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MATHEMATICS
(OPERATOR THEORY)

# Compact Operators on Symmetric Function Spaces

by

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**Summary.** A theorem (see the text) concerning operators in function spaces is proved.

The object of this note is to prove the following theorem.

THEOREM. *Let $X$ be a symmetric $F$-function space on* $(0, 1)$. *Suppose*

(1) *$X$ contains $L_p(0, 1)$ for some $p$, $1 \leqslant p < \infty$.*

(2) *$X$ is not contained in $L_1(0, 1)$.*

(3) *The simple functions are dense in $X$.*

*Suppose $T$ is a linear operator mapping $X$ into a topological vector space $Y$ such that $T(B_\infty)$ is relatively compact, where $B_\infty = \{f \in L_0 \colon \text{ess sup} |f(t)| \leqslant 1\}$; then $T=0$.*

Symmetric $F$-function spaces are defined in [3]. Here $X$ is a lattice subspace of $L_0(0, 1)$ equipped with an $F$-norm $f \mapsto \|f\|$ such that $\|f\| \leqslant \|g\|$ if $|f| \leqslant |g|$ a.e. and $\|f\|$ is re-arrangement invariant. A theorem very similar to the above for Orlicz spaces was proved by the author in [1]. It should be noted that there are spaces $X$ which satisfy the theorem but do not contain any Orlicz space $L_\varphi$ which is not contained in $L_1$. For example, take the weak $L_1$-space, $L_1^*$, of all functions $f$ such that

$$\|f\| = \sup_{t>0} tm\{s \colon |f(s)| \geqslant t\} < \infty .$$

Although this defines only a quasi-norm it is clearly equivalent to a rearrangement invariant $F$-norm. Then $L_1^* \supset L_1$ and $L_1$ is the largest Orlicz space contained in $L_1^*$. Now let $X$ be the closure of the simple functions in $L_1^*$.

For the proof we require the following lemma, which is immediate from Lemma 2.1 of [2].

LEMMA. *Let $B$ be a Banach space and $Y$ an $F$-space; suppose $T \colon B \to Y$ is a continuous linear operator. Suppose $x_n \in B$ and $x_n \to x$ weakly; then if $(Tx_n \colon n \in \mathbf{N})$ is relatively compact in $Y$, $Tx_n \to Tx$.*

Proof of Theorem. We may suppose $Y$ is an $F$-space, $F$-normed by $\|\cdot\|$. If $T \neq 0$, there is a characteristic function $\chi_A$ where $m(A) > 0$ such that $T\chi_A \neq 0$. If $P_A f = \chi_A f$ then $P_A \colon X \to X$ is continuous and $\mathrm{TP}_A(\chi) = 0$ where $\chi$ is characteristic function of $[0, 1]$.

Since $X \not\subset L_1$, it is easy to see that there exists a sequence of simple positive functions $f_n$ such that $\|f_n\|_1 = 1$ for all $n$, but $\|f_n\| \to 0$. For each $n$, let $(g_{n,k}: k \in \mathrm{N})$ be a sequence of independent functions (or random variables!) identically distributed with $f_n$.

As $L_p \subset X$, it follows from the Closed Graph Theorem that $TP_A: L_p \to X$ is continuous. Since

$$\|g_{n,k}\|_\infty = \|f_n\|_\infty \leqslant \infty,$$

we have $\lim_{k\to\infty} g_{n,k} = \chi$ weakly in $L_p$. By the lemma $TP_A\, g_{n,k} \to T\chi_A$ in $Y$.

Hence if $\delta > 0$ is chosen so that $\|f\| < \delta$ implies $\|TP_A f\| < \frac{1}{2}\|T\chi_A\|$, we have

$$\liminf_{k\to\infty} \|g_{n,k}\| \geqslant \delta$$

and so $\|f_n\| \geqslant \delta$, contrary to assumption.

We conclude that $T = 0$.

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## REFERENCES

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Н. Я. Кельтон, **Компакные операторы я мимметрично функцзонал«зыб прострнзтвах**

**Содержание.** Предметом этой работы является доказательство следующей теоремы: *Пусть X — симметричное F — функциональное пространство на (0, 1). Предположим, что* (1) *X содержит L (0, 1) для некоторого p,* $1 \leqslant p < \infty$, (2) *X — не содержится в L (0, 1),* (3) *простые функции являются плотными в X. Предположим, что T является линейным оператором, отображающим X в топологическое векторное пространство У такое, что* $T(B_\infty)$ *является относительно компактным, где* $B_\infty = \{f \in L_0 : \operatorname{ess\,sup} f(t) \leqslant 1\}$; *тогда* $T = 0$.