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GREEDY BASES IN BANACH SPACES

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In connection with the greedy algorithms, S. Konyagin and V. Temlyakov [KT] introduced the notions of greedy and quasi-greedy bases (see also the survey of V. Temlyakov on nonlinear methods of approximation [T]).

Let \((e_n)\) be a basis of a Banach space \(X\) and \((e_n^*)\) be the biorthogonal sequence in \(X^*\). For any \(x \in X\) denote

\[
\sigma_m(x) = \inf \left\{ \left\| x - \sum_{j \in A} \alpha_j e_j \right\| : |A| = m, \quad \alpha_j \in \mathbb{R} \right\}.
\]

Also, for any \(x\) define the greedy ordering for \(x\) as the map \(\rho : \mathbb{N} \to \mathbb{N}\) such that \(\rho(\mathbb{N}) \supset \{j : e^*_j(x) \neq 0\}\) and so that if \(j < k\) then either \(|e^*_{\rho(j)}(x)| > |e^*_{\rho(k)}(x)|\) or \(|e^*_{\rho(j)}(x)| = |e^*_{\rho(k)}(x)|\) and \(\rho(j) < \rho(k)\). The \(m\)-th greedy approximation is given by

\[
G_m(x) = \sum_{j=1}^{m} e^*_{\rho(j)}(x)e_{\rho(j)}.
\]

The basis is called greedy if there is a constant \(C\) such that for every \(x \in X\) and every integer \(m\),

\[
\|x - G_m(x)\| \leq C\sigma_m(x).
\]
The basis is called quasi-greedy if $G_m(x) \to x$ for all $x \in X$. P. Wojtaszczyk [W] proved that a basis is quasi-greedy if and only if there is a constant $C$ so that for any $x \in X$,

$$
\sup_m \|G_m(x)\| \leq C\|x\|.
$$

In the present note we shall survey further development of the study of greedy-type bases in the recent papers [DKKT], [DKK], [DKW].

Konyagin and Temlyakov obtained a nice characterization of greedy bases as unconditional bases with the additional property of being democratic, i.e. there is a constant $C$ such that for any finite subsets of integers $A$, $B$ with $|A| \leq |B|$, we have

$$
\left\| \sum_{j \in A} e_j \right\| \leq C \left\| \sum_{j \in B} e_j \right\|.
$$

A similar condition to democracy was defined for Banach lattices in [K].

In [DKKT] we introduce two natural intermediate properties of a basis between greedy and quasi-greedy. We say that $(e_n)$ is almost greedy if for some constant $C$,

$$
\|x - G_n(x)\| \leq C \inf \left\{ \left\| x - \sum_{j \in A} e_j^*(x)e_j \right\| : |A| = n \right\}, \ x \in X, n \in \mathbb{N}.
$$

The basis is partially greedy if

$$
\|x - G_n(x)\| \leq C \left\| \sum_{k=n+1}^{\infty} e_k^*(x)e_k \right\| \ x \in X, n \in \mathbb{N}.
$$

**Theorem** [DKKT]. Let $(e_n)$ be a basis of a Banach space. The following are equivalent:

1. $(e_n)$ is almost greedy.
2. $(e_n)$ is quasi-greedy and democratic.
3. For any (respectively, every) $\lambda > 1$ there is a constant $C = C_\lambda$ such that for all $x \in X$ and $m \in \mathbb{N}$, $\|x - G_{[\lambda m]}(x)\| \leq C_\lambda \sigma_m(x)$. 

Similarly to the democratic condition, we say that a basis is conservative, if for some constant $C$, \[ \left\| \sum_{j \in A} e_j \right\| \leq C \left\| \sum_{j \in B} e_j \right\| \] whenever $|A| \leq |B|$ and $m < n$ for any $m \in A, n \in B$.

**Theorem [DKKT].** A basis is partially greedy if and only if it is quasi-greedy and conservative.

We also study the duality of greedy-type properties of basic sequences.

**Theorem [DKKT].** Let $(e_n)$ be a greedy basis of a Banach space $X$ with non-trivial (Rademacher) type. Then $(e_n^*)$ is a greedy basis of $X^*$.

We give an example of a Tsirelson-like space $X$ which does not have type and $X$ has a greedy basis $(e_n)$ but $(e_n^*)$ is not a greedy basic sequence in $X^*$.

Wojtaszczyk studied in [W] the existence of quasi-greedy basis in Banach spaces. Note that every unconditional basis is quasi-greedy. It is known that the space $L_1(0, 1)$ does not have an unconditional basis (see e.g. [LT]). Generalizing a result from [W], we have in [DKK] the following corollary.

**Theorem [DKK].** The space $L_1(0, 1)$ has a quasi-greedy basis.

In the theory of Banach spaces it is known a complete description of the spaces with unique (up to equivalence) unconditional basis: up to isomorphism these are the spaces $\ell_1, \ell_2$ and $c_0$ (see [LT]). Examples of conditional quasi-greedy bases were constructed in $\ell_2$ [W] and $\ell_1$ [DM].

**Theorem [DKK].** The only Banach space (up to isomorphism) with unique (up to equivalence) quasi-greedy basis, is $c_0$. 


GREEDY BASES IN BANACH SPACES

It is known that the Haar basis is not quasi-greedy in $L_1(0,1)$. In [DKW] we study subsequences of the Haar system.

Theorem [DKW]. There exists an increasing sequence of integers $(n_j)$ such that the lacunary Haar system $\left( (h_j^{n_j})_{j=1}^{2^n} \right)_{i=0}^{\infty}$ in $L_1(0,1)$ is a quasi-greedy basis for its linear span.

REFERENCES


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