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MATHEMATICS
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Vector Measures of Infinite Variation

by

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Summary. Let X be an infinite-dimensional Banach space. We show the existence of a non-trivial X-valued measure with relatively compact range and not σ-finite variation. This can be regarded as an extension of the Dvoretzky–Rogers Theorem. We also show that the space of Pettis-integrable functions is not complete under the usually cosiderable norm.

Let I be the unit interval and let $\mathscr{B}$ denote the σ-algbera of all Borel subsets of I; let λ denote the usual Lebesgue measure on I. In this note we give a short proof of a result due to Thomas [4], (p. 90). If X is an infinite-dimensional Banach space, we show the existence of a non-trivial vector measure $\mu : \mathscr{B} \to X$ such that $|\mu|(E) = 0$ or ∞ for all $E \in \mathscr{B}$, where $|\mu|$ is the total variation of μ

$$|\mu|(E) = \sup \left\{ \sum_{i=1}^{m} \|\mu(E_i)\| : E_i \text{ disjoint } \bigcup_{i=1}^{m} E_i = E \right\}.$$

We also show that the space P of Pettis-integrable functions on I is not complete under the norm

$$\|f\| = \sup_{E \in \mathscr{B}} \left\| \int_E f d\lambda \right\|.$$

Denote by ca $(X, \mathscr{B}, \lambda)$ the space of all vector measures $\mu : \mathscr{B} \to X$ which are absolutely continuous with respect to λ (i.e. $\lambda(E) = 0$ implies $\mu(E) = 0$). Then ca $(X, \mathscr{B}, \lambda)$ is a Banach space under the norm

$$\|\mu\| = \sup_{E \in \mathscr{B}} \|\mu(E)\|.$$

We shall identify P as a subspace of ca $(X, \mathscr{B}, \lambda)$ by the identification $f \leftrightarrow f.\lambda$. Let $P_0 \subset P$ be the space of simple functions, or measures of the form

$$\mu(B) = \sum_{i=1}^{n} \lambda(A_i \cap B)\, x_i,$$

where $A_1, \ldots A_n$ are disjoint Borel subsets of I and $x_1, .., x_n \in X$.

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The proof of the following theorem has been considerably simplified in the latter stages by a suggestion of D. J. H. Garling.

THEOREM 1. *If X is infinite-dimeniosnal, the closure of P_0 in ca $(X, \mathscr{B}, \lambda)$ contains a measure μ for which $|\mu|$ is not σ-finite.*

Proof. Let M_σ be the subspace of ca $(X, \mathscr{B}, \lambda)$ of all measures μ for which $|\mu|$ is σ-finite. Let $V_n \subset M_\sigma$ ($n \in \mathbf{N}$) be the set of all μ such that $\|\mu\| \leqslant 2^{-n}$ and there exists $E \in \mathscr{B}$ with $\lambda(E) \leqslant 2^{-n}$ and $|\mu|(I-E) \leqslant 2^{-n}$. Then $(V_n: n \in \mathbf{N})$ is a base of a metrizable vector topology in M_σ, denoted by ρ. Furthermore we shall show (M_σ, ρ) is complete. For if $\mu_n \in V_n$ then $\sum \mu_n$ converges in ca $(X, \mathscr{B}, \lambda)$ to some μ. Choose $E_n \in \mathscr{B}$ such that $\lambda(E_n) \leqslant 2^{-n}$ and $|\mu_n|(I-E_n) \leqslant 2^{-n}$ and let $F_n = \bigcup_{n+1}^{\infty} E_k$. Then $\lambda(F_n) \leqslant 2^{-n}$ and $|\mu_k|(I-F_n) \leqslant 2^{-k}$ for $k > n$. For each n there exists $G_n \supset F_n$ such that $\lambda(G_n) \leqslant 2^{-(n-1)}$ and $\sum_{i=1}^{n} |\mu_i|(I-G_n) < \infty$. Thus

$$|\mu|(I-G_n) \leqslant \sum_{=1}^{n} |\mu|_i (I-G_n) + \sum_{i=u+1}^{\infty} |\mu_\tau|(I-G_n) < \infty$$

and so $\mu \in M_\sigma$. Also

$$\left|\sum_{i=n+1}^{\infty} \mu_i\right|(I-F_n) \leqslant \sum_{i=n+1}^{\infty} |\mu_i|(I-F_n) \leqslant 2^{-n}$$

so that $\sum_{i=n+1}^{\infty} \mu_i \in V_n$, and $\sum \mu_n = \mu$ in the topology ρ.

Now suppose $\bar{P}_0 \subset M_\sigma$. Then the inclusion map $\bar{P}_0 \to M_\sigma$ has closed graph if we equip M_σ with the ρ-topology (which is stronger than the norm topology). Hence by the Closed Graph Theorem $V_1 \cap \bar{P}_0$ is a neighbourhood of 0 in $\bar{P}_0$, i.e. there exists $\varepsilon > 0$ such that $\|\mu\| \leqslant \varepsilon$ implies $\mu \in V_1$. Alternatively there exists $C < \infty$ such that if $\|\mu\| \leqslant 1$, then for some $E \in \mathscr{B}$ with $\lambda(E) \leqslant \frac{1}{2}$, $|\mu|(I-E) \leqslant C$.

By the Dvoretzky–Rogers Theorem, (cf. e.g. [1, 2] or [3]). we can find in X elements $x_1, .., x_n$ such that

$$\|t_1 x_1 + \ldots + t_n x_n\| \leqslant 1$$

whenever $\max_{1 \leqslant i \leqslant n} |t_i| \leqslant 1$ and

$$\|x_1\| + \ldots + \|x_n\| \geqslant 3C.$$

Let $J_1, \ldots, J_n$ be disjoint Borel subsets of $[0, 1]$ with measure

$$\lambda(J_i) = \frac{\|x_i\|}{\|x_1\| + \ldots + \|x_n\|} \qquad 1 \leqslant i \leqslant n.$$

Define $\mu \in P_0$ by

$$\mu(B) = \sum_{i=1}^{n} \frac{\lambda(B \cap J_i)}{\lambda(J_i)} x_i.$$

Then $\|\mu\| \leqslant 1$, and for any B

$$|\mu|(B) = \sum_{i=1}^{n} \frac{\lambda(B\cap J_i)}{\lambda(J_i)} \|x_i\| = \lambda(B)\left(\sum_{i=1}^{n} \|x_i\|\right) \geqslant 3C\lambda(B).$$

Hence if $\lambda(E) \leqslant \frac{1}{2}$, $|\mu|(I-E) \geqslant \frac{3}{2}C > C$ which is a contradiction.

The following theorems are now immediate:

THEOREM 2. *If X is an infinite-dimensional Banach space, then there is a measure $\mu\colon \mathscr{B} \to X$ with relatively compact range and $|\mu|(E) = \infty$ for $E \in \mathscr{B}$, with $\lambda(E) > 0$.*

Proof. Choose $\nu \in \bar{P}_0$ with $|\nu|$ non σ-finite; then there exists a set $E_0 \in \mathscr{B}$ with $\lambda(E_0)$ maximal such that $|\nu|$ is σ-finite on E_0. Then ν restricted to $I-E_0$ has the property that $\lambda(F) > 0$ implies $|\nu|(F) = \infty$; by the isomorphism of the measure spaces $(I, \mathscr{B}, \lambda)$ and $(I-E_0, \mathscr{B}, \gamma)$, where $\gamma \doteq \lambda(I-E_0)^{-1}\lambda$, then the theorem is proved.

THEOREM 3. *If X is an infinite-dimensional Banach space then P is not complete.*

Proof. $M_\sigma \supset P \supset \bar{P}_0$ and so the result follows from Theorem 1.

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REFERENCES

[1] A. Dvoretzky, C. A. Rogers, *Absolute and unconditional convergence in normed linear spaces*, Proc. Nat. Acad. Sci. U.S.A., **36** (1950), 192–197.

[2] N. J. Kalton, W. H. Ruckle, *A series characterization of subspaces of $L_p(\mu)$-spaces*, Bull. Amer. Math. Soc., **79** (1973), 1019–1022.

[3] A. Pietsch (translator, W. H. Ruckle), *Nuclear locally convex spaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 66, Springer, Berlin 1972.

[4] E. Thomas, *The Lebesgue–Nikodym Theorem for vector valued random measures*, Mem. Amer. Math. Soc., No. 139 (1974).

Л. Яницка, Н. Дж. Кальтон, **Векторние размерности бесконечных вариаций**

Содержание. Пусть X обозначает произвольное бесконечномерное банахово прстранство. Мы доказываем существование нетривиальной меры со значениями в X, имеющей относительно компактное мномество значений и не имеющей σ-конечной вариации. Кроме того, мы доказываем, что пространство функций, интегрируемых в смысле Петтиса, не полно относительно нормы $\|f\| = \sup_{E\in\mathscr{B}} \left\|\int_E f d\lambda\right\|$.