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A note on galbed spaces

We shall say that an $F$-space (complete metric topological vector space) is strictly galbed if there is a sequence $(C_n)$ of positive real numbers such that whenever $x_n \in X$ and $(C_n x_n)$ is bounded, then $\Sigma x_n$ converges. Such spaces have been studied by Turpin ([7]) who calls them "spaces galbed by $l_2^\infty$. Turpin does consider other galb (or generalized convexity) conditions, but strictly galbed spaces seem to be the most useful. Particularly important are exponentially galbed spaces ($C_n = 2^n$), see [6], p. 91. See [7], p. 141, for examples of strictly galbed spaces.

A locally pseudo-convex $F$-space is strictly galbed by making $C_n = 2^n$. The purpose of this note is to show that if $X$ is not locally bounded (and hence also locally pseudo-convex by [6], p. 61), then $X$ contains an infinite-dimensional locally convex subspace; in fact this subspace can be chosen to be nuclear. We remark that if $X$ is locally convex already, this result reduces to one of Bessaga, Pełczyński and Rolewicz [1], that a locally convex $F$-space which is not a Banach space contains an infinite-dimensional nuclear subspace.

If $X$ is a strictly galbed $F$-space, then we may assume that the corresponding sequence $C_n$ satisfies $C_1 = C_2 = 1$ and $C_{m+n} \geq C_mC_n$. Then we may also select a sequence of symmetric closed neighbourhoods of 0, $V_n$ say, satisfying $C_1^{-1} V_{n+1} + \ldots + C_m^{-1} V_{n+1} \subset V_n$ ($1 \leq m < \infty, 1 \leq n < \infty$).

If we define $\|x\|_n = \inf \{\lambda: x \in \lambda V_n\}$, then

$$\|x_1 + \ldots + x_k\|_n \leq \max_{1 \leq i \leq k} C_i \|x_i\|_{n+1}$$

(1 \leq k < \infty, 1 \leq n < \infty),

and $\|\lambda x\|_n = |\lambda| \|x\|_n$.

If $X$ does not contain arbitrarily short lines we may suppose $V_1$ does not contain any lines and hence for $1 \leq n < \infty$, $\|x\|_n = 0 \leftrightarrow x = 0$.

If $X$ is not locally bounded, we may also suppose $V_{n+1}$ does not absorb $V_n$ for any $n$, i.e.

$$\sup_{\|x\|_n \neq 0} \frac{\|x\|_{n+1}}{\|x\|_n} = \infty.$$
A sequence \((e_n)\) in an \(F\)-space \(X\) is called \(M\)-basic if there are continuous linear functionals \((e_n^*)\) defined on the closed span \(E\) of \((e_n)\) such that \(e_n^*(e_j) = \delta_{ij}\) and if \(x \in E\) with \(e_n^*(x) = 0\) \((1 \leq n < \infty)\), then \(x = 0\). We shall say that \((e_n)\) is equicontinuous if \(e_n^*(x)e_n \to 0\) for \(x \in E\). Our first result is that an \(M\)-basic sequence always contains a subsequence which is equi-
continuous.

**Theorem 1.** Let \((e_n)\) be an \(M\)-basic sequence in an \(F\)-space \(X\). Then \((e_n)\) has a subsequence \((f_n)\) which is an equicontinuous \(M\)-basic sequence.

**Proof.** We can suppose that \((e_n)\) is fundamental in \(X\) and hence that \(X\) is separable. Let us denote by \(\theta\) the \(F\)-space topology of \(X\). We construct, similar to Section 6 of [4], a transfinite sequence of topologies \(\tau \alpha (1 \leq \alpha \leq \Omega)\), where \(\Omega\) is the first uncountable ordinal. Let \(\tau_0\) be the topology induced by the linear functionals \(x \to e_n^*(x)\) \((n \in \mathbb{N})\). Then for \(\alpha = \beta + 1\), let \(\tau_\alpha\) be the topology whose base consists of all \(\tau_\beta\)-closed \(\theta\)-neighbourhoods of \(0\); for \(\alpha\) a limit ordinal, \(\tau_\alpha = \sup (\tau_\beta; \beta < \alpha)\). Then as in Lemma 6.3 of [4], there exists a countable ordinal \(\eta\) such that \(\tau_\eta = \theta\).

Consider the following property:

\((P_\alpha)\) There exists a subsequence \((f_n)\) of \((e_n)\) such that whenever \(a_nf_n \to O(\tau_\alpha)\) then \(a_nf_n \to O(\theta)\).

If \(P_0\) holds, then \(a_nf_n \to O(\theta)\) for any sequence \((a_n)\). Thus \((f_n)\) has a subsequence equivalent to the usual basis of \(\omega\) and the theorem follows easily. Otherwise let \(\beta\) be the first ordinal such that \(P_\beta\) holds; clearly \(\beta \leq \eta\) is countable. If \(\beta\) is not a limit ordinal \(\beta = \alpha + 1\), and there exists \(a_n\) such \(a_nf_n \to O(\tau_\alpha)\) but \(a_nf_n \not\to O(\tau_\beta)\). By [3], Proposition 3.2, \((f_n)\) has a subsequence \((g_n)\) which is \(\tau_\beta\)-basic. If \(x \in F\), the closed linear span of \((g)\), then

\[x = \sum_{n=1}^{\infty} g_n^*(x)g_n(\tau_\beta)\]

and hence \(g_n^*(x)g_n \to O(\tau_\beta)\) and so by \((P_\beta)\) \(g_n^*(x)g_n \to O(\theta)\).

Now suppose \(\beta\) is a limit ordinal, say \(\beta = \sup \alpha_n\). Then there exist \(a_n\) such that \(a_nf_n \to O(\tau_\alpha)\) but \(a_nf_n \not\to O(\tau_\beta)\). Hence there exists \(\gamma_1, \alpha_1 \leq \gamma_1 < \beta\), such that \(a_nf_n \to O(\tau_\gamma)\) but \(a_nf_n \not\to O(\tau_{\gamma_1 + 1})\). Now again by [3], Proposition 3.2, there is a subsequence \((h_n)\) of \((f_n)\) which is \(\tau_{\gamma_1 + 1}\)-basic. Now replace \((f_n)\) by \((g_n)\) and \(\alpha_1\) by \(\alpha_2\) and repeat the process to obtain a subsequence \((h_n)\) of \((g)\) which is \(\tau_{\gamma_2 + 1}\)-basic where \(\alpha_2 \leq \gamma_2 < \beta\). Repeating inductively and using a diagonal argument, we obtain a subsequence \((u_n)\) of \((f_n)\) which is \(\tau_{\gamma_n + 1}\)-basic for all \(n\), where \(\alpha_n \leq \gamma_n < \beta\). Clearly, \((u_n)\) is also \(\tau_\beta\)-basic since \(\beta = \sup \gamma_n\). The remainder of the argument is as for the case where \(\beta\) is a non-limit ordinal.

**Theorem 2.** Let \(X\) be a strictly galbed \(F\)-space; then either \(X\) is locally
bounded or \( X \) has an infinite-dimensional closed subspace which is locally convex and nuclear.

Proof. If \( X \) contains arbitrarily short lines, then \( X \) contains a subspace isomorphic to \( \omega \), which is nuclear and locally convex ([6], p. 114). Hence, we suppose \( X \) non-locally bounded and without arbitrarily short lines. We then choose a base of neighbourhoods \((V_n)\) of 0 as described in the introduction with \( V_1 \) linearly bounded.

Let \( m \) be fixed, where \( 1 \leq m < \infty \). By induction we may pick \( u_n \) (\( 1 \leq n \leq m \)) so that
\[
\|u_n\|_{3n+1} \leq C_{n+1}^{-1}, \quad \|u_n\|_{3n+2} \geq C_m (\|u_1 + \ldots + u_{n-1}\|_{3n+3} + 1).
\]
Then let \( e_m = u_1 + \ldots + u_m \). If \( 1 \leq n \leq m \)
\[
\|e_m\|_{3n} \leq \max (C_1 \|u_1 + \ldots
\ldots + u_{n-1}\|_{3n+1}, C_2 \|u_n\|_{3n}, C_3 \|u_{n+1}\|_{3n+1}, \ldots, C_{m-n+2} \|u_m\|_{3n+1}).
\]
However, for \( k \geq n \)
\[
C_{k-n+2} \|u_k\|_{3n+1} \leq C_{k-n+2} \|u_k\|_{3k+1} \leq C_{k-n+2} C_{k+1}^{-1} \leq 1.
\]
Hence
\[
\|e_m\|_{3n} \leq \max (1, \|u_1 + \ldots + u_{n-1}\|_{3n+1}).
\]
On the other hand,
\[
\|u_n\|_{3n+2} \leq \max (C_1 \|u_1 + \ldots
\ldots + u_{n-1}\|_{3n+3}, C_2 \|e_m\|_{3n+3}, C_3 \|u_{n+1}\|_{3n+1}, \ldots, C_{m-n+2} \|u_m\|_{3n+3})
\]
and
\[
C_{k-n+2} \|u_k\|_{3n+3} \leq C_{k-n+2} \|u_k\|_{3k+1} \leq 1 \quad (k \geq n+1).
\]
Hence
\[
\|e_m\|_{3n+3} \geq C_m (\|u_1 + \ldots + u_{n-1}\|_{3n+3} + 1) \geq C_m \|e_m\|_{3n} \quad (1 \leq n \leq m).
\]
Now let \( b_{m,n} = \|e_m\|_m \); then
\[
\sum_{n=1}^{\infty} \frac{b_{m,n}}{b_{m+6,n}} \leq \sum_{n=1}^{m-1} \frac{b_{m,n}}{b_{m+6,n}} + \sum_{n=m}^{\infty} C_n^{-1} < \infty
\]
so that the Köthe sequence space \( l_1 (b_{m,n}) \) of all sequences \((\xi_n)\) such that
\[
\sum_{n=1}^{\infty} b_{m,n} |\xi_n| < \infty, \quad m = 1, 2, \ldots,
\]
is nuclear in its natural topology (the Grothendieck–Pietsch criterion, cf. [2], p. 59).
Suppose $\xi_n \in l_1(b_{m,n})$. Then
\[ \sup_n b_{m,n} |\xi_n| < \infty, \quad 1 \leq m < \infty, \]
and hence
\[ \sup_n |\xi_n| \|e_n\|_{3m+3} < \infty, \quad 1 \leq m < \infty. \]
Thus
\[ \sup_n C_n |\xi_n| \|e_n\|_{3m} < \infty, \quad 1 \leq m < \infty, \]
i.e. $(C_n \xi_n e_n)$ is bounded. Hence $\sum \xi_n e_n$ converges, and we may define
\[ T: l_1(b_m, n) \to X \]
by
\[ T(\xi) = \sum_{n=1}^{\infty} \xi_n e_n, \]
and $T$ is continuous by the Banach–Steinhaus Theorem.

We shall show that the range of $T$ includes an infinite-dimensional closed subspace $E$; then since $E$ is isomorphic to a quotient of $T^{-1}(E)$, $E$ is also a nuclear locally convex $F$-space.

Let $\gamma$ be the vector topology on $X$ with a base of neighbourhoods
\[ W(a_k) = \bigcup_{n=1}^{\infty} \bigcap_{k=1}^{n} a_k V_3, \]
where $(a_k)$ ranges over all sequence $a_k > 0$. It is easy to see that $\gamma$ is indeed a vector topology. Also $W(C_k^{-1}) = V_2$ and is thus linearly bounded; hence $\gamma$ is Hausdorff. Also $\|e_m\|_{e^{-1}} e_m \to O(\gamma)$ since $\|e_m\|_{e^{-1}} \|e_m\|_3 \leq C^{-1}_m$. However, $\|e_m\|_{e^{-1}} e_m \to 0$ in $X$ and hence $(e_m)$ has a subsequence which is $M$-basic ([3], [5]) and by Theorem 1 a further subsequence $(e_{m_k})$ which is equicontinuous and $M$-basic; let $E$ be the closed linear span of this sequence.

If $x \in E$, then $e_{m_k}^*(x) e_{m_k} \to 0$, where $e_{m_k}^*$ are the dual functionals on $E$. Define $\xi_{m_k} = e_{m_k}^*(x)$ and $\xi_n = 0 (n \notin m_k)$. Then $\sup_n b_{m,n} |\xi_n| < \infty, \quad 1 \leq m < \infty$, and hence $\xi \in l_1(b_{m,n})$. Clearly, $T\xi = x$, and so $E \subset T(l_1(b_{m,n}))$; thus we are home.

Remark. Suppose $\varphi(0) = 0$ and
\[ \varphi(x) = \left(1 + \log \frac{1}{x}\right)^{-1}, \quad 0 < x \leq 1, \]
\[ \varphi(x) = 1, \quad 1 \leq x < \infty. \]
Then the Orlicz sequence space $l_\varphi$ has no infinite-dimensional locally pseudo-convex subspace. Indeed, any infinite-dimensional closed subspace contains a basic sequence equivalent to a block basis $(u_n)$ of the standard unit
vector basis. Now pick $a_n$ so that
\[ \sum_{k=1}^{\infty} \varphi(|a_n u_n(k)|) = \frac{1}{\log(n+1)}, \quad 1 \leq n < \infty, \]
so that $a_n u_n \to 0$. Then
\[ \sum_{k=1}^{\infty} \varphi(|e^{-n} a_n u_n(k)|) \geq \frac{\varphi(e^{-n})}{\log(n+1)}, \quad 1 \leq n < \infty, \]
where $\varphi$ is supermultiplicative. As
\[ \sum_{n=1}^{\infty} \frac{\varphi(e^{-n})}{\log(n+1)} = \sum_{n=1}^{\infty} \frac{1}{(n+1) \log(n+1)} = \infty, \]
$\sum e^{-n} a_n u_n$ does not converge, and hence the closed linear span of $(a_n u_n)$ is not locally pseudo-convex. Of course, $l_\infty$ is not strictly galbed (cf. [7]); indeed it has no strictly galbed subspace by the above result.

References