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A note on galbed spaces

We shall say that an F-space (complete metric topological vector space) is *strictly galbed* if there is a sequence (C_n) of positive real numbers such that whenever $x_n \in X$ and $(C_n x_n)$ is bounded, then Σx_n converges. Such spaces have been studied by Turpin ([7]) who calls them "spaces galbed by l_t^0". Turpin does consider other galb (or generalized convexity) conditions, but strictly galbed spaces seem to be the most useful. Particularly important are exponentially galbed spaces ($C_n = 2^n$), see [6], p. 91. See [7], p. 141, for examples of strictly galbed spaces.

A locally pseudo-convex F-space is strictly galbed by making $C_n = 2^n$. The purpose of this note is to show that if X is not locally bounded (and hence also locally pseudo-convex by [6], p. 61), then X contains an infinite-dimensional locally convex subspace; in fact this subspace can be chosen to be nuclear. We remark that if X is locally convex already, this result reduces to one of Bessaga, Pełczyński and Rolewicz [1], that a locally convex F-space which is not a Banach space contains an infinite-dimensional nuclear subspace.

If X is a strictly galbed F-space, then we may assume that the corresponding sequence C_n satisfies $C_1 = C_2 = 1$ and $C_{m+n} \geqslant C_m C_n$. Then we may also select a sequence of symmetric closed neighbourhoods of 0, V_n say, satisfying $C_1^{-1} V_{n+1} + \ldots + C_m^{-1} V_{n+1} \subset V_n$ $(1 \leqslant m < \infty, 1 \leqslant n < \infty)$.

If we define $\|x\|_n = \inf\{\lambda\colon x \in \lambda V_n\}$, then

$$\|x_1 + \ldots + x_k\|_n \leqslant \max_{1 \leqslant i \leqslant k} C_i \|x_i\|_{n+1} \quad (1 \leqslant k < \infty, 1 \leqslant n < \infty),$$

and $\|\lambda x\|_n = |\lambda| \, \|x\|_n$.

If X does not contain arbitrarily short lines we may suppose V_1 does not contain any lines and hence for $1 \leqslant n < \infty$, $\|x\|_n = 0 \Leftrightarrow x = 0$.

If X is not locally bounded, we may also suppose V_{n+1} does not absorb V_n for any n, i.e.

$$\sup_{\|x\|_n \neq 0} \frac{\|x\|_{n+1}}{\|x\|_n} = \infty.$$

A sequence (e_n) in an F-space X is called *M-basic* if there are continuous linear functionals (e_n^*) defined on the closed span E of (e_n) such that $e_i^*(e_j) = \delta_{ij}$ and if $x \in E$ with $e_n^*(x) = 0$ $(1 \leqslant n < \infty)$, then $x = 0$. We shall say that (e_n) is equicontinuous if $e_n^*(x)e_n \to 0$ for $x \in E$. Our first result is that an M-basic sequence always contains a subsequence which is equicontinuous.

THEOREM 1. *Let (e_n) be an M-basic sequence in an F-space X. Then (e_n) has a subsequence (f_n) which is an equicontinuous M-basic sequence.*

Proof. We can suppose that (e_n) is fundamental in X and hence that X is separable. Let us denote by θ the F-space topology of X. We construct, similar to Section 6 of [4], a transfinite sequence of topologies τ_α $(1 \leqslant \alpha \leqslant \Omega)$, where Ω is the first uncountable ordinal. Let τ_0 be the topology induced by the linear functionals $x \to e_n^*(x)$ $(n \in N)$. Then for $\alpha = \beta + 1$, let τ_α be the topology whose base consists of all τ_β-closed θ-neighbourhoods of 0; for α a limit ordinal, $\tau_\alpha = \sup(\tau_\alpha\colon \beta < \alpha)$. Then as in Lemma 6.3 of [4], there exists a countable ordinal η such that $\tau_\eta = \theta$.

Consider the following property:

(P_α) There exists a subsequence (f_n) of (e_n) such that whenever $a_n f_n \to O(\tau_\alpha)$ then $a_n f_n \to O(\theta)$.

If P_0 holds, then $a_n f_n \to O(\theta)$ for any sequence (a_n). Thus (f_n) has a subsequence equivalent to the usual basis of ω and the theorem follows easily. Otherwise let β be the first ordinal such that P_β holds; clearly $\beta \leqslant \eta$ is countable. If β is not a limit ordinal $\beta = \alpha + 1$, and there exists a_n such $a_n f_n \to O(\tau_\alpha)$ but $a_n f_n \nrightarrow O(\tau_\beta)$. By [3], Proposition 3.2, (f_n) has a subsequence (g_n) which is τ_β-basic. If $x \in F$, the closed linear span of (g), then

$$x = \sum_{n=1}^{\infty} g_n^*(x)\, g_n(\tau_\beta)$$

and hence $g_n^*(x)g_n \to O(\tau_\beta)$ and so by (P_β) $g_n^*(x)g_n \to O(\theta)$.

Now suppose β is a limit ordinal, say $\beta = \sup_n \alpha_n$. Then there exist a_n such that $a_n f_n \to O(\tau_{\alpha_1})$ but $a_n f_n \nrightarrow O(\tau_\beta)$. Hence there exists $\gamma_1, \alpha_1 \leqslant \gamma_1 < \beta$, such that $a_n f_n \to O(\gamma_1)$ but $a_n f_n \nrightarrow O(\tau_{\gamma_1+1})$. Now again by [3], Proposition 3.2, there is a subsequence (g_n) of (f_n) which is τ_{γ_1+1}-basic. Now replace (f_n) by (g_n) and α_1 by α_2 and repeat the process to obtain a subsequence (h_n) of (g_n) which is τ_{γ_2+1}-basic where $\alpha_2 \leqslant \gamma_2 < \beta$. Repeating inductively and using a diagonal argument, we obtain a subsequence (u_n) of (f_n) which is τ_{γ_n+1}-basic for all n, where $\alpha_n \leqslant \gamma_n < \beta$. Clearly, (u_n) is also τ_β-basic since $\beta = \sup \gamma_n$. The remainder of the argument is as for the case where β is a non-limit ordinal.

THEOREM 2. *Let X be a strictly galbed F-space; then either X is locally* *bounded or X has an infinite-dimensional closed subspace which is locally convex and nuclear.*

Proof. If X contains arbitrarily short lines, then X contains a subspace isomorphic to ω, which is nuclear and locally convex ([6], p. 114). Hence, we suppose X non-locally bounded and without arbitrarily short lines. We then choose a base of neighbourhoods (V_n) of 0 as described in the introduction with V_1 linearly bounded.

Let m be fixed, where $1 \leqslant m < \infty$. By induction we may pick u_n $(1 \leqslant n \leqslant m)$ so that

$$\|u_n\|_{3n+1} \leqslant C_{n+1}^{-1}, \qquad \|u_n\|_{3n+2} \geqslant C_m(\|u_1+\ldots+u_{n-1}\|_{3n+3}+1).$$

Then let $e_m = u_1+\ldots+u_m$. If $1 \leqslant n \leqslant m$

$$\|e_m\|_{3n} \leqslant \max(C_1\|u_1+\ldots \\ \ldots+u_{n-1}\|_{3n+1}, C_2\|u_n\|_{3n}, C_3\|u_{n+1}\|_{3n+1}, \ldots, C_{m-n+2}\|u_m\|_{3n+1}).$$

However, for $k \geqslant n$

$$C_{k-n+2}\|u_k\|_{3n+1} \leqslant C_{k-n+2}\|u_k\|_{3k+1} \leqslant C_{k-n+2}C_{k+1}^{-1} \leqslant 1.$$

Hence

$$\|e_m\|_{3n} \leqslant \max(1, \|u_1+\ldots+u_{n-1}\|_{3n+1}).$$

On the other hand,

$$\|u_n\|_{3n+2} \leqslant \max(C_1\|u_1+\ldots \\ \ldots+u_{n-1}\|_{3n+3}, C_2\|e_m\|_{3n+3}, C_3\|u_{n+1}\|_{3n+1}, \ldots, C_{m-n+2}\|u_m\|_{3n+3})$$

and

$$C_{k-n+2}\|u_k\|_{3n+3} \leqslant C_{k-n+2}\|u_k\|_{3k+1} \leqslant 1 \qquad (k \geqslant n+1).$$

Hence

$$\|e_m\|_{3n+3} \geqslant C_m(\|u_1+\ldots+u_{n-1}\|_{3n+3}+1) \geqslant C_m\|e_m\|_{3n} \qquad (1 \leqslant n \leqslant m).$$

Now let $b_{m,n} = \|e_n\|_m$; then

$$\sum_{n=1}^{\infty} \frac{b_{m,n}}{b_{m+6,n}} \leqslant \sum_{n=1}^{m-1} \frac{b_{m,n}}{b_{m+6,n}} + \sum_{n=m}^{\infty} C_n^{-1} < \infty$$

so that the Köthe sequence space $l_1(b_{m,n})$ of all sequences (ξ_n) such that

$$\sum_{n=1}^{\infty} b_{m,n}|\xi_n| < \infty, \qquad m = 1, 2, \ldots,$$

is nuclear in its natural topology (the Grothendieck–Pietsch criterion, cf. [2], p. 59).

Suppose $\xi_n \in l_1(b_{m,n})$. Then

$$\sup_n b_{m,n}|\xi_n| < \infty, \quad 1 \leqslant m < \infty,$$

and hence

$$\sup_n |\xi_n| \, \|e_n\|_{3m+3} < \infty, \quad 1 \leqslant m < \infty.$$

Thus

$$\sup_n C_n|\xi_n| \, \|e_n\|_{3m} < \infty, \quad 1 \leqslant m < \infty,$$

i.e. $(C_n \xi_n e_n)$ is bounded. Hence $\sum \xi_n e_n$ converges, and we may define

$$T\colon \; l_1(b_m, n) \to X$$

by

$$T(\xi) = \sum_{n=1}^{\infty} \xi_n e_n,$$

and T is continuous by the Banach–Steinhaus Theorem.

We shall show that the range of T includes an infinite-dimensional closed subspace E; then since E is isomorphic to a quotient of $T^{-1}(E)$, E is also a nuclear locally convex F-space.

Let γ be the vector topology on X with a base of neighbourhoods $W(a_k) = \bigcup_{n=1}^{\infty} \sum_{k=1}^{n} a_k V_3$, where (a_k) ranges over all sequence $a_k > 0$. It is easy to see that γ is indeed a vector topology. Also $W(C_k^{-1}) \subset V_2$ and is thus linearly bounded; hence γ is Hausdorff. Also $\|e_m\|_6^{-1} e_m \to O(\gamma)$ since $\|e_m\|_6^{-1} \|e_m\|_3 \leqslant C_m^{-1}$. However, $\|e_m\|_6^{-1} e_m \nrightarrow 0$ in X and hence (e_m) has a subsequence which is M-basic ([3], [5]) and by Theorem 1 a further subsequence (e_{m_k}) which is equicontinuous and M-basic; let E be the closed linear span of this sequence.

If $x \in E$, then $e_{m_k}^*(x) e_{m_k} \to 0$, where $e_{m_k}^*$ are the dual functionals on E. Define $\xi_{m_k} = e_{m_k}^*(x)$ and $\xi_n = 0$ $(n \notin m_k)$. Then $\sup_n b_{m,n}|\xi_n| < \infty$, $1 \leqslant m < \infty$, and hence $\xi \in l_1(b_{m,n})$. Clearly, $T\xi = x$, and so $E \subset T\big(l_1(b_{m,n})\big)$; thus we are home.

Remark. Suppose $\varphi(0) = 0$ and

$$\varphi(x) = \left(1 + \log \frac{1}{x}\right)^{-1}, \quad 0 < x \leqslant 1,$$

$$\varphi(x) = 1, \quad 1 \leqslant x < \infty.$$

Then the Orlicz sequence space l_φ has no infinite-dimensional locally pseudo-convex subspace. Indeed, any infinite-dimensional closed subspace contains a basic sequence equivalent to a block basis (u_n) of the standard unit vector basis. Now pick a_n so that

$$\sum_{k=1}^{\infty} \varphi(|a_n u_n(k)|) = \frac{1}{\log(n+1)}, \quad 1 \leqslant n < \infty,$$

so that $a_n u_n \to 0$. Then

$$\sum_{k=1}^{\infty} \varphi(|e^{-n} a_n u_n(k)|) \geqslant \frac{\varphi(e^{-n})}{\log(n+1)}, \quad 1 \leqslant n < \infty,$$

since φ is supermultiplicative. As

$$\sum_{n=1}^{\infty} \frac{\varphi(e^{-n})}{\log(n+1)} = \sum_{n=1}^{\infty} \frac{1}{(n+1)\log(n+1)} = \infty,$$

$\sum e^{-n} a_n u_n$ does not converge, and hence the closed linear span of $(a_n u_n)$ is not locally pseudo-convex. Of course, l_φ is not strictly galbed (cf. [7]); indeed it has no strictly galbed subspace by the above result.

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