APPROXIMATION THEORY:

A Volume Dedicated to Blagovest Sendov

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DARBA SOFIA

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Sofia, BULGARIA

Printed in Bulgaria

ISBN 954-90126-5-4

GREEDY BASES IN BANACH SPACES

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In connection with the greedy algorithms, S. Konyagin and V. Temlyakov [KT] introduced the notions of greedy and quasi-greedy bases (see also the survey of V. Temlyakov on nonlinear methods of approximation [T]).

Let (e_n) be a basis of a Banach space X and (e_n^*) be the biorthogonal sequence in X^*. For any $x \in X$ denote

$$\sigma_m(x) = \inf \left\{ \left\| x - \sum_{j \in A} \alpha_j e_j \right\| : |A| = m, \quad \alpha_j \in \mathbb{R} \right\}.$$

Also, for any x define the greedy ordering for x as the map $\rho : \mathbb{N} \to \mathbb{N}$ such that $\rho(\mathbb{N}) \supset \{j : e_j^*(x) \neq 0\}$ and so that if $j < k$ then either $|e^*_{\rho(j)}(x)| > |e^*_{\rho(k)}(x)|$ or $|e^*_{\rho(j)}(x)| = |e^*_{\rho(k)}(x)|$ and $\rho(j) < \rho(k)$. The m-th greedy approximation is given by

$$G_m(x) = \sum_{j=1}^{m} e^*_{\rho(j)}(x) e_{\rho(j)}.$$

The basis is called greedy if there is a constant C such that for every $x \in X$ and every integer m,

$$\|x - G_m(x)\| \leq C\sigma_m(x).$$

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

The basis is called quasi-greedy if $G_m(x) \to x$ for all $x \in X$. P. Wojtaszczyk [W] proved that a basis is quasi-greedy if and only if there is a constant C so that for any $x \in X$,

$$\sup_m \|G_m(x)\| \leq C\|x\|.$$

In the present note we shall survey further development of the study of greedy-type bases in the recent papers [DKKT], [DKK], [DKW].

Konyagin and Temlyakov obtained a nice characterization of greedy bases as unconditional bases with the additional property of being democratic, i.e. there is a constant C such that for any finite subsets of integers A, B with $|A| \leq |B|$, we have $\left\|\sum_{j\in A} e_j\right\| \leq C\left\|\sum_{j\in B} e_j\right\|$. A similar condition to democracy was defined for Banach lattices in [K].

In [DKKT] we introduce two natural intermediate properties of a basis between greedy and quasi-greedy. We say that (e_n) is almost greedy if for some constant C,

$$\|x - G_n(x)\| \leq C \inf\left\{\left\|x - \sum_{j\in A} e_j^*(x)e_j\right\| : |A| = n\right\}, \quad x \in X, n \in \mathbb{N}.$$

The basis is partially greedy if

$$\|x - G_n(x)\| \leq C\left\|\sum_{k=n+1}^{\infty} e_k^*(x)e_k\right\| \quad x \in X, n \in \mathbb{N}.$$

Theorem [DKKT]. *Let (e_n) be a basis of a Banach space. The following are equivalent:*

(1) *(e_n) is almost greedy.*

(2) *(e_n) is quasi-greedy and democratic.*

(3) *For any (respectively, every) $\lambda > 1$ there is a constant $C = C_\lambda$ such that for all $x \in X$ and $m \in \mathbb{N}$, $\|x - G_{[\lambda m]}(x)\| \leq C_\lambda \sigma_m(x)$.*

Similarly to the democratic condition, we say that a basis is conservative, if for some constant C, $\left\|\sum_{j\in A} e_j\right\| \le C\left\|\sum_{j\in B} e_j\right\|$ whenever $|A| \le |B|$ and $m < n$ for any $m \in A$, $n \in B$.

Theorem [DKKT]. *A basis is partially greedy if and only if it is quasi-greedy and conservative.*

We also study the duality of greedy-type properties of basic sequences.

Theorem [DKKT]. *Let (e_n) be a greedy basis of a Banach space X with non-trivial (Rademacher) type. Then (e_n^*) is a greedy basis of X^*.*

We give an example of a Tsirelson-like space X which does not have type and X has a greedy basis (e_n) but (e_n^*) is not a greedy basic sequence in X^*.

Wojtaszczyk studied in [W] the existence of quasi-greedy basis in Banach spaces. Note that every unconditional basis is quasi-greedy. It is known that the space $L_1(0,1)$ does not have an unconditional basis (see e.g. [LT]). Generalizing a result from [W], we have in [DKK] the following corollary.

Theorem [DKK]. *The space $L_1(0,1)$ has a quasi-greedy basis.*

In the theory of Banach spaces it is known a complete description of the spaces with unique (up to equivalence) unconditional basis: up to isomorphism these are the spaces ℓ_1, ℓ_2 and c_0 (see [LT]). Examples of conditional quasi-greedy bases were constructed in ℓ_2 [W] and ℓ_1 [DM].

Theorem [DKK]. *The only Banach space (up to isomorphism) with unique (up to equivalence) quasi-greedy basis, is c_0.*

It is known that the Haar basis is not quasi-greedy in $L_1(0,1)$. In [DKW] we study subsequences of the Haar system.

Theorem [DKW]. *There exists an increasing sequence of integers (n_j) such that the lacunary Haar system $((h_j^{n_i})_{j=1}^{2^{n_i}})_{i=0}^{\infty}$ in $L_1(0,1)$ is a quasi-greedy basis for its linear span.*

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