Vector Measures of Infinite Variation

by

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Summary. Let $X$ be an infinite-dimensional Banach space. We show the existence of a non-trivial $X$-valued measure with relatively compact range and not $\sigma$-finite variation. This can be regarded as an extension of the Dvoretzky–Rogers Theorem. We also show that the space of Pettis-integrable functions is not complete under the usually considerable norm.

Let $I$ be the unit interval and let $B$ denote the $\sigma$-algebra of all Borel subsets of $I$; let $\lambda$ denote the usual Lebesgue measure on $I$. In this note we give a short proof of a result due to Thomas [4], (p. 90). If $X$ is an infinite-dimensional Banach space, we show the existence of a non-trivial vector measure $\mu : B \to X$ such that $|\mu|(E) = 0$ or $\infty$ for all $E \in B$, where $|\mu|$ is the total variation of $\mu$

$$|\mu|(E) = \sup \left\{ \sum_{i=1}^{m} \|\mu(E_i)\| : E_i \text{ disjoint} \bigcup_{i=1}^{m} E_i = E \right\}.$$ 

We also show that the space $P$ of Pettis-integrable functions on $I$ is not complete under the norm

$$\|f\| = \sup_{E \in B} \left\| \int_{E} f d\lambda \right\|.$$ 

Denote by $ca(X, B, \lambda)$ the space of all vector measures $\mu : B \to X$ which are absolutely continuous with respect to $\lambda$ (i.e. $\lambda(E) = 0$ implies $\mu(E) = 0$). Then $ca(X, B, \lambda)$ is a Banach space under the norm

$$\|\mu\| = \sup_{E \in B} \|\mu(E)\|.$$ 

We shall identify $P$ as a subspace of $ca(X, B, \lambda)$ by the identification $f \leftrightarrow f \lambda$. Let $P_0 \subset P$ be the space of simple functions, or measures of the form

$$\mu(B) = \sum_{i=1}^{n} \lambda(A_i \cap B) x_i,$$

where $A_1, \ldots, A_n$ are disjoint Borel subsets of $I$ and $x_1, \ldots, x_n \in X$.

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The proof of the following theorem has been considerably simplified in the latter stages by a suggestion of D. J. H. Garling.

**Theorem 1.** If \( X \) is infinite-dimenional, the closure of \( P_0 \) in \( \text{ca} (X, \mathcal{B}, \lambda) \) contains a measure \( \mu \) for which \( |\mu| \) is not \( \sigma \)-finite.

**Proof.** Let \( M_\sigma \) be the subspace of \( \text{ca} (X, \mathcal{B}, \lambda) \) of all measures \( \mu \) for which \( |\mu| \) is \( \sigma \)-finite. Let \( V_n \subset M_\sigma \) \((n \in \mathbb{N})\) be the set of all \( \mu \) such that \( |\mu| \leq 2^{-n} \) and there exists \( E \in \mathcal{B} \) with \( \lambda (E) \leq 2^{-n} \) and \( |\mu| (I - E) \leq 2^{-n} \). Then \( (V_n : n \in \mathbb{N}) \) is a base of a metrizable vector topology in \( M_\sigma \), denoted by \( \rho \). Furthermore we shall show \((M_\sigma, \rho)\) is complete. For if \( \mu_n \in V_n \) then \( \sum \mu_n \) converges in \( \text{ca} (X, \mathcal{B}, \lambda) \) to some \( \mu \).

Choose \( E_n \in \mathcal{B} \) such that \( \lambda (E_n) \leq 2^{-n} \) and \( |\mu_n| (I - E_n) \leq 2^{-n} \) and let \( F_n = \bigcup_{n+1}^{\infty} E_k \).

Then \( \lambda (F_n) \leq 2^{-n} \) and \( |\mu_k| (I - F_n) \leq 2^{-k} \) for \( k > n \). For each \( n \) there exists \( G_n \supseteq F_n \) such that \( \lambda (G_n) \leq 2^{-(n-1)} \) and \( \sum_{i=1}^{n} |\mu_i| (I - G_n) < \infty \). Thus

\[
|\mu| (I - G_n) \leq \sum_{i=1}^{n} |\mu_i| (I - G_n) + \sum_{i=n+1}^{\infty} |\mu_i| (I - G_n) < \infty
\]

and so \( \mu \in M_\sigma \). Also

\[
\left| \sum_{i=n+1}^{\infty} \mu_i (I - F_n) \right| \leq \sum_{i=n+1}^{\infty} |\mu_i| (I - F_n) \leq 2^{-n}
\]

so that \( \sum_{i=n+1}^{\infty} \mu_i \in V_n \) and \( \sum \mu_n = \mu \) in the topology \( \rho \).

Now suppose \( P_0 \subset M_\sigma \). Then the inclusion map \( P_0 \to M_\sigma \) has closed graph if we equip \( M_\sigma \) with the \( \rho \)-topology (which is stronger than the norm topology). Hence by the Closed Graph Theorem \( V_1 \cap P_0 \) is a neighbourhood of \( 0 \) in \( P_0 \), i.e. there exists \( \varepsilon > 0 \) such that \( |\mu| \leq \varepsilon \) implies \( \mu \in V_1 \). Alternatively there exists \( C < \infty \) such that if \( |\mu| \leq 1 \), then for some \( E \in \mathcal{B} \) with \( \lambda (E) \leq \frac{1}{2} \), \( |\mu| (I - E) \leq C \).

By the Dvoretzky–Rogers Theorem, (cf. e.g. [1, 2] or [3]). we can find in \( X \) elements \( x_1, \ldots, x_n \) such that

\[
||x_1 + \ldots + x_n|| \leq 1
\]

whenever \( \max_{1 \leq i \leq n} |x_i| \leq 1 \) and

\[
||x_1|| + \ldots + ||x_n|| \geq 3C.
\]

Let \( J_1, \ldots, J_n \) be disjoint Borel subsets of \([0, 1]\) with measure

\[
\lambda (J_i) = \frac{||x_i||}{||x_1|| + \ldots + ||x_n||}, \quad 1 \leq i \leq n.
\]

Define \( \mu \in P_0 \) by

\[
\mu (B) = \sum_{i=1}^{n} \frac{\lambda (B \cap J_i)}{\lambda (J_i)} x_i.
\]
Then \(\|\mu\| \leq 1\), and for any \(B\)

\[
|\mu|(B) = \sum_{i=1}^{n} \frac{\lambda(B \cap J_i)}{\lambda(J_i)} \|x_i\| = \lambda(B) \left( \sum_{i=1}^{n} \|x_i\| \right) \geq 3C \lambda(B).
\]

Hence if \(\lambda(E) \leq \frac{1}{2}\), \(|\mu|(I - E) \geq \frac{3}{2} C > C\) which is a contradiction.

The following theorems are now immediate:

**Theorem 2.** If \(X\) is an infinite-dimensional Banach space, then there is a measure \(\mu : \mathcal{B} \rightarrow X\) with relatively compact range and \(|\mu|(E) = \infty\) for \(E \in \mathcal{B}\), with \(\lambda(E) > 0\).

**Proof.** Choose \(v \in \mathcal{P}_0\) with \(|v|\) non \(\sigma\)-finite; then there exists a set \(E_0 \in \mathcal{B}\) with \(\lambda(E_0)\) maximal such that \(|v|\) is \(\sigma\)-finite on \(E_0\). Then \(v\) restricted to \(I - E_0\) has the property that \(\lambda(F) > 0\) implies \(|v| (F) = \infty\); by the isomorphism of the measure spaces \((I, \mathcal{B}, \lambda)\) and \((I - E_0, \mathcal{B}, \gamma)\), where \(\gamma = \lambda(I - E_0)^{-1} \lambda\), then the theorem is proved.

**Theorem 3.** If \(X\) is an infinite-dimensional Banach space then \(P\) is not complete.

**Proof.** \(M_0 \supseteq \mathcal{P}_0\) and so the result follows from Theorem 1.

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References


Л. Янцка, Н. Дж. Кальтон, Векторные размерности бесконечных вариаций

Содержание. Пусть \(X\) обозначает произвольное бесконечномерное банахово пространство. Мы доказываем существование непрерывной меры со значениями в \(X\), имеющей относительно компактное множество значений и не имеющей \(\sigma\)-конечной вариации. Кроме того, мы доказываем, что пространство функций, интегрируемых в смысле Петтиса, не полна относительно нормы \(\|f\| = \sup_{E \in \mathcal{B}} \| \int_E f d\lambda \|\).