On rearrangements of vector-valued H_1-functions

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1. Introduction[1]

In [4], B. Davis characterized those real functions in $L_1(\mathbf{T})$ which are rearrangements of an H_1-function. Let us identify $\mathbf{T}$ with $\mathbf{R}/\mathbf{Z}$ i.e. with $[0, 1)$ under addition modulo one. Then f can be rearranged to be in $\Re H_1$ if and only if $f_d \in \Re H_1$ where f_d is the decreasing rearrangement of f on $[0, 1)$, and this is if and only if

(1) $$\int_0^1 \frac{|M_1(t)|}{t}\,dt < \infty$$

where

$$M_1(t) = \int_{-t}^{t} f_d(s)\,ds.$$

Here $f_d(s) = f_d(s+1)$, $s<0$. Davis' original proof uses probabilistic methods. Later J. L. Lewis (unpublished) gave an analytic proof. See also [6] for another proof, and also see [5], for related work.

After the initial preparation of the paper, Professor Davis informed the author of the existence of another solution of the rearrangement problem due to O. D. Ceretelli [3], who shows that f has a rearrangement in $\Re H_1$ if and only if

(2) $$\int_1^\infty \frac{|M_2(t)|}{t}\,dt < \infty$$

where

$$M_2(t) = \int_{|f(s)|>t} f(s)\,ds.$$

Ceretelli's results seem to have escaped attention in the West until quite recently; see [7].

In the course of preparing [6] we also found yet another proof which has the virtue of extending naturally to an arbitrary Banach space. In this note we there-

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fore prove a theorem which extends Davis's result to an arbitrary Banach space. Our proof is self-contained, although the ideas intersect those of [6]. It turns out that our proof in fact combines both the Ceretelli and Davis solutions in the scalar case.

We will work with an atomic definition of $H_1(X)$ (cf. [8]). A Bochner measurable function $a\colon \mathbf{T}\to X$ is called an *atom* provided we can find an interval $I\subset\mathbf{T}$ such that $\operatorname{supp} a\subset I$, $\|a(t)\|\leqq|I|^{-1}$, $(t\in I)$ and $\int a(t)\,dt=0$. We then define $H_1^a(X)$ to be the space of all $f\in L_1(X)=L_1(\mathbf{T},X)$ such that

$$f=\sum_{n=1}^{\infty} c_n a_n$$

where each a_n is an atom and $\sum|c_n|<\infty$. We then define $\|f\|_{aH}$ to be the infimum of $\sum|c_n|$ over all such representations.

If $f,g\in L_1(X)$ we say that f is a rearrangement of g if $\mu_f=\mu_g$ where μ_f is the Borel measure on X defined by $\mu_f(B)=\lambda(f^{-1}(B))$ (λ denotes normalized Haar measure on $\mathbf{T}$).

Finally f is said to be *almost decreasing* if for $0<s,t<1$ we have $\|f(t)\|\leqq\|f(s)\|$ whenever $t\geqq 2s$. Any function $f\in L_1(X)$ has an almost decreasing rearrangement.

We can now state our main theorem.

Theorem 1. *Let $f\in L_1(\mathbf{T},X)$ have mean zero and let g be an almost decreasing rearrangement of f. Let*

$$M_1(t)=\int_0^t g(s)\,ds$$

$$M_2(t)=\int_{\|f(s)\|>t} f(s)\,ds.$$

Then the following conditions on f are equivalent:

(i) *f has a rearrangement in $H_1^a(X)$*

(ii) $g\in H_1^a(X)$

(iii) $\displaystyle\int_0^\infty \|M_2(t)\|\frac{dt}{t}<\infty$

(iv) $\displaystyle\int_0^1 \|M_1(t)\|\frac{dt}{t}<\infty.$

Remarks. Note that M_2 is unchanged by passing to a rearrangement of f, while M_1 depends on g. In the scalar case $X=\mathbf{R}$, it is not difficult to show that (iv) is equivalent to the Davis condition (1) (see [6] for details). However, in the vector-valued case there is no apparent analogue of the decreasing rearrangement f_d.

Condition (iii) in the scalar case is Ceretelli's criterion; it is shown equivalent to (iv) in [6] (see also [7] for the probabilistic setting).

Let us also note the connection with the regular analytic definition of $H_1(X)$ when X is complex. Define $H_{1,0}(X)$ to be the subspace of $L_1(X)$ of all f such that

$$\int_0^1 f(t)e^{2ni\pi t}\,dt = 0 \quad n \geqq 0.$$

Then by a result of Bourgain [2] and J. Garcia-Cuerva, $H_{1,0}(X)\subset H_1^a(X)$. $H_1^a(X)$ coincides with $H_{1,0}(X)\oplus\bar{H}_{1,0}(X)$ exactly when X is a UMD-space (see [1]); here $\bar{H}_{1,0}(X)$ is the space of all f such that

$$\int_0^1 f(t)e^{-2ni\pi t}\,dt = 0 \quad n \geqq 0.$$

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2. Proof of Theorem 1

Clearly (ii)$\Rightarrow$(i). We complete the proof by showing that (iv)$\Rightarrow$(ii), (iii)$\Rightarrow$(iv) and then (i)$\Rightarrow$(iii).

Proof of (iv)$\Rightarrow$(ii): Let f^* be the decreasing rearrangement of $\|f(t)\|$. Then $\|g(t)\|\leqq f^*(\frac{1}{2}t)$. Let $\tau_0=1$ and then let τ_n be a point in $[2^{-n}, 2^{1-n}]$, $(n\geqq 1)$ so that $\|M_1(t)\|$ is minimized. Thus if $n\geqq 1$,

$$\|M_1(\tau_n)\| \leqq 2^n\int_{2^{-n}}^{2^{1-n}} \|M_1(t)\|\,dt \leqq 2\int_{2^{-n}}^{2^{1-n}} \|M_1(t)\|\frac{dt}{t}.$$

For $n\geqq 0$, let $I_n=[\tau_{n+1}, \tau_n]$, and let

$$\gamma_n = \frac{1}{|I_n|}\int_{I_n} g(t)\,dt$$

($=0$ if I_n is trivial). Let $a_n=(g-\gamma_n)1_{I_n}$ and $b_n=\gamma_n(1_{I_n}-|I_n|1_{[0,1]})$. Then

$$g = \sum_{n=1}^{\infty} a_n+\sum_{n=1}^{\infty} b_n$$

in $L_1(X)$.

Now

$$\|a_n\|_{aH} \leqq 2|I_n|\sup_{t\in I_n}\|g(t)\| \leqq 2(\tau_n-\tau_{n+1})f^*\left(\frac{1}{2}\tau_{n+1}\right) \leqq 3.2^{-n}f^*(2^{-(n+2)}).$$

Hence

$$\sum \|a_n\|_{aH} \leqq 24 \sum 2^{-(n+3)} f^*(2^{-(n+2)}) \leqq 24\|f\|_1.$$

Also

$$\|b_n\|_{aH} \leqq |\gamma_n| \leqq \|M_1(\tau_{n+1})\| + \|M_1(\tau_n)\|.$$

Thus

$$\sum \|b_n\|_{aH} \leqq 4 \int_0^1 \frac{\|M_1(t)\|}{t}\, dt.$$

Hence $g \in H_1^a$ and

$$\|g\|_{aH} \leqq 24\|f\|_1 + 4\int_0^1 \frac{\|M_1(t)}{t}\, dt,$$

completing the proof of (iv)$\Rightarrow$(ii).

Before continuing we introduce some notation. Denote by $\mathscr{L}$ the set of all maps $\varphi\colon \mathbf{R} \to X^*$ which are bounded and satisfy

$$\|\varphi(t) - \varphi(s)\| \leqq |t-s| \quad s, t \in \mathbf{R}.$$

Let $\mathscr{C}$ be the subset of $\mathscr{L}$ of all φ such that φ is continuously differentiable and φ' has compact support. If $\varphi \in \mathscr{L}$ then one can find a sequence $\varphi_n \in \mathscr{C}$ such that

$$\sup_{n \in N} \sup_{t \in \mathbf{R}} \|\varphi(t) - \varphi_n(t)\| < \infty, \quad \lim_{n \to \infty} \varphi_n(t) = \varphi(t) \quad t \in \mathbf{R}.$$

To see this first approximate φ by a function $\psi \in \mathscr{L}$ which is constant outside a compact set and then smooth by convolution with a suitable "bump" function.

For $\varphi \in \mathscr{L}$ we define $\Omega_\varphi\colon L_1(X) \to \mathbf{C}$ (or $\mathbf{R}$) by

$$\Omega_\varphi(f) = \int_0^1 \langle f(t), \varphi(\log \|f(t)\|)\rangle\, dt$$

where the integrand vanishes if $f(t)=0$. Similarly define Γ_φ by

$$\Gamma_\varphi(f) = \int_0^1 \langle f(t), \varphi(\log t)\rangle\, dt.$$

Now under the hypotheses of the theorem, we claim:

Lemma 2.

$$\sup_{\varphi \in \mathscr{L}} |\Omega_\varphi(f)| = \sup_{\varphi \in \mathscr{C}} |\Omega_\varphi(f)| = \int_0^\infty \|M_2(t)\| \frac{dt}{t},$$

$$\sup_{\varphi \in \mathscr{L}} |\Gamma_\varphi(g)| = \sup_{\varphi \in \mathscr{C}} |\Gamma_\varphi(g)| = \int_0^1 \|M_1(t)\| \frac{dt}{t}.$$

Proof. In each case the first equality is routine based on the comments preceding the lemma. The second equalities are similar to each other and we prove only the first. Note that if $\varphi \in \mathscr{C}$, $\Omega_\varphi(f)$ is unchanged by replacing φ by $\varphi + x^*$ for any fixed

$x^*\in X^*$ since f has mean zero. We therefore assume that $\lim_{t\to-\infty}\varphi(t)=0$. Then

$$\Omega_\varphi(f)=\int_0^1\int_{-\infty}^{\log\|f(t)\|}\langle f(t),\varphi'(s)\rangle\,ds\,dt=\int_{-\infty}^{\infty}\langle M_2(e^s),\varphi'(s)\rangle\,ds$$

by Fubini's theorem. Hence

$$\sup_{\varphi\in\mathscr{C}}|\Omega_\varphi(f)|=\int_{-\infty}^{\infty}\|M_2(e^s)\|\,ds=\int_0^\infty\|M_2(t)\|\frac{dt}{t}.$$

Proof of (iii)⇒(iv). By Lemma 2 we have

$$\int_0^1\|M_1(t)\|\frac{dt}{t}=\sup_{\psi\in\mathscr{L}}\left|\int_0^1\langle g(t),\psi(\log t)\rangle dt\right|.$$

Now define for $0<t\leqq 1$,

$$\beta(t)=\sup_{t\leqq s\leqq 1}\left(\frac{s}{t}\right)^{1/2}\|g(s)\|$$

and notice that if $t_1\leqq t_2$,

$$\log\beta(t_2)\leqq\log\beta(t_1)+\tfrac{1}{2}(\log t_1-\log t_2).$$

It follows that, given $\psi\in\mathscr{L}$, there exists $\varphi\in\mathscr{L}$ with

$$\varphi(\log\beta(t))=\tfrac{1}{2}\psi(\log t)$$

as long as $\beta(t)>0$.

We now estimate $\int\beta(t)dt$. Since g is almost decreasing, if $2^{-(n+1)}<s\leqq 2^{-n}$, $(n\geqq 0)$ we have $\|g(s)\|\leqq\|g(2^{-(n+2)})\|$ and so if $t\leqq s\leqq 1$

$$\left(\frac{s}{t}\right)^{1/2}\|g(s)\|\leqq 2\left(\frac{2^{-(n+2)}}{t}\right)^{1/2}\|g(2^{-(n+2)})\|.$$

Hence

$$\beta(t)\leqq\sup_n 2\left(\frac{2^{-(n+2)}}{t}\right)^{1/2}\|g(2^{-(n-2)})\|1_{[0,2^{-n})}(t)$$

$$\leqq\sum_{n=0}^\infty 2\left(\frac{2^{-(n+2)}}{t}\right)^{1/2}\|g(2^{-(n+2)})\|1_{[0,2^{-n})}(t)$$

and

$$\int\beta(t)\,dt\leqq 2\sum_{n=0}^\infty 2^{-n}\|g(2^{-(n+2)})\|\leqq 32\sum_{n=0}^\infty\int_{2^{-(n+4)}}^{2^{-(n+3)}}\|g(t)\|\,dt\leqq 32\|g\|_1.$$

Now if $\psi\in\mathscr{L}$,

$$\int_0^1\langle g(t),\psi(\log t)\rangle dt=2\int_0^1\langle g(t),\varphi(\log\beta(t))\rangle\,dt$$

and, using the inequality $x \log 1/x \leqq 1/e$, $0<x\leqq 1$,

$$\left|\int_0^1 \langle g(t), \varphi(\log \beta(t)) - \varphi(\log \|g(t)\|)\rangle\, dt\right| \leqq \int_0^1 \|g(t)\| \log \frac{\beta(t)}{\|g(t)\|}\, dt$$

$$\leqq \frac{1}{e}\int_0^1 \beta(t)\, dt \leqq \frac{32}{e}\|g\|_1.$$

Hence

$$\int_0^1 \langle g(t), \psi(\log t)\rangle dt \leqq \int_0^\infty \|M_2(t)\| \frac{dt}{t} + \frac{64}{e}\|g\|_1$$

so that

$$\int_0^1 \|M_1(t)\| \frac{dt}{t} \leqq \int_0^\infty \|M_2(t)\| \frac{dt}{t} + \frac{64}{e}\|g\|_1 < \infty,$$

i.e. (iii)$\Rightarrow$(iv).

In order to complete the proof we need to introduce the dyadic H_1-space, $H_1^d(X)$. A *dyadic atom* is an atom a such that the associated interval I is a dyadic interval $D(n, k) = [(k-1)2^{-n}, k\cdot 2^n)$ for $1 \leqq k \leqq 2^n$. We say that $f \in H_1^d(X)$ if it can be written $f = \sum c_n a_n$ where each a_n is a dyadic atom and $\sum |c_n| < \infty$. We set $\|f\|_{dH}$ to be the infimum of $\sum |c_n|$ over all such representations.

The following lemma is essentially known (cf. [2]).

Lemma 3. *Let $f \in H_1^a(X)$. Then there exists θ, $0<\theta<1$ such that $f_\theta \in H_1^d(X)$ and $\|f_\theta\|_{dH} \leqq 32\|f\|_{aH}$ where $f_\theta(t) = f(t+\theta)$.*

Proof. First suppose a is an atom supported on an interval I. Let $J=[\alpha, \beta)$ be a half-open interval containing I with $|J| \leqq 2|I|$ and $|J| = 2^{-m}$ for some $m \geqq 0$.

If $m=0$ then we note that $\|a_\theta\|_{dH} \leqq 2$ for all θ as $|I| \geqq \frac{1}{2}$. Suppose then $m \geqq 1$. Define $k=k(\theta)$ to be the unique k so that $1 \leqq k \leqq 2^m$ and $J_\theta \subset D(m, k) \cup D(m, k+1)$. Here $J_\theta = \{t\colon t+\theta \in J\}$ and we interpret $D(m, 2^m+1)$ as $D(m, 1)$. Let $r=r_k=r(\theta)$ be the greatest integer such that 2^r divides k. If $r=0$ then $D(m, k)\cup D(m, k+1)$ is a dyadic interval and so $\|a_\theta\|_{dH} \leqq 2\cdot 2^{-m}|I|^{-1} \leqq 4$.

If $r>0$ set

$$x = 2^m \int_{D(m,k)} a_\theta(t)\, dt = -2^m \int_{D(m,k+1)} a_\theta(t)\, dt.$$

Then $\|x\| \leqq |I|^{-1} \leqq 2^{m+1}$. Now

$$\|a_\theta 1_{D(m,k)} - \tfrac{1}{2} x 1_{D(m-1,k/2)}\|_{dH} \leqq 6.$$

If $1 \leqq s \leqq r-1$,

$$\left\|\frac{x}{2^s} 1_{D(m-s,2^{-s}k)} - \frac{x}{2^{s+1}} 1_{D(m-s-1,2^{-(s+1)}k)}\right\|_{dH} \leqq 2$$

with similar equations on the other side of $2^{-m}k$. Also

$$\left\|\frac{x}{2^r}1_{D(m-r,2^{-r}k)}-\frac{x}{2^r}1_{D(m-r,2^{-r}k+1)}\right\|_{dH}\leqq 4.$$

Thus

$$\|a_\theta\|_{dH}\leqq 2(6+2(r-1))+4=4r+12.$$

Now

$$\int_0^1 r(\theta)\,d\theta=2^{-m}\sum_{k=1}^{2m}(4r_k+12)=\sum_{r=0}^{m}2^{-(r+1)}(4r+12)\leqq 16.$$

Now if $f\in H_1^a(X)$ we can write $f=\sum c_n a_n$ where each a_n is an atom and $\sum|c_n|\leqq 2\|f\|_{aH}$. Then

$$\|a_{n,\theta}\|_{dH}\leqq \varrho_n(\theta)$$

where $\int \varrho_n(\theta)\,d\theta\leqq 16$. Thus there exists θ such that

$$\sum_{n=1}^{\infty}|c_n|\varrho_n(\theta)\leqq 32\|f\|_{aH}$$

and so

$$\|f\|_{dH}\leqq 32\|f\|_{aH}.$$

Completion of proof. (i)$\Rightarrow$(iii): It suffices to show that if $f\in H_1^d(X)$ then (iii) holds. We suppose $f=\sum c_n a_n$ where each a_n is a dyadic atom and $c_n\geqq 0$ with $\sum c_n<\infty$. We may further suppose that each a_n is supported on a distinct dyadic interval I_n and that the sequence $\{|I_n|\}$ is nonincreasing.

Let $\sigma=\sum c_n$. Define also $\sigma_0=0$ and then $\sigma_n=c_1+\ldots+c_n$ for $n\in\mathbf{N}$. We define a piecewise linear map $F\colon[0,\sigma]\to L_1(X)$ by

$$F(0)=0,$$
$$F(\sigma_n)=\sum_{k=1}^{n}c_k a_k \quad n\in\mathbf{N},$$
$$F(\sigma)=f$$

and such that F is linear on each interval $[\sigma_{n-1},\sigma_n]$. Similarly define a piecewise linear map $G\colon[0,\sigma]\to L_1$ by

$$G(0)=0,$$
$$G(\sigma_n)=\sum_{k=1}^{n}c_k|I_k|^{-1}1_{I_k}\quad n\in\mathbf{N},$$
$$G(\sigma)=\sum_{k=1}^{\infty}c_k|I_k|^{-1}1_{I_k},$$

and G is linear on each $[\sigma_{n-1},\sigma_n]$. By construction, if $\sigma_{n-1}<\tau<\sigma_n$ we have $F'(\tau)=a_n$ and $G'(\tau)=|I_n|^{-1}1_{I_n}$. For convenience we write $F(\tau)(t)=F(\tau,t)$ and $G(\tau)(t)=G(\tau,t)$ and we similarly define $F'(\tau,t)$, $G'(\tau,t)$. Notice that for all $0\leqq\tau\leqq\sigma$, $0\leqq t<1$, we have $\|F(\tau,t)\|\leqq G(\tau,t)$.

For any $\varphi\in\mathscr{C}$ we define

$$\alpha(\tau)=\int_0^1\langle F(\tau,t),\ \varphi(\log G(\tau,t))\rangle\,dt,$$

where the integrand vanishes whenever $F(\tau,t)=0$. Then α is continuous on $[0,\sigma]$, $\alpha(0)=0$ and if $\sigma_{n-1}<\tau<\sigma_n$,

$$\alpha'(\tau)=\int_{I_n}\langle a_n(t),\ \varphi(\log G(\tau,t))\rangle\,dt+\int_{I_n}\langle F(\tau,t),\ \varphi'(\log G(\tau,t))\rangle\frac{G'(\tau,t)}{G(\tau,t)}\,dt.$$

Note here that $G(\tau,t)$ cannot vanish on I_n.

Now $G(\tau,t)$ is constant, as a function of t, on I_n since for $k\leqq n$, either $I_n\subset I_k$ or $I_n\cap I_k=\emptyset$. Thus the first integral in the above formula vanishes and we have

$$|\alpha'(\tau)|\leqq\int_{I_n}\|F(\tau,t)\|\,\frac{G'(\tau,t)}{G(\tau,t)}\,dt\leqq\int_{I_n}G'(\tau,t)\,dt=1.$$

We conclude that $|\alpha(\sigma)|\leqq\sigma$. Now

$$\alpha(\sigma)-\Omega_\varphi(f)=\int_0^1\langle f(t),\ \varphi(\log G(\sigma,t))-\varphi(\log\|f(t)\|)\rangle\,dt$$

so that

$$|\alpha(\sigma)-\Omega_\varphi(f)|\leqq\int_0^1\|f(t)\|\log\frac{G(\sigma,t)}{\|f(t)\|}\,dt\leqq\frac{1}{e}\int_0^1G(\sigma,t)\,dt\leqq\frac{1}{e}\sigma$$

again using $x\log\frac{1}{x}\leqq\frac{1}{e}$ for $0<x\leqq 1$. Thus

$$|\Omega_\varphi(f)|\leqq\left(1+\frac{1}{e}\right)\sigma.$$

Now by Lemma 1 we obtain

$$\int_0^\infty\|M_2(t)\|\frac{dt}{t}\leqq\left(1+\frac{1}{e}\right)\sigma<\infty$$

and the theorem is proved.

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