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Compact and strictly singular operators on certain function spaces

By

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1. Introduction. This paper improves and completes results proved about Orlicz function spaces in [1]. It was shown in [1], for example, that if ϕ is an Orlicz function satisfying the Δ_2-condition then for any non-zero operator T: $L_\phi \to Y$, either T factors through the containing Banach space of L_ϕ or there is a Hilbertian subspace H of L_ϕ so that $T|H$ is an isomorphism; if L_ϕ has trivial dual, the first alternative is impossible. Other results were obtained on the existence of non-zero compact operators. Part of the motivation of this paper is to replace Orlicz function spaces by general symmetric function spaces (e.g. Lorentz spaces); such an extension was obtained in the trivial dual case for compact operators in [3], by a very simple argument. For convenience of exposition we only consider the locally bounded case, i.e. quasi-Banach spaces.

As we shown in Section 3, the methods of [1] can be adapted to give a very general theorem concerning operators on spaces $L_p(X)$ where $0 < p \leqq \infty$ and X is an arbitrary quasi-Banach space. We apply this result in two ways.

In Section 4 we deduce the non-existence of "averaging projections" on $L_p(X)$ for a wide class of space X. We conjecture that if X is a non-locally convex quasi-Banach space then for $p < \infty$ there cannot be a projection of $L_p(X)$ onto its subspace of constants. This is related to the problem of whether $L_p (0 < p < 1)$ is prime.

In Section 5 we apply our results to symmetric function spaces. If X is a separable symmetric function space with trivial dual and $X \supset L_p$ for some $p < \infty$ then any non-zero operator T: $X \to Y$ preserves a copy of l_2, as for Orlicz spaces. If X has non-trivial dual the statement of the theorem must be modified somewhat and the containing Banach space of X does not in general play the same role.

In [1] it is shown that an Orlicz function space with a basis is locally convex. We conclude by establishing a necessary and sufficient condition for a separable symmetric function space to have a basis. We show in fact that if X has a basis (or even embeds in a space with a basis) then the Haar system in a basis. We show that X can be non-locally convex and have a basis; in fact the Lorentz spaces $L(p, q)$ where $p > 1$ and $q < 1$ are examples. We also show that the spaces $L(1, q)$ for $q < 1$ do not have a basis.

2. Preliminaries. We recall that a quasi-Banach space X is a complete metrizable topological vector space whose topology may be given by a quasi-norm, i.e. a map

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$x \to \|x\|$ $(X \mapsto \mathbb{R})$ so that

(i) $\|x\| > 0$ $\quad x \neq 0$
(ii) $\|\alpha x\| = |\alpha| \, \|x\|$ $\quad \alpha \in \mathbb{R}, \; x \in X$
(iii) $\|x + y\| \leqq C(\|x\| + \|y\|)$ $\quad x, y \in X$

where C is independent of x and y. We shall always suppose that the quasi-norm is lower-semi-continuous (or that $\{x\colon \|x\| \leqq 1\}$ is closed). X is called an r-Banach space $(0 < r \leqq 1)$ if in addition we have

(iv) $\|x + y\|^r \leqq \|x\|^r + \|y\|^r$ $\quad x, y \in X.$

Every quasi-Banach can be equivalently re-normed as an r-Banach space for some $r \leqq 1$.

On any quasi-Banach space X we define $\| \;\|_c$ to be the greatest semi-norm so that

$$\|x\|_c \leqq \|x\| \qquad x \in X.$$

Alternatively $\|x\|_c \leqq 1$ if and only if x lies in the closed convex hull of the unit ball of X. The containing Banach space $\hat{X}$ of X is the Banach space obtained by completing the Hausdorff quotient of $(X, \| \;\|_c)$.

For $0 < p \leqq \infty$ we define $L_p(X)$ to be the space of all Borel measurable, separably valued, functions $f\colon [0,1] \to X$ so that

$$\|f\|_p = \left\{ \int_0^1 \|f(t)\|^p \, dt \right\}^{1/p} < \infty$$

(for $p < \infty$) or

$$\|f\|_\infty = \text{ess. sup} \, \|f(t)\| < \infty.$$

If $\phi \in L_p$ and $x \in X$ we write $\phi \otimes x$ for the function $f(s) = \phi(s)\, x$.

We denote Lebesgue measure on $(0,1)$ by λ. For $f \in L_0(0,1)$ we define its *decreasing rearrangement* f^* by

$$f^*(t) = \inf_{\lambda(A) = t} \; \sup_{s \in (0,1) \setminus A} |f(s)|.$$

A *symmetric function space* X is a quasi-Banach space of measurable functions on $(0,1)$ (where functions equal almost everywhere are identified) so that

(i) If $f^* \leqq g^*$ and $g \in X$ then $f \in X$ and $\|f\| \leqq \|g\|$.
(ii) If $0 \leqq f_n \leqq 1$ and $f_n \to 0$ a.e. then $\|f_n\| \to 0$.

If X is symmetric function space then $X([0,1]^2)$ denote the space of all $f \in L_0([0,1]^2)$ so that $f^* \in X$ where f^* is defined in the obvious way.

We define, for $0 < s < \infty$ the dilation operators $D_s\colon X \to X$ by

$$\begin{aligned} D_s f(t) &= f(t s^{-1}) \qquad & 0 < t < \min(1, s) \\ &= 0 & s \leqq t < 1 \end{aligned}$$

and define the Boyd indices of X by

$$p_X = \lim_{s \to \infty} \frac{\log s}{\log \|D_s\|}$$

$$q_X = \lim_{s \to 0} \frac{\log s}{\log \|D_s\|}$$

(see [6]).

We also introduce for $f \in L_1$ the function

$$f^{**}(t) = \frac{1}{t} \int_0^t f^*(s)\, ds.$$

The dyadic intervals $D(n,k)$ denote the intervals $((k-1)\, 2^{-n}, k \cdot 2^{-n}) \subset [0,1]$.

3. Operators on $L_p(X)$ spaces. We shall need a lemma which is probably well-known. Essentially the same lemma is proved in [1].

Lemma 3.1. *Let $\phi\colon \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the growth condition*

$$|\phi(x)| \leqq A + B\,|x|^p \qquad x \in \mathbb{R}$$

where $A, B > 0$. Let (Ω, P) be a probability measure space and suppose $\eta\colon \Omega \to \mathbb{R}$ is normally distributed with mean zero and variance one. Suppose further that every $\varepsilon > 0$ and $x \in \mathbb{R}$ we have

$$\mathscr{E}(\phi(x + \varepsilon\eta)) \geqq \phi(x).$$

Then ϕ is convex.

Proof. We need only show that ϕ is midpoint convex i.e. for $x, y \in \mathbb{R}$

$$\phi(x+y) + \phi(x-y) \geqq 2\phi(x).$$

Fix $y \in \mathbb{R}$, and define $\psi\colon \mathbb{R} \to \mathbb{R}$ by

$$\psi(x) = \int_0^y (y-t)(\phi(x+t) + \phi(x-t))\, dt.$$

From the hypotheses we deduce that

$$\mathscr{E}(\psi(x + \varepsilon\eta)) \geqq \psi(x) \qquad \varepsilon > 0,\ x \in \mathbb{R}.$$

However ψ is twice-differentiable and indeed

$$\psi''(x) = \phi(x+y) + \phi(x-y) - 2\phi(x).$$

Now from Taylor's theorem we have

$$\psi(x+t) + \psi(x-t) - 2\psi(x) = \tfrac{1}{2} t^2 (\psi''(x + \theta t) + \psi''(x - \theta t))$$

where $0 < \theta < 1$. Hence

$$|\psi(x+t) + \psi(x-t) - 2\psi(x)| \leqq 4[A + B(|x| + |y| + |t|)^p]\, t^2.$$

Thus if $0 < \varepsilon < 1$,

$$\varepsilon^{-2} |\psi(x + \varepsilon\eta) + \psi(x - \varepsilon\eta) - 2\psi(x)| \leqq 4[A + B(|x| + |y| + |\eta|)^p]\eta^2$$

and

$$\int_\Omega (A + B(|x| + |y| + |\eta|)^p)\eta^2 \, dP < \infty.$$

Hence by the Dominated Convergence Theorem of Lebesgue,

$$\begin{aligned}\psi''(x) &= \lim_{\varepsilon \to 0} \int \frac{\psi(x + \varepsilon\eta) + \psi(x - \varepsilon\eta) - 2\psi(x)}{\varepsilon^2} \, dP \\ &= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} [2(\mathscr{E}(\psi(x + \varepsilon\eta)) - \psi(x))] \\ &\geqq 0,\end{aligned}$$

i.e. ϕ is midpoint convex as required.

Theorem 3.2. *Let X be any quasi-Banach space and let Y be an r-Banach space where $r > 0$. Suppose $0 < p < \infty$ and $T\colon L_p(X) \to Y$ is a bounded linear operator. Then either*:

(i) *There is a subspace H of $L_p(X)$ isomorphic to l_2 so that $T|H$ is an isomorphism or*
(ii) $\| Tf \| \leqq \| T \| \, \{\int \| f(s) \|_c^p \, ds\}^{1/p} \quad f \in L_p(X)$.

Corollary 3.3. *If X has trivial dual and $T\colon L_p(X) \to Y$ is a non-zero bounded linear operator then there is a subspace $\mathscr{H}$ of $L_p(X)$ with $H \cong l_2$ so that $T|H$ is an isomorphism.*

Corollary 3.4. *If $1 \leqq p < \infty$ the containing Banach space of $L_p(X)$ can be naturally identified with $L_p(\hat{X})$.*

Corollaries 3.3 and 3.4 are automatic from Theorem 3.2, which we now prove.

Proof of Theorem 3.2. Clearly we may suppose $0 < r < p$. Now let Γ be the collection of all r-subadditive semi-quasi-norms γ on $L_p(X)$ so that $\gamma(f) \leqq \| f \|_p$ for $f \in L_p(X)$ and, whenever $H \subset L_p(X)$ is isomorphic to l_2 then

$$\inf_{\| f \|_p = 1, f \in H} \gamma(f) = 0.$$

The latter condition here is equivalent to insisting that the identity map $i\colon H \to (H, \gamma)$ is strictly singular for every infinite-dimensional Hilbertian subspace H of $L_p(X)$.

Now let

$$|\!|\!| f |\!|\!| = \sup_{\gamma \in \Gamma} \gamma(f).$$

Clearly $|\!|\!| \cdot |\!|\!|$ is an r-subadditive semi-quasi-norm on $L_p(X)$, and for the particular operator T in the statement of the theorem, if T fails condition (i) then

$$\| Tf \| \leqq \| T \| \, |\!|\!| f |\!|\!| \quad f \in L_p(X).$$

We now deduce two properties of $|||\cdot|||$. First note that if $E: L_p(X) \to L_p(X)$ is any non-zero endomorphism then if $\gamma \in \Gamma$ then $\|E\|^{-1}\gamma(Ef)$ is also in Γ. Hence

$$|||Ef||| \leqq \|E\| \, |||f||| \qquad f \in L_p(X).$$

For $x \in X$ let us set

$$|||x||| = |||1 \otimes x|||.$$

Using the above property twice we see that if B is a Borel subset of $[0, 1]$ of positive measure

$$|||1_B \otimes x||| = \lambda(B)^{1/p} \, |||1 \otimes x||| = \lambda(B)^{1/p} \, |||x|||.$$

The other property we shall need is that if $\gamma, \delta \in \Gamma$ then if A and B are disjoint Borel subsets of $[0, 1]$, $\beta \in \Gamma$ where

$$\beta(f) = (\gamma(1_A \cdot f)^p + \delta(1_B \cdot f)^p)^{1/p}.$$

Thus

$$|||f|||^p \geqq |||1_A \cdot f|||^p + |||1_B \cdot f|||^p.$$

Conversely using the case $\gamma = \delta$ we deduce

$$|||f|||^p = |||1_A \cdot f|||^p + |||1_B \cdot f|||^p.$$

Now, combining these two properties we see that if f is simple

$$|||f|||^p = \int_0^1 |||f(s)|||^p \, ds$$

and by continuity this extends to all $f \in L_p(X)$.

Let $B = \{x \in X: |||x||| \leqq 1\}$. We shall show that B is convex and hence it will follow that $|||x||| \leqq \|x\|_c$, since $|||x||| \leqq \|x\|$. Let $x, y \in B$ with $x \neq y$ and define $\phi: \mathbb{R} \to \mathbb{R}$ by

$$\phi(t) = |||x + t(y - x)|||^p \qquad -\infty < t < \infty.$$

Let $\{\eta_n: n \in \mathbb{N}\}$ be a sequence of independent random variables each with normal distribution, mean zero and variance one. Then since $y - x \neq 0$ the sequence $\eta_n \otimes (y - x)$ spans a subset of $L_p(X)$ isomorphic to H. Thus for any $\varepsilon > 0$, $v > 0$ and $\gamma \in \Gamma$ there exists $\eta \in L_p(0, 1)$ with distribution $N(0, 1)$ so that

$$\gamma(\varepsilon\eta \otimes (y - x)) \leqq v.$$

Hence

$$\gamma(1 \otimes (x + t(y - x)) + \varepsilon\eta \otimes (y - x))^r \geqq \gamma(1 \otimes (x + t(y - x)))^r - v^r.$$

We conclude that

$$\int_0^1 \phi(t + \varepsilon\eta(s)) \, ds \geqq (\phi(t)^{r/p} - v^r)^{p/r}.$$

As $v > 0$ is arbitrary we have

$$\int_0^1 \phi(t + \varepsilon\eta(s)) \, ds \geqq \phi(t),$$

Now by Lemma 3.1 ϕ is convex, for

$$|\phi(t)| \leqq (|||x|||^r + |t|^r\, |||y - x|||)^{p/r}.$$

In particular $\phi(t) \leqq 1$ for $0 \leqq t \leqq 1$ and the Theorem is proved.

We shall need an L_∞-version of the above theorem. This theorem can be compared with results in [1].

Theorem 3.5. *Let X be any quasi-Banach space and let Y be an r-Banach space where $r > 0$. Suppose $T\colon L_\infty(X) \to Y$ is a compact linear operator such that $\| Tf_n \| \to 0$ whenever f_n is a uniformly bounded sequence such that $\| f_n(s) \| \to 0$ a.e. Then*

$$\| Tf \| \leqq \| T \| \text{ ess. sup } \| f(s) \|_c \qquad f \in L_\infty(X).$$

Proof. The proof is very similar. This time let Γ be the collection of r-subadditive semi-quasi-norms γ on $L_\infty(X)$ so that $\gamma(f) \leqq \| f \|_\infty$, $\gamma(f_n) \to 0$ whenever $\| f_n \|_\infty \leqq 1$ and $\| f_n(s) \| \to 0$ a.e. and the identity map $I\colon L_\infty(X) \to (L_\infty(X), \gamma)$ is compact. Let

$$|||f||| = \sup_{\gamma \in \Gamma} \gamma(f).$$

Now by arguments analogous to the proof of Theorem 3.2 it can be shown that

$$|||f||| = \text{ess. sup } |||f(s)|||$$

where for $x \in X$,

$$|||x||| = |||1 \otimes x|||.$$

We conclude, as before, by showing that $B = \{x \in X\colon |||x||| \leqq 1\}$ is convex. Suppose $x, y \in B$ with $y \neq x$. Let $\{\sigma_n\colon n \in \mathbb{N}\}$ be a sequence of independent random variables with common distribution $\lambda(\sigma_n = 1) = \lambda(\sigma_n = -1) = 1/2$. Let $\gamma \in \Gamma$; then by passing to a subsequence we may suppose $\gamma(\sigma_n \otimes (y - x) - \sigma_{n+1} \otimes (y - x)) \leqq 2^{-n}$. It follows quickly that

$$\lim_{n \to \infty} \gamma(\sigma_n \otimes (y - x) - \frac{1}{n} \sum_{i=1}^{n} \sigma_i \otimes (y - x)) = 0.$$

However

$$\frac{1}{n} \sum_{i=1}^{n} \sigma_i \otimes (y - x) \to 0 \qquad \text{a.e.}$$

and hence

$$\lim_{n \to \infty} \gamma(\sigma_n \otimes (y - x)) = 0.$$

Thus

$$\gamma(1 \otimes \tfrac{1}{2}(x + y) + \sigma_n \otimes \tfrac{1}{2}(y - x)) \to \gamma(1 \otimes \tfrac{1}{2}(x + y))$$

so that since $|||1 \otimes \frac{1}{2}(x + y) + \sigma_n \otimes \frac{1}{2}(y - x)|||$ is independent of n

$$|||1 \otimes \tfrac{1}{2}(x + y) + \sigma_n \otimes \tfrac{1}{2}(y - x)||| \geqq |||\tfrac{1}{2}(x + y)|||$$

or

$$\max(|||x|||, |||y|||) \geqq \tfrac{1}{2} |||x + y|||.$$

Thus B is convex and the theorem is proved. Finally we shall also note that we can deduce a similar result from Theorem 3.4 if we assume that T extends to an operator on $L_p(X)$ for $p < \infty$.

Theorem 3.6. *Let X be a quasi-Banach space and let Y be an r-Banach space where $r > 0$. Suppose $T\colon L_p(X) \to Y$ is a bounded linear operator which is not an isomorphism on any subspace of $L_p(X)$ isomorphic to l_2. Then for $f \in L_\infty(X)$*

$$\| Tf \| \leqq \| T \|_\infty \text{ ess. sup } \| f(s) \|_c$$

where $\| T \|_\infty$ is the norm of the operator $T\colon L_\infty(X) \to Y$.

Proof. By Theorem 3.4 if $p \leqq q < \infty$, and $f \in L_q(X)$,

$$\| Tf \| \leqq \| T \|_q \{\int \| f(s) \|_c^q \, ds\}^{1/q}$$

where $\| T \|_q$ the norm of $T\colon L_q(X) \to Y$. Now if $f \in L_q(X)$, $\| f \|_q \leqq 1$ and $\varepsilon > 0$ we can write f as a disjoint sum,

$$f = g + h$$

where $\| g \|_\infty \leqq 1 + \varepsilon$ and either $\| h(s) \| \geqq 1 + \varepsilon$ or $\| h(s) \| = 0$. Thus

$$\| h(s) \|^p \leqq (1 + \varepsilon)^{p-q} \| h(s) \|^q$$

and

$$\| h \|_p \leqq (1 + \varepsilon)^{1 - q/p}.$$

Thus

$$\| Tf \|^r \leqq (1 + \varepsilon)^r \| T \|_\infty^r + (1 + \varepsilon)^{r - \frac{rq}{p}} \| T \|_p^r$$

so that

$$\| T \|_q \leqq ((1 + \varepsilon)^r \| T \|_\infty^r + (1 + \varepsilon)^{r - \frac{rq}{p}} \| T \|_\infty^r)^{1/r}.$$

Hence

$$\limsup_{q \to \infty} \| T \|_q \leqq (1 + \varepsilon) \| T \|_\infty$$

and then $\lim\limits_{q \to \infty} \| T \|_q = \| T \|_\infty$.

Thus if $f \in L_\infty(X)$

$$\| Tf \| \leqq \| T \|_\infty \text{ ess. sup } \| f(s) \|_c.$$

4. Averaging projections. X can be naturally embedded in $L_p(X)$ as the space of constant functions. We shall say that there is an averaging projection on $L_p(X)$ if there exists a projection of $L_p(X)$ onto X. Of course if X is a Banach space and $p \geqq 1$ there is an averaging projection given by

$$Pf = \int_0^1 f(s) \, ds.$$

Note also that the existence of an averaging projection on $L_p(X)$ implies the existence of an averaging projection on $L_q(X)$ where $p < q \leqq \infty$.

Theorem 4.1. *Let X be a quasi-Banach space and suppose $0 < p < \infty$. Suppose there is an averaging projection on $L_p(X)$ and that either*

(a) *X embeds into a space with a basis or*
(b) *X contains no copy of l_2.*

Then X is locally convex, i.e. a Banach space.

Proof. (a) By [2] Theorem 2 if X is not locally convex there is a non-zero compact operator $C: X \to Z$ so that $C^{-1}(0)$ is weakly dense in X. Let P be a projection of $L_p(X)$ onto X. Then $CP: L_p(X) \to Z$ is a compact operator and hence

$$\| CPf \| \leqq \| CP \| \, \{\int \| f(s) \|_c^p \, ds\}^{1/p}$$

for $f \in L_p(X)$. For $f = 1 \otimes x$ we obtain

$$\| Cx \| \leqq \| CP \| \, \| x \|_c$$

so that $C^{-1}(0)$ is also weakly closed, contrary to our assumptions. Thus X is locally convex.

(b) Here we simply argue by Theorem 3.2 that

$$\| Pf \|_p \leqq \| P \| \, \{\int \| f(s) \|_c^p \, ds\}^{1/p}$$

so that for $f = 1 \otimes x$ we have

$$\| x \| \leqq \| P \| \, \| x \|_c$$

i.e. X is locally convex.

Conjecture. If there is an averaging projection on $L_p(X)$, where $0 < p < \infty$, then $1 \leqq p < \infty$ and X is locally convex.

Remarks. (1) This is related to the question whether $L_p (0 < p < 1)$ is prime. In [4] it is shown that if L_p is not prime there is a complemented subspace Z of L_p, such that every complemented subspace of L_p is isomorphic either to Z or to L_p. It can also be shown that $L_p(Z)$ admits an averaging projection. However it can be shown that $L_q(L_p)$ does not admit an averaging projection if $p \leqq q < \infty$.

(2) If we replace $[0, 1]$ by an arbitrary measure space then the conjecture holds. Indeed in Theorem 3.2, if we replace $[0, 1]^\Gamma$ for some uncountable set Γ then in condition (i) we can change l_2 to $l_2(\Gamma)$. Hence for any fixed space X we can choose $\Gamma > \operatorname{card} X$ and then the existence of an averaging projection on $L_p([0, 1]^\Gamma; X)$ implies that X is locally convex.

In a similar spirit we add the following result.

Theorem 4.2. *Suppose X is a separable quasi-Banach space and $1 \leqq p < \infty$. Suppose $L_p(X)$ embeds into a quasi-Banach space Y with a basis. Then X is locally convex.*

Proof. We may suppose Y is an r-Banach space. Then there exist finite-rank operators $A_n: L_p(X) \to Y$ so that $\| A_n \| \leqq C (n \in \mathbb{N})$ and

$$\| f \| \leqq \sup_n \| A_n f \| \qquad f \in L_p(X).$$

For $x \in X$

$$\|1 \otimes x\|_p \leqq \sup_n \|A_n(1 \otimes x)\|$$

$$\leqq \sup_n \|A_n\| \, \|x\|_c$$

by Theorem 3.2. Thus

$$\|x\| \leqq C \, \|x\|_c$$

i.e. X is locally convex.

5. Applications to function spaces. Let $h \in L_\infty$ with $h \geqq 0$. We shall let $\Pi(h) = \{g \in L_0 \colon g^* \leqq h^*\}$ and $\Sigma(h) = \{g \in L_0 \colon g^{**} \leqq h^{**}\}$.

Lemma 5.1. *$\Pi(h)$ is closed in L_∞ and $\Sigma(h)$ is the closed convex hull of $\Pi(h)$.*

Proof. This is essentially known (cf. [6], pp. 124–125). $\Pi(h)$ and $\Sigma(h)$ are clearly closed sets. Let $F \in L_\infty^*$. Then $F = F_1 + F_2$ where

$$F_1(f) = \int_0^1 fg \, dt$$

for some $g \in L_1$ and F_2 is such that given $\varepsilon > 0$ there is a Borel set B of measure $1 - \varepsilon$ such that

$$|F_2(f)| \leqq \varepsilon \, \|f\|_\infty$$

whenever $\operatorname{supp} f \subset B$. It is now easy to verify that

$$\sup_{f \in \Pi(h)} |F(f)| = \int_0^1 h^* g^* \, dt + \|F_2\| \, \|h\|_\infty .$$

Now suppose $f_0^{**} \leqq h^{**}$. Then for all $t \in [0, 1]$

$$\int_0^t f_0^*(s) \, ds \leqq \int_0^t h^*(s) \, ds$$

and hence for every monotone decreasing function u on $[0, 1]$

$$\int_0^1 u(s) \, f_0^*(s) \, ds \leqq \int_0^1 u(s) \, h^*(s) \, ds.$$

In particular

$$\int_0^1 f_0^* g^* \, dt \leqq \int_0^1 h^* g^* \, dt.$$

Thus

$$|F(f_0)| \leqq \sup_{f \in \Pi(h)} |F(f)|$$

and by the Hahn-Banach theorem $f_0 \in \overline{co} \; \Pi(h)$ i.e. $\Sigma(h) \subset \overline{co} \; \Pi(h)$. However $\Sigma(h) \supset \Pi(h)$ and is closed and convex.

Theorem 5.2. *Let Y be an r-Banach space and suppose $T: L_\infty[0,1] \to Y$ is a compact operator such that whenever f_n is uniformly bounded and $f_n(s) \to 0$ a.e. then $\| Tf_n \| \to 0$. Suppose $h \in L_\infty$ and $h \geqq 0$. Then*

$$\sup_{f \in \Sigma(h)} \| Tf \| = \sup_{f \in \Pi(h)} \| Tf \|.$$

Before proving this result we state its companion for L_p where $p < \infty$.

Theorem 5.3. *Let Y be an r-Banach space and suppose $T: L_p[0,1] \to Y$ is a bounded linear operator with the property that whenever $H \subset L_p$ is a subspace isomorphic to l_2 then $T|H$ fails to be a isomorphism. Let $h \in L_\infty$ with $h \geqq 0$. Then*

$$\sup_{f \in \Sigma(h)} \| Tf \| = \sup_{f \in \Pi(h)} \| Tf \|.$$

Proofs of 5.2 and 5.3. Suppose $f \in \Sigma(h)$ is a simple function. We shall show in either case that

$$\| Tf \| \leqq \sup_{g \in \Pi(h)} \| Tg \|$$

and the theorems will follow by a density argument. Since f is simple we can find a measure preserving Borel map $\sigma: [0,1] \to [0,1]^2$ so that if $J_\sigma \phi(s) = \phi(\sigma(s))$ then $J_\sigma(1 \otimes f) = f$. Here of course $(1 \otimes f)(s,t) = f(t)$ for $0 \leqq s, t \leqq 1$. Now consider the map $T_0: L_\infty(L_\infty) \to Y$ given by

$$T_0 \phi = T(\phi \circ \sigma) \qquad \phi \in L_\infty(L_\infty).$$

We identify here $\phi \in L_\infty(L_\infty)$ with a corresponding $\phi \in L_\infty[0,1]^2$ in the normal way. Now the inclusion $L_\infty(L_\infty) \hookrightarrow L_\infty[0,1]^2$ is not surjective; however $T_0(1 \otimes f) = Tf$.

For $\varepsilon > 0$ we let $\Pi(h + \varepsilon)$ the unit ball of a quasi-norm $||| \cdot |||$ on L_∞ which is lower-semi-continuous and equivalent to the usual norm. Since $f \in \Sigma(h)$, $||| f |||_c \leqq 1$ for this quasi-norm.

In the case of 5.2, we can apply Theorem 3.5 to T_0 to deduce that for $\phi \in L_\infty(L_\infty)$

$$\| T_0 \phi \| \leqq ||| T_0 ||| \text{ ess. sup } ||| \phi(s) |||_c.$$

Now $||| T_0 ||| \leqq \sup_{g \in \Pi(h+\varepsilon)} \| Tg \|$. Thus letting $\phi = 1 \otimes f$,

$$\| Tf \| \leqq \sup_{g \in \Pi(h+\varepsilon)} \| Tg \|.$$

Letting $\varepsilon \to 0$, we quickly obtain the result.

In the case of 5.3 we note that T_0 extends continuously to $L_p(L_\infty) \subset L_p[0,1]^2$ and apply the same argument, using instead Theorem 3.6.

Now let X be a separable symmetric function space. If X^* is non-trivial then $X \subset L_1$. For $f \in X$ we shall define

$$\| f \|_d = \inf\{\| g \| : g^{**} \geqq f^{**}\}.$$

$\|\cdot\|_d$ is a quasi-norm on X. In general

$$\|f\|_d \geqq \|f\|_c.$$

We say X has *property* (*d*) if $\|\ \|_d$ is equivalent to $\|\cdot\|$ i.e. for some constant C

$$\|f\| \leqq C\|g\|$$

whenever $f^{**} \leqq g^{**}$.

Theorem 5.4. *Let X be a separable symmetric function space and let Y be an r-Banach space where $r > 0$. Let T: $X \to Y$ be an operator carrying the unit ball of L_∞ into a compact set. Then*

(i) *If $X^* = \{0\}$, $T = 0$.*
(ii) *If X^* is non-trivial then for $f \in X$*

$$\|Tf\| \leqq \|T\|\,\|f\|_d.$$

Theorem 5.5. *Let X be a separable symmetric function space containing L_p for some $p < \infty$, and let Y be an r-Banach space where $r > 0$. Then T: $X \to Y$ be an operator such that for every subspace $\mathscr{H}$ of X isomorphic to l_2, $T|H$ fails to be an isomorphism. Then*

(i) *If $X^* = \{0\}$, $T = 0$.*
(ii) *If X^* is non-trivial then for $f \in X$*

$$\|Tf\| \leqq \|T\|\,\|f\|_d.$$

Proofs of 5.4 and 5.5. These results follow from 5.2 and 5.3. For example in Theorem 5.5 we deduce that $T|L_p$ fails to be an isomorphism on any Hilbertian subspace of L_p and hence if $f \in L_p$ is simple

$$\|Tf\| \leqq \inf_{g^{**} \geq f^{**}} \sup_{h \in \Pi(g)} \|Th\|.$$

If $X^* = \{0\}$, then there exist simple $g_n \geqq 0$ so that $\|g_n\| \to 0$ but $\|g_n\|_1 = 1$. Hence if $\|f\|_\infty \leqq 1$, $\|Tf\| \leqq \|T\|\,\|g_n\|$ for all n, i.e. $Tf = 0$. Otherwise we obtain

$$\|Tf\| \leqq \|T\|\,\|f\|_d.$$

Theorem 5.6. *Let X be a separable symmetric function space. The following are equivalent.*

(i) *X can be embedded into a quasi-Banach space with a basis.*
(ii) *The Haar system is a basis of X.*
(iii) *X has property* (*d*).

Proof. (i) $\Rightarrow$ (iii). If X can be embedded in a space with a basis there exist finite-rank operators A_n: $X \to Y$ (where Y is an r-Banach space) so that $\sup \|A_n\| = C < \infty$ and

$$\|f\| \leqq \sup_n \|A_n f\| \qquad f \in X.$$

Thus since each A_n is finite-rank, by Theorem 5.4,

$$\sup_n \|A_n f\| \leqq C \|f\|_d$$

and so X has property (d).

(iii) $\Rightarrow$ (ii). Let h_n be the Haar system normalized in L_2 and define P_n: $L_2 \to L_2$ by

$$P_n f = \sum_{i=1}^{n} (h_i, f)\, h_i.$$

Then for $f \in L_2$

$$(P_n f)^{**} \leqq f^{**}.$$

Hence if $f \in L_2 \cap X$

$$\|P_n f\| \leqq C \|f\|$$

so that the operators P_n extend to an equicontinuous family P_n: $X \to X$. By a density argument it follows that (h_n) is a basis of X.

(ii) $\Rightarrow$ (i). Trivial.

To illustrate this result we prove two further results.

Lemma 5.7. *Suppose X has property (d) and $X \neq L_1$. Then*

$$\lim s^{-1} \|1_{[0,s]}\| = \infty.$$

Proof. Suppose $h \in L_\infty$ with $\|h\|_1 \leqq 1$ and $|h| \leqq M$. Then $h^{**} \leqq (M\, 1_{[0, M^{-1}]})^{**}$. Hence if

$$\liminf_{s \to 0} s^{-1} \|1_{[0,s]}\| = B < \infty$$

then $\|h\| \leqq CB$, whenever $\|h\|_1 \leqq 1$ i.e. $X = L_1$.

Theorem 5.8. *Suppose X is a separable symmetric function space for which $p_X > 1$. Then X has property (d).*

Proof. In this case $\|D_s\| \leqq c s^{1/p} (1 \leqq s < \infty)$ where $c > 0$ and $p > 1$. Now for $f \in X$

$$f^{**}(t) \leqq \sum_{k=1}^{\infty} 2^{-k} f^*(t/2^k).$$

Suppose X is r-normable i.e. for some $\gamma \geqq 1$,

$$\|f_1 + \ldots + f_n\| \leqq \gamma (\|f_1\|^r + \ldots + \|f_n\|^r)^{1/r}$$

for any $f_1, \ldots, f_n \in X$.

Then let $g_k(t) = f^*(t/2^k)$ for $k \geqq 1$. We have

$$\|g_k\| \leqq \|D_{2^k}\| \, \|f\| \leqq c\, 2^{k/p} \|f\|.$$

Hence

$$\left\|\sum_{k=1}^{n} 2^{-k} g_k\right\| \leqq c\gamma \|f\| \left\{\sum_{k=1}^{n} 2^{kr\left(\frac{1}{p}-1\right)}\right\}^{1/r}.$$

As $\frac{1}{p} - 1 < 0$ we see that $\Sigma 2^{-k} g_k$ converges in X and so $f^{**} \in X$ with

$$\|f^{**}\| \leqq \beta \|f\|$$

where β is independent of f.

Hence if $g^{**} \leqq f^{**}$ we have

$$\|g\| \leqq \|g^{**}\| \leqq \|f^{**}\| \leqq \beta \|f\|$$

i.e. X has property (d).

Examples. We consider the Lorentz spaces $L(p,q)$ where $0 < p, q < \infty$. Here $f \in L(p,q)$ if and only if

$$\|f\|_{p,q} = \left\{\int_0^1 t^{q/p-1} f^*(t)^q \, dt\right\}^{1/q} < \infty.$$

It is well-known that $L(p,q)$ has non-trivial dual if either $p > 1$ or $p = 1$ and $q \leqq 1$. $L(p,q)$ is locally convex if either $p > 1$ and $q \geqq 1$ or $p = q = 1$. We shall see that $L(p,q)$ has a basis (equivalently has property (d)) if either $p > 1$ or $p = q = 1$.

In fact since $p_X = p$ for $L(p,q)$ if $p > 1$ then X has property (d) by Theorem 5.8; if $p = q = 1$ then $L(p,q) = L_1$ has a basis. Conversely if $L(p,q)$ has a basis then either $p > 1$ or $p = 1$ and $q \leqq 1$ since $L(p,q)$ must have non-trivial dual. Suppose $p = 1$ and $q < 1$. Then

$$\|1_{[0,s]}\| = \theta s$$

where $\theta = q^{-(1/q)}$. By Lemma 5.7 $L_{(p,q)}$ does not then have a basis.

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