x^* \in X^*$ with $\|x^*\| \leqslant 1$ then

$$\begin{aligned} |x^*(T((y_n)_{n \geqslant 1})| &= \lim_{n \to \infty} |x^*(\textstyle\sum_{i=1}^n y_i)| \\ &\leqslant \liminf_{n \to \infty} \|\textstyle\sum_{i=1}^n y_i\| \\ &\leqslant |||(y_n)_{n \geqslant 1}|||. \end{aligned}$$

This shows that T is bounded. We claim that T is onto. To see that let $v \in \beta_1(X)$ and choose a sequence $(x_n)_{n \geqslant 1}$ in X so that $v = \text{weak*-lim}_{n \to \infty} x_n$. Since X has the property (u), one can choose a weakly unconditionally Cauchy series $\sum_{i=1}^\infty y_i$ so that $(x_n - \sum_{i=1}^n y_i)$ converges weakly to zero. This implies that $v = \text{weak*-lim}_{n \to \infty} \sum_{i=1}^n y_i$. Hence $T((y_n)_{n \geqslant 1}) = v$. Apply now the Open Mapping Theorem to find a constant C so that for every $v \in \beta_1(X)$ with $\|v\| \leqslant 1$ one can find an element $(y_n)_{n \geqslant 1} \in E$ such that $|||(y_n)_{n \geqslant 1}||| \leqslant C$ and $v = \text{weak*-lim}_{n \to \infty} \sum_{i=1}^n y_i$. To finish the proof, let $(x_n)_{n \geqslant 1}$ be a weakly Cauchy sequence in X and choose $w \in \beta_1(X)$ such that $w = \text{weak*-lim}_{n \to \infty} x_n$. Let $\alpha = \liminf_{n \to \infty} \|x_n\|$. Without loss of generality we can suppose that $\alpha > 0$. Let $v = w/\alpha$, then $\|v\| \leqslant 1$. By the above, there is a sequence $(v_n)_{n \geqslant 1} \in E$ with $|||(v_n)_{n \geqslant 1}||| \leqslant C$ and

$$v = \frac{w}{\alpha} = \text{weak*-lim}_{n \to \infty} \sum_{i=1}^n v_i$$

Putting $y_n = \alpha v_n$ for every $n \geqslant 1$ gives

$$\sup_{\sigma \in \mathcal{F}} \|\sum_{i \in \sigma} y_i\| \leqslant \alpha C = C \liminf_{n \to \infty} \|x_n\|.$$

Since

$$w = \text{weak*-lim}_{n \to \infty} x_n = \text{weak*-lim}_{n \to \infty} \sum_{i=1}^n y_i,$$

it follows that $(x_n - \sum_{i=1}^n y_i)$ converges weakly to zero in X.

Let X be a separable Banach space having the property (u). Let C be the constant found for X in Proposition 1.

Consider now the following subset H of $X^{\mathbf{N}} \times X^{\mathbf{N}}$.

An element $((x_n)_{n\geqslant 1}, (y_n)_{n\geqslant 1})$ belongs to H if and only if the following conditions are satisfied:

(i) $\sup_{\sigma\in\mathcal{F}}\|\sum_{i\in\sigma}y_i\|\leqslant C\liminf_{n\to\infty}\|x_n\|$;

(ii) For every $n\geqslant 1$ there exist a natural number m_n and rational numbers $t_n, t_{n+1}, \ldots, t_{n+m_n}$ so that

(*a*) $t_i\geqslant 0$ for every i, and $\sum_{i=n}^{n+m_n} t_i=1$;

(*b*) $\|\sum_{i=n}^{n+m_n} t_i(x_i-\sum_{j=1}^{i} y_j)\|\leqslant 1/n$.

PROPOSITION 2. — *The set H is a Borel subset of $X^{\mathbf{N}}\times X^{\mathbf{N}}$ when the latter is equipped with the product topology.*

Proof. – Let

$$A=\{((x_n)_{n\geqslant 1}, (y_n)_{n\geqslant 1})\in X^{\mathbf{N}}\times X^{\mathbf{N}};\ \sup_{\sigma\in\mathcal{F}}\|\sum_{i\in\sigma}y_i\|\leqslant C\liminf_{n\to\infty}\|x_n\|\}.$$

It is clear that A is a Borel subset of $X^{\mathbf{N}}\times X^{\mathbf{N}}$. For each $m\geqslant 0$, let $\mathbf{Q}^{m+1}$ be the product of $m+1$ copies of the rationals and let

$$T_m=\{t=(t_0, t_1, t_2, \ldots, t_m)\in\mathbf{Q}^{m+1};\ t_i\geqslant 0 \text{ for each } i \text{ and } \textstyle\sum_{i=0}^{m} t_i=1\}$$

For $t\in T_m$ and $n\geqslant 1$, let

$$H_{t,n}=\left\{((x_n)_{n\geqslant 1}, (y_n)_{n\geqslant 1})\in X^{\mathbf{N}}\times X^{\mathbf{N}};\ \|\textstyle\sum_{i=n}^{n+m} t_{i-n}(x_i-\sum_{j=1}^{i} y_j)\|\leqslant \dfrac{1}{n}\right\}.$$

Each set $H_{t,n}$ is closed and

$$H=A\cap(\textstyle\bigcap_{n=1}^{\infty}\bigcup_{m=1}^{\infty}\bigcup_{t\in T_m}H_{t,n})$$

so H is a Borel subset of $X^{\mathbf{N}}\times X^{\mathbf{N}}$.

PROPOSITION 3. — *For every weakly Cauchy sequence $(x_n)_{n\geqslant 1}$ in X there is a weakly unconditionally Cauchy series $\sum_{i=1}^{\infty} y_i$ in X so that*

$$((x_n)_{n\geqslant 1}, (y_n)_{n\geqslant 1})\in H.$$

Proof. – By Proposition 1, one can choose a weakly unconditionally Cauchy series $\sum_{i=1}^{\infty} y_i$ so that

(i) $(x_n-\sum_{i=1}^{n} y_i)$ converges weakly to zero in X;

(ii) $\sup_{\sigma\in\mathcal{F}}\|\sum_{i\in\sigma}y_i\|\leqslant C\liminf_{n\to\infty}\|x_n\|$.

It follows from Mazur's theorem that zero must be in the norm closed convex hull of $\{x_i - \sum_{j=1}^{i} y_j\}_{i=n}^{\infty}$ for any $n \geqslant 1$.To finish the proof use this and the fact that the rationals form a dense subset of the reals to conclude that $((x_n)_{n \geqslant 1}, (y_n)_{n \geqslant 1}) \in H$.

In the sequel we let W be the set

$$W = \{(x_n)_{n \geqslant 1} \in X^{\mathbf{N}};\ (x_n)_{n \geqslant 1} \text{ is a weakly Cauchy sequence in } X\}.$$

THEOREM 4. — *There is an analytic subset M of $X^{\mathbf{N}}$ that contains W and a sequence $(S_i)_{i \geqslant 1}$ of universally measurable maps from M to X such that if $x = (x_n)_{n \geqslant 1}$ is a weakly Cauchy sequence in X we have*

(i) $(x_n - \sum_{i=1}^{n} (S_i(x))$ *converges weakly to zero in X;*

(ii) $\sup_{\sigma \in \mathscr{F}} \| \sum_{i \in \sigma} S_i(x) \| \leqslant C \liminf_{n \to \infty} \| x_n \|$.

Proof. — Let $L : H \to X^{\mathbf{N}}$ be the restriction to H of the first projection of $X^{\mathbf{N}} \times X^{\mathbf{N}}$ onto $X^{\mathbf{N}}$. Let $M = L(H)$, then M is analytic since it is the continuous image of a Borel set in the Polish space $X^{\mathbf{N}} \times X^{\mathbf{N}}$. Proposition 3 shows that M contains W. By [2] (Theorem 8.5.3, p. 286), there exists a universally measurable map $S : M \to X^{\mathbf{N}}$ such that for every $(x_n)_{n \geqslant 1} \in M$ we have $((x_n)_{n \geqslant 1}, S((x_n)_{n \geqslant 1})) \in H$. For each $i \geqslant 1$, let $\theta_i : X^{\mathbf{N}} \to X$ be defined by $\theta_i((y_n)_{n \geqslant 1}) = y_i$ and let $S_i = \theta_i \circ S$. To finish the proof let $x = (x_n)_{n \geqslant 1} \in W$. Since $((x_n)_{n \geqslant 1}, S((x_n)_{n \geqslant 1})) \in H$, it follows that

$$\sup_{\sigma \in \mathscr{F}} \| \sum_{i \in \sigma} S_i(x) \| \leqslant C \liminf_{n \to \infty} \| x_n \|$$

and for every $n \geqslant 1$ there exist m_n and $t_n, t_{n+1}, \ldots, t_{n+m_n}$ of rationals so that

(*a*) $t_i \geqslant 0$ for every i, and $\sum_{i=n}^{n+m_n} t_i = 1$;

(*b*) $\| \sum_{i=n}^{n+m_n} t_i (x_i - \sum_{j=1}^{i} S_j(x)) \| \leqslant 1/n$.

This together with the fact that $(x_n)_{n \geqslant 1}$ is a weakly Cauchy sequence imply that the sequence $(x_i - \sum_{j=1}^{i} S_j(x))$ converges weakly to zero.

We are now ready to prove our main Theorem.

THEOREM 5. — *Let X be a Banach space having the property (u) then $L^p(X)$ enjoys the same property.*

Proof. — We will give the proof for $p = 1$, the proof for any $p \in (1, \infty)$ can be obtained with obvious and minor changes. Without loss of generality we can and do assume that X is separable. Let $(h_n)_{n \geqslant 1}$ be a weakly Cauchy sequence in $L^1(X)$. By [13], the sequence $(h_n)_{n\geqslant 1}$ can be decomposed as

$$h_n = g_n + w_n \text{ for every } n \geqslant 1$$

where for almost all ω, the sequence $(g_n(\omega))_{n\geqslant 1}$ is a weakly Cauchy sequence in X and w_n converges weakly to zero in $L^1(X)$. We can suppose that for every ω the sequence $(g_n(\omega))_{n\geqslant 1}$ is a weakly Cauchy sequence in X. Let W, M and H as in Theorem 4 and consider the map

$$g: \quad \Omega \to W$$

defined by

$$g(\omega) = (g_n(\omega))_{n\geqslant 1}.$$

For each $i \geqslant 1$ let

$$S_i : M \to X$$

be the universally measurable map obtained by Theorem 4 and let $f_i = S_i \circ g$. Since S_i is universally measurable, it follows that f_i is λ-measurable. The properties of the sequence $(S_i)_{i\geqslant 1}$ imply that

$$(*) \qquad \sup_{\sigma \in \mathscr{F}} \| \textstyle\sum_{i \in \sigma} f_i(\omega) \| \leqslant C \liminf_{n \to \infty} \| g_n(\omega) \|$$

and

$$(**) \qquad (g_n(\omega) - \textstyle\sum_{i=1}^{n} f_i(\omega)) \text{ converges weakly to zero in } X$$

for every $\omega \in \Omega$. The sequence $(g_n)_{n\geqslant 1}$ is a weakly Cauchy sequence in $L^1(X)$ and therefore it is uniformly integrable. Let $h \in (L^1(X))^*$. By [7] (Theorem 7, page 94), we have that h is a map from Ω to X^* that is essentially bounded and weak* scalarly measurable. This map h acts on an element f of $L^1(X)$ as follows:

$$h(f) = \int_\Omega \langle f(\omega), h(\omega) \rangle \, d\lambda.$$

For every $n \geqslant 1$ and $\omega \in \Omega$, let

$$s_n(\omega) = \langle g_n(\omega) - \textstyle\sum_{i=1}^{n} f_i(\omega), h(\omega) \rangle.$$

Let $\varepsilon > 0$, choose a $\delta > 0$ so that if $\lambda(B) < \delta$ then

$$\int_B \| g_n(\omega) - \sum_{i=1}^n f_i(\omega) \| \, d\lambda < \varepsilon$$

for every $n \geq 1$.

This implies that if $\lambda(B) < \delta$ then $\int_B |s_n(\omega) \, d\lambda < \| h \| \varepsilon$ for every $n \geq 1$. This shows that the sequence $(s_n)_{n \geq 1}$ is uniformly integrable in L^1. Since this sequence converges pointwise to zero by (**), then it converges in L^1 and consequently

$$\lim_{n \to \infty} \int_\Omega s_n(\omega) \, d\lambda = 0.$$

This shows that the sequence

$$(g_n - \sum_{i=1}^n f_i) \text{ converges weakly to zero in } L^1(X).$$

Now (*) implies that $\sum_{i=1}^\infty f_i$ is weakly unconditionally Cauchy in $L^1(X)$. To finish the proof, notice that

$$h_n - \sum_{i=1}^n f_i = g_n - \sum_{i=1}^n f_i + w_n$$

and the sequence $(w_n)_{n \geq 1}$ converges weakly to zero in $L^1(X)$.

Now, we would like to offer another application of Theorem 4. Before doing that, let us recall some definitions

Let $T: E \to F$ be a bounded linear operator from a Banach space E into a Banach space F. We say that T is **weakly completely continuous** (w. c. c) (also called a Dieudonné operator) if for every weakly Cauchy sequence (x_n) in E, the sequence (Tx_n) converges weakly in F, and we say that T is **unconditionally converging** if for every weakly unconditionally Cauchy series $\sum_n x_n$ in E, the series $\sum_n Tx_n$ converges unconditionally in F. It is clear that T weakly compact implies T weakly completely continuous which in turn implies T unconditionally converging. It is also evident that if a Banach space E has property (u) then every unconditionally converging operator on E is weakly completely continuous. A. Pełczynski [11] considered a class of Banach spaces on which every unconditionally converging operator is weakly compact, such space are said to have Pelczynski's property (V). In [11], Pełczynski showed that if K is a compact Hausdorff space then $C(K)$ has property (V). Let us agree to say that a Banach space E has the property semi-(V) if every unconditionally converging operator is weakly completely continuous. It is clear that if has E has property (V), then it has property semi-(V), the converse is of course not true (*i.e.* $E=l_1$). In the sequel we denote by $C(K, E)$ the Banach space of all E-valued continuous functions on K under sup norm.

The following theorem extends and strengthens Theorem 3 of [12].

THEOREM 6. — *If X has the property* (u) *then $C(K, X)$ has the property semi-*(V).

Proof. – As in [12], we can and do assume that X is separable and K is metrizable. Let $(f_n)_{n\geqslant 1}$ be a weakly Cauchy sequence in $C(K, X)$. Let W, M and H be as in Theorem 4. Consider the following map $g: K \rightarrow W$ defined by $g(k)=(f_n(k))_{n\geqslant 1}$. For $i\geqslant 1$ let $S_i: M\rightarrow X$ be the universally measurable map whose existence is guaranteed by Theorem 4. For every $n\geqslant 1$, let $\psi_n(k)=S_n(g(k))$ for every $k\in K$. From now on the theorem goes on as in [12] (Theorem 3) using along the way the properties of the functions $(S_n)_{n\geqslant 1}$.

In connection to Theorem 6, it is worth mentioning that it was shown in [1] that if X has the property (u) and l_1 is not isomorphic to a closed subspace of X then $C(K, X)$ has the property (V). It is still an open problem whether $C(K, X)$ has the property (V) whenever X does. The following question seems now appropriate:

QUESTION. – *Does $C(K, X)$ have semi-*V *whenever X does?*

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