Bull. London Math. Soc. 37 (2005) 265–274
doi:10.1112/S0024609304003881

UNCONDITIONALLY CONVERGENT SERIES OF OPERATORS AND NARROW OPERATORS ON L_1

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Abstract

A class of operators is introduced on L_1 that is stable under taking sums of pointwise unconditionally convergent series, contains all compact operators and does not contain isomorphic embeddings. It follows that any operator from L_1 into a space with an unconditional basis belongs to this class.

1. *Introduction*

A famous theorem due to A. Pełczyński [**7**] states that $L_1[0,1]$ cannot be embedded in a space with an unconditional basis. A somewhat stronger version is also true [**4**]: if an operator $J : L_1[0,1] \to X$ is bounded from below, then it cannot be represented as a pointwise unconditionally convergent series of compact operators. This last theorem in fact also holds for embedding operators $J : E \to X$ if E has the Daugavet property; see [**5**].

We rephrase the theorem using the following definition.

Definition 1.1. Let $\mathcal{U}$ be a linear subspace of $\mathcal{L}(E, X)$, the space of bounded linear operators from E into X. By $\mathrm{unc}(\mathcal{U})$ we denote the set of all operators that can be represented by pointwise unconditionally convergent series of operators from $\mathcal{U}$.

In terms of this definition, the above theorem says that an isomorphic embedding operator $J : L_1[0,1] \to X$ does not belong to $\mathrm{unc}(\mathcal{K}(L_1[0,1], X))$, where $\mathcal{K}(E, X)$ stands for the space of compact operators from E into X.

Clearly, one can iterate the operation 'unc' and consider the classes

$$\mathrm{unc}(\mathrm{unc}(\mathcal{K}(L_1[0,1], X))), \qquad \mathrm{unc}(\mathrm{unc}(\mathrm{unc}(\mathcal{K}(L_1[0,1], X)))),$$

and so on. Thus the question arises as to whether one can obtain an isomorphic embedding operator through such a chain of iterations; indeed, it is not clear at the outset whether possibly $\mathrm{unc}(\mathrm{unc}(\mathcal{K}(E, X))) = \mathrm{unc}(\mathcal{K}(E, X))$.

A natural approach to generalising Pełczyński's theorem in this direction is to find a large class of operators $T : L_1[0,1] \to X$ which is stable under taking sums of pointwise unconditionally convergent series, contains all compact operators, and does not contain isomorphic embeddings.

Received 17 November 2003.

2000 *Mathematics Subject Classification* 46B04 (primary), 46B15, 46B25, 47B07 (secondary).

The work of the first author was supported by a fellowship from the *Alexander-von-Humboldt Stiftung*. The second author was supported by NSF grant DMS-9870027.

It was shown by R. Shvidkoy in his PhD thesis [**10**] and independently in [**3**] that in the case $X = L_1[0,1]$, the PP-narrow operators on $L_1[0,1]$ form such a class. Here is the definition.

Let (Ω, Σ, μ) be a fixed nonatomic probability space, and let $L_p = L_p(\Omega, \Sigma, \mu)$. By Σ^+ we denote the collection of all measurable subsets of Ω having nonzero measure.

Definition 1.2. Let $A \in \Sigma^+$.

(a) A function $f \in L_p$ is said to be a *sign supported on* A if $f = \chi_{B_1} - \chi_{B_2}$, where B_1 and B_2 form a partition of A into two measurable subsets of equal measure.

(b) An operator $T \in \mathcal{L}(L_p, X)$ is said to be *PP-narrow* if for every set $A \in \Sigma^+$ and every $\varepsilon > 0$ there is a sign f supported on A with $\|Tf\| \leqslant \varepsilon$.

The concept of a PP-narrow operator was introduced by Plichko and Popov in [**8**] under the name *narrow operator*. We use the term 'PP-narrow' in order to distinguish such operators from a related concept of a narrow operator given in [**6**], where, incidentally, PP-narrow operators were called L_1*-narrow*. It should be noted that PP-narrow operators appear implicitly in Rosenthal's papers on sign embeddings (for example, [**9**]), where an operator on L_1 is called *sign preserving* if it is not PP-narrow.

Obviously, no embedding operator is PP-narrow. On the other hand, it is clear that a compact operator T is PP-narrow. Indeed, let (r_n) be a Rademacher sequence supported on a set $A \in \Sigma^+$; that is, the r_n are stochastically independent with respect to the probability space $(A, \Sigma\,|_A, \mu/\mu(A))$ and $\mu(\{r_n = 1\}) = \mu(\{r_n = -1\}) = \mu(A)/2$. Then $r_n \to 0$ weakly and hence $Tr_n \to 0$ in norm. The same argument shows that weakly compact operators on L_1 are PP-narrow, since L_1 has the Dunford–Pettis property.

The aim of this paper is to find a class of operators with the above properties that works for general X rather than just for $X = L_1[0,1]$. For this purpose, we shall introduce the class of hereditarily PP-narrow operators in Section 2. We show that they form a linear space of operators (which is false for PP-narrow operators, at least for $p > 1$), and in Section 3 we derive a factorisation scheme for unconditional sums of such operators. This enables us to give an example of a Banach space X for which $\operatorname{unc}(\operatorname{unc}(\mathcal{K}(X,X))) \neq \operatorname{unc}(\mathcal{K}(X,X))$ (Theorem 3.3). In Section 4 we specialise to the case $p = 1$, and show that a pointwise unconditionally convergent series of hereditarily PP-narrow operators on L_1 is hereditarily PP-narrow (Theorem 4.3). As a result, it follows that no embedding operator is in any of the spaces $\operatorname{unc}(\ldots(\operatorname{unc}(\mathcal{K}(L_1,X))))$. A further consequence is that every operator from L_1 into a space with an unconditional basis is hereditarily PP-narrow and in particular PP-narrow; this implies that L_1 does not even sign-embed into a space with an unconditional basis. These last results are due to Rosenthal (in unpublished work).

In this paper we deal with real Banach spaces.

2. *Haar-like systems and hereditarily PP-narrow operators*

We start by introducing some notions that will be used throughout the paper. Denote

$$\mathcal{A}_0 = \{\varnothing\}, \qquad \mathcal{A}_n = \{-1,1\}^n, \qquad \mathcal{A}_\infty = \bigcup_{n=0}^{\infty} \mathcal{A}_n.$$

The elements of $\mathcal{A}_n$ are n-tuples of the form $(\alpha_1, \dots, \alpha_n)$ with $\alpha_k = \pm 1$. For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathcal{A}_n$ and $\alpha_{n+1} \in \{-1, 1\}$, denote by α, α_{n+1} the $(n+1)$-tuple $(\alpha_1, \dots, \alpha_n, \alpha_{n+1}) \in \mathcal{A}_{n+1}$; also, put $\emptyset, \alpha_1 = (\alpha_1)$. The elements of $\mathcal{A}_\infty$ can be written as a sequence in the following *natural order*:

$$\emptyset,\ -1,\ 1,\ (-1,-1),\ (-1,1),\ (1,-1),\ (1,1),\ (-1,-1,-1),\ \dots.$$

DEFINITION 2.1. Let $A \in \Sigma^+$.

(a) A collection $\{A_\alpha\colon \alpha \in \mathcal{A}_\infty\}$ of subsets of A is said to be a *tree of subsets on* A if $A_\emptyset = A$ and if for every $\alpha \in \mathcal{A}_\infty$ the subsets $A_{\alpha,1}$ and $A_{\alpha,-1}$ form a partition of A_α into two measurable subsets of equal measure.

(b) The collection of functions $\{h_\alpha\colon \alpha \in \mathcal{A}_\infty\}$ defined by $h_\alpha = \chi_{A_{\alpha,1}} - \chi_{A_{\alpha,-1}}$ is said to be a *Haar-like system on* A (corresponding to the tree of subsets A_α, $\alpha \in \mathcal{A}_\infty$).

It is easy to see that after deleting the constant function, the classical Haar system is an example of a Haar-like system. Moreover, every Haar-like system is equivalent to this example. In particular we make the following observations.

REMARK 2.2. (a) Let $\{h_\alpha\colon \alpha \in \mathcal{A}_\infty\}$ be a Haar-like system on A corresponding to a tree of subsets A_α, and let $1 \leqslant p < \infty$. Denote by Σ_1 the σ-algebra on A generated by the subsets A_α. Then the system $\{h_\alpha\colon \alpha \in \mathcal{A}_\infty\}$ in its natural order forms a monotone Schauder basis for the subspace $L_p^0(A, \Sigma_1, \mu)$ of $L_p(A, \Sigma_1, \mu)$ consisting of all $f \in L_p(A, \Sigma_1, \mu)$ with $\int_A f\, d\mu = 0$. Note that, for $\alpha \in \mathcal{A}_n$, $\|h_\alpha\| = \big(2^{-n}\mu(A)\big)^{1/p}$ for every Haar-like system on A.

(b) Therefore, if $\varepsilon > 0$ and $\{\varepsilon_\alpha\colon \alpha \in \mathcal{A}_\infty\}$ is a family of positive numbers such that $\sum_\alpha \varepsilon_\alpha / \|h_\alpha\| \leqslant \varepsilon/2$ and if $\{x_\alpha\colon \alpha \in \mathcal{A}_\infty\}$ is a family of vectors in a Banach space X such that $\|x_\alpha\| \leqslant \varepsilon_\alpha$, then the mapping $h_\alpha \mapsto x_\alpha$ extends to a bounded linear operator from $L_p^0(A, \Sigma_1, \mu)$ to X of norm at most ε.

LEMMA 2.3. *Let $1 \leqslant p < \infty$ and let $T\colon L_p \to X$ be a PP-narrow operator.*

(a) *For every $A \in \Sigma^+$ and every family of numbers $\varepsilon_\alpha > 0$, there is a Haar-like system $\{h_\alpha\colon \alpha \in \mathcal{A}_\infty\}$ on A such that $\|Th_\alpha\| \leqslant \varepsilon_\alpha$ for $\alpha \in \mathcal{A}_\infty$.*

(b) *For every $\varepsilon > 0$ and every $A \in \Sigma^+$, there is a σ-algebra $\Sigma_\varepsilon \subset \Sigma$ on A such that $(A, \Sigma_\varepsilon, \mu)$ is a nonatomic measure space and the restriction of T to $L_p^0(A, \Sigma_\varepsilon, \mu)$ has norm at most ε.*

Proof. To construct a tree of subsets and the corresponding Haar-like system for (a), we repeatedly apply the definition of a PP-narrow operator. That is, we let $h_\emptyset$ be a sign supported on A with $\|Th_\emptyset\| \leqslant \varepsilon_\emptyset$. Using the notation $\{h = x\} = \{\omega\colon h(\omega) = x\}$, put

$$A_{-1} = \{h_\emptyset = -1\}, \qquad A_1 = \{h_\emptyset = 1\}.$$

Let h_{-1} and h_1 be signs supported on A_{-1} and A_1 respectively, with $\|Th_{\pm 1}\| \leqslant \varepsilon_{\pm 1}$; put

$$A_{-1,-1} = \{h_{-1} = -1\}, \qquad A_{-1,1} = \{h_{-1} = 1\},$$
$$A_{1,-1} = \{h_1 = -1\}, \qquad A_{1,1} = \{h_1 = 1\}$$

and continue in the above fashion. This yields part (a).

Part (b) follows from (a) and Remark 2.2(b). □

For $1 < p < \infty$, the class of PP-narrow operators on L_p is not stable under taking sums (see [**8**, p. 59]); this is why we have to consider a smaller class of operators, which we introduce next. Incidentally, the stability of PP-narrow operators on L_1 under sums is still an open problem.

DEFINITION 2.4. An operator $T\colon L_p \to X$ is said to be *hereditarily PP-narrow* if for every $A \in \Sigma^+$ and every nonatomic sub-σ-algebra $\Sigma_1 \subset \Sigma$ on A, the restriction of T to $L_p(A, \Sigma_1, \mu)$ is PP-narrow.

Since every compact operator on L_p is PP-narrow and compactness is inherited by restrictions, compact operators on L_p are hereditarily PP-narrow. On the other hand, the operator

$$T\colon L_p([0,1]^2) \to L_p[0,1], \qquad (Tf)(s) = \int_0^1 f(s,t)\,dt$$

shows that a PP-narrow operator need not be hereditarily PP-narrow.

We now show that the set of hereditarily PP-narrow operators forms a subspace of $\mathcal{L}(L_p, X)$.

PROPOSITION 2.5. *Let $1 \leqslant p < \infty$ and let $U, V\colon L_p \to X$.*

(a) *If U is PP-narrow and V is hereditarily PP-narrow, then $U+V$ is PP-narrow.*

(b) *If U and V are both hereditarily PP-narrow, then $U + V$ is hereditarily PP-narrow as well.*

Proof. (a) Let $A \in \Sigma^+$ and $\varepsilon > 0$. By Lemma 2.3(b) there is a σ-algebra $\Sigma_\varepsilon \subset \Sigma$ on A such that $(A, \Sigma_\varepsilon, \mu)$ is a nonatomic measure space and the restriction of U to $L_p^0(A, \Sigma_\varepsilon, \mu)$ has norm at most ε. Since V is hereditarily PP-narrow, there is a Σ_ε-measurable sign f supported on A for which $\|Vf\| \leqslant \varepsilon$. Then $\|(U+V)f\| \leqslant \varepsilon\mu(A)^{1/p} + \varepsilon \leqslant 2\varepsilon$.

(b) This follows from (a). □

3. *Unconditionally convergent series of hereditarily PP-narrow operators*

In this section we give an example of a Banach space X for which

$$\mathrm{Id} \in \mathrm{unc}(\mathrm{unc}(\mathcal{K}(X,X))) \setminus \mathrm{unc}(\mathcal{K}(X,X)).$$

We begin with a factorisation lemma for unconditional sums of hereditarily PP-narrow operators.

LEMMA 3.1. *Let $1 \leqslant p < \infty$, let X be a Banach space, let $T_n : L_p \to X$ be hereditarily PP-narrow operators with $\sum_{n=1}^\infty T_n$ converging pointwise unconditionally to an operator T, and let $M = \sup_\pm \|\sum_{n=1}^\infty \pm T_n\|$. Given $0 < \varepsilon < 1/2$, there exist a Banach space Y and a factorisation as in Figure 1, with $\|\tilde{T}\| \leqslant M$, $\|W\| \leqslant 1$. There are also a nonatomic sub-σ-algebra $\Sigma_1 \subset \Sigma$, a Haar-like system $\{h_\alpha\}$ forming a basis for $L_p^0(\Omega, \Sigma_1, \mu)$ and operators $U, V\colon L_p^0(\Omega, \Sigma_1, \mu) \to Y$ with $U + V = \tilde{T}$ on $L_p^0(\Omega, \Sigma_1, \mu)$ such that U maps $\{h_\alpha\}$ to a 1-unconditional basic sequence and $\|V\| \leqslant \varepsilon$.*

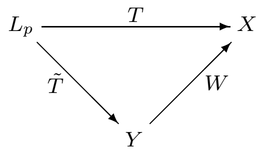

FIGURE 1.

Proof. Define Y as the space of all sequences $y = (y_1, y_2, \dots)$, $y_n \in X$, such that $\sum_{n=1}^{\infty} y_n$ converges unconditionally in X. Equip Y with the natural norm

$$\|y\| = \sup_{\pm} \Big\| \sum_{n=1}^{\infty} \pm y_n \Big\|.$$

Put $\tilde{T}f = (T_1 f, T_2 f, \dots)$ and $Wy = \sum_{n=1}^{\infty} y_n$. Then Y, $\tilde{T}$ and W satisfy the desired factorisation scheme.

Our main task is now to define, for this $\tilde{T}$, a Haar-like system $\{h_\alpha\}$ and operators U, V as claimed in the lemma. To do this, one uses a standard blocking technique and the stability of hereditarily PP-narrow operators under summation (Proposition 2.5). That is, for every $1 \leqslant n < m \leqslant \infty$, we define a projection operator $P_{n,m}\colon Y \to Y$ as follows:

$$P_{n,m}(y_1, y_2, \dots) = (0, 0, \dots, 0, y_n, y_{n+1}, \dots, y_{m-1}, 0, 0, \dots).$$

Let (ε_α) be positive numbers. Select an arbitrary sign $h_\emptyset$ supported on Ω, and find $n_\emptyset \in \mathbb{N}$ for which

$$\|P_{n_\emptyset,\infty} \tilde{T} h_\emptyset\| \leqslant \varepsilon_\emptyset.$$

Put

$$U h_\emptyset = P_{1,n_\emptyset} \tilde{T} h_\emptyset, \qquad V h_\emptyset = P_{n_\emptyset,\infty} \tilde{T} h_\emptyset.$$

The sign $h_\emptyset$ generates a partition of Ω; that is,

$$A_{-1} = \{h_\emptyset = -1\}, \qquad A_1 = \{h_\emptyset = 1\}.$$

Since the operator $P_{1,n_\emptyset}\tilde{T}$ is PP-narrow by Proposition 2.5, there is a sign h_{-1} supported on A_{-1} for which

$$\|P_{1,n_\emptyset} \tilde{T} h_{-1}\| \leqslant \frac{1}{2}\varepsilon_{-1}.$$

Find $n_{-1} > n_\emptyset$ such that

$$\|P_{n_{-1},\infty} \tilde{T} h_{-1}\| \leqslant \frac{1}{2}\varepsilon_{-1}.$$

Put

$$U h_{-1} = P_{n_\emptyset,n_{-1}} \tilde{T} h_{-1}, \qquad V h_{-1} = (P_{1,n_\emptyset} + P_{n_{-1},\infty}) \tilde{T} h_{-1}.$$

Continuing in this fashion, we obtain a Haar-like system $\{h_\alpha\}$ and operators U, V: $\overline{\mathrm{lin}}\{h_\alpha\} \to Y$ such that $U + V = \tilde{T}$ on $\overline{\mathrm{lin}}\{h_\alpha\}$, U maps $\{h_\alpha\}$ to disjoint elements of the sequence space Y (and hence to a 1-unconditional basic sequence) and V maps $\{h_\alpha\}$ to elements whose norms are controlled by the numbers ε_α; therefore $\|V\| \leqslant \varepsilon$ by Remark 2.2(b) if $\varepsilon_\alpha \to 0$ sufficiently fast. $\square$

LEMMA 3.2. *Under the conditions of Lemma 3.1 assume in addition that the operator T is bounded from below by a constant c; that is,*

$$\|Tf\| \geqslant c\|f\| \qquad \forall f \in L_p.$$

Then

$$M = \sup_{\pm} \Bigl\| \sum_{n=1}^{\infty} \pm T_n \Bigr\| \geqslant \beta_p c,$$

where β_p is the unconditional constant of the Haar system in L_p.

Proof. Let $0 < \varepsilon < 1/2$. Under the above conditions, the operator U from Lemma 3.1 maps a Haar-like system $\{h_\alpha\}$ to a 1-unconditional basic sequence. This implies that if U is considered as acting from $\overline{\mathrm{lin}}\{h_\alpha\}$ into $\overline{\mathrm{lin}}\{Uh_\alpha\}$, then $\|U\|\|U^{-1}\| \geqslant \beta_p$. On the other hand,

$$\|U\| \leqslant \|\tilde{T}\| + \|V\| \leqslant M + \varepsilon$$

and

$$\|Uf\| \geqslant \|\tilde{T}f\| - \varepsilon\|f\| \geqslant \|Tf\| - \varepsilon\|f\| \geqslant (c-\varepsilon)\|f\|$$

for all $f \in \overline{\mathrm{lin}}\{h_\alpha\}$, so $\|U^{-1}\| \leqslant (c-\varepsilon)^{-1}$. Hence we have $(M+\varepsilon)(c-\varepsilon)^{-1} \geqslant \beta_p$, which yields the desired inequality since $\varepsilon > 0$ was arbitrary. □

It is known that $\beta_p \to \infty$ if $p \to 1$ or $p \to \infty$; in fact, Burkholder [**2**] has shown that

$$\beta_p = \max\Bigl\{p-1, \frac{1}{p-1}\Bigr\}.$$

THEOREM 3.3. *There exists a Banach space X for which*

$$\mathrm{Id} \in \mathrm{unc}(\mathrm{unc}(\mathcal{K}(X,X))) \setminus \mathrm{unc}(\mathcal{K}(X,X)).$$

Proof. Consider the space $X = L_{p_1} \oplus_2 L_{p_2} \oplus_2 \dots$ where $1 < p_n < \infty$ and $p_n \to 1$.

Suppose that $\mathrm{Id} = \sum_{n=1}^{\infty} T_n$ pointwise unconditionally with compact operators T_n. The restrictions of T_n to L_{p_j} are also compact and hence hereditarily PP-narrow, so by the previous lemma

$$\sup_{\pm} \Bigl\| \sum_{n=1}^{\infty} \pm T_n \Bigr\| \geqslant \sup_{\pm} \Bigl\| \sum_{n=1}^{\infty} \pm T_n \restriction_{L_{p_j}} \Bigr\| \geqslant \beta_{p_j} \to \infty.$$

Thus the assumption of pointwise unconditional convergence of $\sum_{n=1}^{\infty} T_n$ leads to a contradiction, and hence Id does not belong to $\mathrm{unc}(\mathcal{K}(X,X))$.

On the other hand, all the natural projections $P_j\colon X \to L_{p_j}$ belong to $\mathrm{unc}(\mathcal{K}(X,X))$ since each L_{p_j} has an unconditional basis. Taking into account the unconditional representation $\mathrm{Id} = \sum_{n=1}^{\infty} P_n$, we find that $\mathrm{Id} \in \mathrm{unc}(\mathrm{unc}(\mathcal{K}(X,X)))$. □

4. *Hereditarily PP-narrow operators on L_1*

In this section we prove the main result of the paper, namely that the sum of a pointwise unconditionally convergent series of hereditarily PP-narrow operators on L_1 is again a hereditarily PP-narrow operator.

The following lemma implies that the operator U from Lemma 3.1 factors through c_0.

LEMMA 4.1. *Let $\{h_\alpha\}$ be a Haar-like system in L_1, and let $U\colon L_1 \to X$ be an operator that maps $\{h_\alpha\}$ into an unconditional basic sequence. Then there is a constant C such that for every element of the form $f = \sum_\alpha a_\alpha h_\alpha$, one has*

$$\|Uf\| \leqslant C \sup_\alpha |a_\alpha|. \tag{4.1}$$

Proof. Without loss of generality we can assume that $\|U\| = 1$, $\|h_\emptyset\| = 1$ and that the unconditional constant of $\{Uh_\alpha\}$ also equals 1. (One can achieve all these goals by an equivalent renorming of X and by multiplication of μ by a constant.)

Let us first remark that for every $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathcal{A}_n$,

$$\|\alpha_1 h_\emptyset + 2\alpha_2 h_{\alpha_1} + 4\alpha_3 h_{\alpha_1,\alpha_2} + \dots + 2^{n-1}\alpha_n h_{\alpha_1,\dots,\alpha_{n-1}}\| \leqslant 2;$$

indeed, it is easy to check by induction over n that this sum equals

$$2^n \chi_{A_{\alpha_1,\dots,\alpha_n}} - \chi_{A_\emptyset}.$$

Hence

$$\|\alpha_1 U h_\emptyset + 2\alpha_2 U h_{\alpha_1} + \dots + 2^{n-1}\alpha_n U h_{\alpha_1,\dots,\alpha_{n-1}}\| \leqslant 2,$$

and, since $\{Uh_\alpha\}$ is a 1-unconditional basic sequence,

$$\|U h_\emptyset + 2 U h_{\alpha_1} + \dots + 2^{n-1} U h_{\alpha_1,\dots,\alpha_{n-1}}\| \leqslant 2.$$

Passing from $n-1$ to n in the last inequality and averaging over $\alpha \in \mathcal{A}_n$, we obtain

$$2 \geqslant \Big\| \frac{1}{2^n} \sum_{\alpha \in \mathcal{A}_n} (U h_\emptyset + 2 U h_{\alpha_1} + \dots + 2^{n-1} U h_{\alpha_1,\dots,\alpha_n}) \Big\| = \Big\| \sum_{k=0}^{n} \sum_{\alpha \in \mathcal{A}_k} U h_\alpha \Big\|.$$

Again by the 1-unconditionality of $\{Uh_\alpha\}$, the last inequality implies that for all $a_\alpha \in [-1, 1]$,

$$\Big\| \sum_{k=0}^{n} \sum_{\alpha \in \mathcal{A}_k} a_\alpha U h_\alpha \Big\| \leqslant 2,$$

which gives (4.1) with $C = 2$. □

An inspection of the proof shows that

$$\|Uf\| \leqslant 2\|U\|\beta^2 \sup_\alpha |a_\alpha|,$$

where β denotes the unconditional constant of the basic sequence (Uh_α).

LEMMA 4.2. *For every Haar-like system $\{h_\alpha\}$ in L_1 supported on A, and every $\delta > 0$, there is a sign*

$$f = \sum_{k=0}^{\infty} \sum_{\alpha \in \mathcal{A}_k} a_\alpha h_\alpha \tag{4.2}$$

supported on A with $\sup_\alpha |a_\alpha| \leqslant \delta$.

Proof. Fix an $m \in \mathbb{N}$ such that $1/m \leqslant \delta$, and define

$$f_k = \sum_{\alpha \in \mathcal{A}_k} a_\alpha h_\alpha$$

as follows: $f_0 = (1/m)\, h_\emptyset$, and for every $\alpha \in \mathcal{A}_n$ put $a_\alpha = 1/m$ if $|\sum_{k=0}^{n-1} f_k| < 1$ on $\operatorname{supp} h_\alpha$ and $a_\alpha = 0$ if $|\sum_{k=0}^{n-1} f_k| = 1$ on $\operatorname{supp} h_\alpha$. Under this construction, all the partial sums of the series $\sum_{k=0}^{\infty} f_k$ are bounded by 1 in modulus. Since $\{f_k\}_{k=0}^{\infty}$ is an orthogonal system, the series $\sum_{k=0}^{\infty} f_k$ converges in L_2 (and hence in L_1) to a function f supported on A that can be represented as in (4.2) with $\sup_\alpha |a_\alpha| \leqslant \delta$. We shall prove that f is a sign.

Obviously, $\int_A f\, d\mu = 0$. Consider $B = \{t \in A\colon |f(t)| \neq 1\}$. By our construction, we have, for each $n \in \mathbb{N}$,

$$B \subset \{t \in A\colon f_n(t) \neq 0\} = \Big\{t \in A\colon |f_n(t)| = \frac{1}{m}\Big\},$$

so $\mu(B) \leqslant m\|f_n\|$, and since $\|f_n\| \to 0$, we conclude that $\mu(B) = 0$. Therefore f is a sign. □

The previous lemma can also be proved by means of abstract martingale theory. For simplicity of notation let us work with the classical Haar system $h_1, h_2, \dots$ on $[0,1]$. Let $\xi_n = \sum_{k=1}^{n} h_k$ and $T = \inf\{n\colon |\xi_n| \geqslant m\}$. Then (ξ_n) is a martingale, T is a stopping time and $(\xi'_n) = (\xi_{n\wedge T})$ is a uniformly bounded martingale. Hence (ξ'_n) converges almost surely and in L_1 to a limit ξ that takes only the values $\pm m$ on $\{T < \infty\}$, but since (ξ_n) fails to converge pointwise, the event $\{T = \infty\}$ has probability 0. This shows that $\xi = \pm m$ almost surely and $\mathbb{E}\xi = 0$. Hence $f = \xi/m$ is the sign that we are seeking.

We are now ready for the main result of this paper. An analogous theorem for operators on $C(K)$-spaces is proved in [**1**].

THEOREM 4.3. *Let $T_n\colon L_1 \to X$ be hereditarily PP-narrow operators, and suppose that $\sum_{n=1}^{\infty} T_n$ converges pointwise unconditionally to some operator T. Then T is hereditarily PP-narrow.*

Proof. Let $A \in \Sigma^+$, and let $\tilde{\Sigma}$ be a nonatomic sub-σ-algebra of $\Sigma\,|_A$. We have to show that for every $\varepsilon > 0$ there is a sign $f \in L_1(A, \tilde{\Sigma}, \mu)$ supported on A with $\|Tf\| \leqslant \varepsilon$.

Applying Lemma 3.1 to the restrictions of T_n and T to $L_1(A, \tilde{\Sigma}, \mu)$, we obtain a Haar-like system $\{h_\alpha\}$ forming a basis for some $L_1^0(A, \Sigma_1, \mu)$ and we obtain operators $U, V\colon L_1^0(A, \Sigma_1, \mu) \to Y$, $W\colon Y \to X$ such that $\|W\| \leqslant 1$, $T = W(U+V)$ on $L_1^0(A, \Sigma_1, \mu)$, $\|V\| \leqslant \varepsilon/2$ and U maps $\{h_\alpha\}$ to a 1-unconditional basic sequence.

Let C be the constant from (4.1). Taking a sign

$$f = \sum_{k=0}^{\infty} \sum_{\alpha \in \mathcal{A}_k} a_\alpha h_\alpha$$

supported on A with $\sup_\alpha |a_\alpha| \leqslant \varepsilon/(2C)$ (Lemma 4.2), we obtain from (4.1) that $\|Uf\| \leqslant \varepsilon/2$. Therefore $\|Tf\| \leqslant \|Uf\| + \|Vf\| \leqslant \varepsilon$. □

COROLLARY 4.4. *For any Banach space X, no embedding operator is contained in* $\mathrm{unc}(\ldots(\mathrm{unc}(\mathcal{K}(L_1, X))))$.

Proof. Compact operators are hereditarily PP-narrow. □

The next corollary is due to Rosenthal (unpublished).

COROLLARY 4.5. *Every operator T from L_1 into a Banach space X with an unconditional basis is hereditarily PP-narrow; in particular, it is PP-narrow. Consequently, L_1 does not even sign-embed into a space with an unconditional basis.*

Proof. If P_n, $n = 1, 2, \ldots$, are the partial sum projections associated to an unconditional basis of X, then $T = \sum_{n=1}^{\infty}(P_n - P_{n-1})T$ is a pointwise unconditionally convergent series of rank-1 operators. □

5. *Questions*

(1) Can one describe $\mathrm{unc}(\mathcal{K}(L_1, \mathcal{X}))$ for general X? What about $X = L_1$?

(2) Describe the smallest class of operators $\mathcal{M} \subset \mathcal{L}(L_1, X)$ that contains the compact operators and is stable under pointwise unconditional sums. In particular, is $\mathrm{unc}(\mathcal{K}(L_1, L_1)) = \mathrm{unc}(\mathrm{unc}(\mathcal{K}(L_1, L_1)))$? Note that X does not embed into a space with an unconditional basis if $\mathcal{M} \neq \mathcal{L}(L_1, X)$.

(3) Can one develop a similar theory for operators on the James space, or other spaces that do not embed into spaces with unconditional bases?

(4) Is there a space X with the Daugavet property such that

$$\mathrm{Id} \in \mathrm{unc}(\ldots(\mathrm{unc}(\mathcal{K}(X, X))))?$$

(5) Suppose that E is a Banach space with the Daugavet property, on which the set of narrow operators from E to X is a linear space. (This is not always the case; for example, it is not so for $E = X = C([0,1], \ell_1)$; see [**1**].) If $T = \sum T_n$ is a pointwise unconditionally convergent series of narrow operators from E into X, must T also be narrow? It is known that under these conditions $\|\mathrm{Id} + T\| \geqslant 1$; see [**5**]. The answer is positive for $E = C([0,1], \ell_p)$ if $1 < p < \infty$; see [**1**].

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