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VALUES IN DIFFERENTIAL GAMES

BY ROBERT J. ELLIOTT AND NIGEL J. KALTON[1]

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1. **Introduction.** Two person zero-sum differential games can be considered as control problems with two opposing controllers or players. One player seeks to maximize and one to minimize the pay-off function. The greatest pay-off that the maximizing player can force is termed the lower value of the game and similarly the least value which the minimizing player can force is called the upper value. Our objective is to determine conditions under which these values coincide.

In the case of two person zero-sum matrix games, von Neumann showed that if the players are allowed "mixed strategies," i.e., probability measures over the pure strategies, then the values of the game will coincide. By analogy, the authors in collaboration with L. Markus [**1**] introduced relaxed controls into differential game theory; for a full discussion of relaxed controls the reader is referred to [**1**].

In this announcement, we define notions of strategy and values and relate these to the approaches adopted by Fleming [**3**], [**4**] and Friedman [**5**]. Using relaxed controls we are able to show that if the "Isaacs condition" (3) is satisfied then the upper and lower values are equal, so that the game has value. In particular, if the players are allowed relaxed controls then the game always has value. Detailed proofs of these results will appear in a later publication [**2**].

2. **Notation.** A differential game G played by two players J_1 and J_2 for the fixed time interval $I = [0, 1]$ is considered. At each time $t \in I$, J_1 picks an element $y(t)$ from a compact metric space Y and J_2 picks $z(t)$ from a similar space Z in such a way that the functions $t \to y(t)$ and $t \to z(t)$ are measurable. The dynamics are given by the differential equation

$$\dot{x} = dx/dt = f(t, x, y(t), z(t)). \tag{1}$$

Here $x \in \mathbf{R}^m$ and $f: I \times \mathbf{R}^m \times Y \times Z \to \mathbf{R}^m$ is a continuous function. For simplicity of exposition we assume f satisfies constant Lipschitz conditions in t and x. Assuming $x(0) = 0$, the above conditions ensure that for any pair of functions $(y(t), z(t))$ there is a trajectory $x(t)$. At the end of the game a pay-off to J_1,

$$P(y, z) = g(x(1)) + \int_0^1 h(t, x(t), y(t), z(t))\, dt, \tag{2}$$

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is computed. Here g is a differentiable function $\mathbf{R}^m \to \mathbf{R}$ which, together with its first two derivatives satisfies Lipschitz conditions in x. Also $h: I \times \mathbf{R}^m \times Y \times Z \to \mathbf{R}$ is continuous and satisfies Lipschitz conditions in x and t.

As the game is zero-sum the aim of J_1 is to maximize the pay-off whilst J_2 is trying to minimize the pay-off.

3. **Strategies.** Denote by $\mathscr{M}_1$ the set of all measurable functions $y: I \to Y$. The elements of $\mathscr{M}_1$ are called control functions for J_1. $\mathscr{M}_2$, the set of control functions for J_2, is similarly defined with Z replacing Y.

Any map $\alpha: \mathscr{M}_2 \to \mathscr{M}_1$ is called a pseudo-strategy for J_1. Each pseudo-strategy has a value $u(\alpha) = \inf_{z \in \mathscr{M}_2} P(\alpha z, z)$, giving the worst possible result for J_1 if he adopts pseudo-strategy α. Not all pseudo-strategies are reasonable as they imply a foreknowledge of the opponent's choice of control function. Thus $\alpha: \mathscr{M}_2 \to \mathscr{M}_1$ is called a strategy if, whenever $0 < T \leqq 1$ and $z_1(t) = z_2(t)$ a.e., $0 \leqq t \leqq T$, then $\alpha z_1(t) = \alpha z_2(t)$ a.e., $0 \leqq t \leqq T$. Similar definitions are made for J_2. The set of strategies for J_1 is denoted by Γ and the set of strategies for J_2 by Δ. The value U of the game to J_1 is then

$$U = \sup_{\alpha \in \Gamma} u(\alpha).$$

Similarly the value V to J_2 is

$$V = \inf_{\beta \in \Delta} v(\beta) = \inf_{\beta \in \Delta} \sup_{y \in \mathscr{M}_1} P(y, \beta y).$$

If $U = V$ the game is said to have value.

4. **Friedman's approach.** For any integer N we shall define a game E_N^+. Let $\delta = 2^{-N}$, $I_1 = [0, \delta]$, $I_j = ((j-1)\delta, j\delta]$, $j = 2, \ldots, 2^N$. The game E_N^+ has the same dynamics and pay-off as G but it is played as follows: J_2 selects a control function on I_1 and then J_1 selects a control function on I_1, and the players then play alternately, J_2 selecting his function on I_j before J_1. From the theory of alternate move games it follows that the game E_N^+ has a value V_N^+. For the details of this approach see Friedman [**5**].

It follows immediately that $V_N^+ \geqq V_{N+1}^+$ for all N, and so we may define $V^+ = \lim_{N \to \infty} V_N^+$. V^+ is the upper value of the game in the sense of Friedman. The game E_N^- is defined like E_N^+ except that J_1 plays first at each step. E_N^- has a value V_N^- and $V^- = \lim_{N \to \infty} V_N^-$ is called the lower value of the game. As $V_N^- \leqq V_N^+$ for all N we have $V^- \leqq V^+$, and G is said to have a value in the sense of Friedman if $V^+ = V^-$.

Using pseudo-strategies with certain delays we can relate the approach of §3 and that of Friedman and show, in particular,

THEOREM 1. $V^- \leqq U$, $V \leqq V^+$.

In reference [5] Friedman shows that $V^+ = V^-$ if the dynamics are of the form

$$\dot{x} = f_1(t, x, y) + f_2(t, x, z)$$

and the function h in the pay-off is of the form $h_1(t, x, y) + h_2(t, x, z)$.

5. **Fleming's approach.** Again let N be a positive integer and let $\delta = 2^{-N}$. Fleming [3], [4] considered discrete games K_N^+ and K_N^- as follows. In K_N^+, J_1 and J_2 select constant control functions alternately on the intervals I_j, and, as in E_N^+, J_2 plays first at each step. The trajectory is determined by $x(0) = 0$ and

$$x(t_j) = x(t_{j-1}) + \delta f(t_{j-1}, x(t_{j-1}), y_j, z_j)$$

$(t_j = j\delta)$. The pay-off is given by

$$P = g(x(1)) + \delta \sum_{j=1}^{2^N} h(t_{j-1}, x(t_{j-1}), y_j, z_j).$$

By the theory of alternate move games this has a value W_N^+. Similarly, the game K_N^-, in which J_1 plays first at each step, has a value W_N^-. We may also consider K_N^+ starting at time t_k with initial condition $x(t_k) = x$. This game has a value $W_N^+(t_k, x)$ and a dynamic programming argument shows

$$W_N^+(t_k, x) = \min_z \max_y \{W_N^+(t_{k+1}, x') + \delta h(t_k, x, y, z)\},$$

where $x' = x + \delta f(t_k, x, y, z)$ and $W^+(1, x) = g(x)$. Considerations of this kind led Isaacs [7] to derive heuristically the Isaacs-Bellman differential equation for the "upper value" $R(t, x)$ of G. This equation is

$$\frac{\partial R}{\partial t} + F^+(t, x, \nabla R) = 0$$

where

$$F^+(t, x, p) = \min_z \max_y (p \cdot f + h),$$

for $p \in \boldsymbol{R}^m$. R also must satisfy $R(1, x) = g(x)$. Unfortunately there are no theorems guaranteeing the existence or uniqueness of solutions of this equation. However, introducing "white noise" into the dynamics, Fleming considers a nonlinear parabolic equation for the expectation of R,

$$\frac{\lambda^2}{2} \nabla^2 R + \frac{\partial R}{\partial t} + F^+(t, x, \nabla R) = 0,$$

for which solutions are known to exist. Letting $N \to \infty$ and $\lambda \to 0$, Fleming [4] is able to show in particular that $W^+ = \lim_{N\to\infty} W_N^+$ exists. From the differential equation it is seen that this Fleming value W^+

depends only on $F^+(t, x, p)$. Similar considerations prove the existence of $W^- = \lim_{N\to\infty} W_N^-$, and this depends only on

$$F^-(t, x, p) = \max_y \min_z (p \cdot f + h).$$

Therefore, if

$$F^+(t, x, p) = F^-(t, x, p) \tag{3}$$

then $W^+ = W^-$. We call (3) the Isaacs condition.

6. **Relaxed controls.** Relaxed controls were introduced into differential games in [1], to which we refer for a detailed discussion. Denote by $\Lambda(Y)$ and $\Lambda(Z)$ the sets of regular probability measures on Y and Z. $\Lambda(Y)$ can be considered as a subset of the dual of the space of continuous functions on Y, and with the weak* topology $\Lambda(Y)$ is a compact metrizable space. The domains of f and h may then be extended thus:

$$f: I \times \boldsymbol{R}^m \times \Lambda(Y) \times \Lambda(Z) \to \boldsymbol{R}^m,$$

where $f_i(t, x, \sigma, \tau) = \int_Z \int_Y f_i(t, x, y, z)\, d\sigma(y)\, d\tau(z)$, $i = 1 \ldots m$, and $h(t, x, \sigma, \tau)$ is defined similarly.

Using relaxed controls it may be shown that $W^- \leqq V^-$ and $W^+ \geqq V^+$. It follows from these inequalities that

THEOREM 2. *If the Isaacs condition* (3) *is satisfied* $W^+ = W^- = V^+ = V^- = U = V$.

We remark that this result includes the previous result of Friedman [**5**] (see above) for if the functions f and h split in y and z, then the Isaacs condition is automatically satisfied. From a result of Wald [**8**] on games played over compact metric spaces (generalizing the original result of von Neumann on finite matrix games), we may also show

THEOREM 3. *If both players use relaxed controls then the Isaacs condition is satisfied.*

Thus relaxed controls provide for differential games the analogue of von Neumann's mixed strategies for finite matrix games.

By using an approximation technique we may weaken the restrictions on the functions f, g and h in Theorem 2. Although we no longer can guarantee the existence of W^+ and W^- we can still define the quantities V^-, V^+, U and V.

THEOREM 4. *Suppose f satisfies a Lipschitz condition of the form*

$$\|f(t, x, y, z) - f(t, x', y, z)\| \leqq k(t)\|x - x'\|$$

where $\int_0^1 k(t)\,dt < \infty$, *and* h *is continuous. Suppose the pay-off is given by*

$$P = \mu(x(t)) + \int_0^1 h(t, x, y(t), z(t))\,dt$$

where μ *is a continuous functional on the space of trajectories (considered as a subspace of the Banach space of continuous functions* $x: I \to \boldsymbol{R}^m$*). Then* $V^- = V^+ = U = V$, *if* (3) *is satisfied.*

Finally we note that by using the theory of stochastic differential equations, A. Friedman [**6**] has recently been able to replace the inequalities preceding Theorem 2, $W^+ \geqq V^+$, $W^- \leqq V^-$, by equalities.

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Department of Mathematics, University of Warwick, Coventry, England

Department of Mathematics, University College, Swansea, Wales