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Intersection bodies and L_p-spaces

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Abstract

We prove that the convex intersection bodies are isomorphically equivalent to unit balls of subspaces of L_q for each $q \in (0,1)$. This is done by extending to negative values of p the factorization theorem of Maurey and Nikishin which states that for any $0 < p < q < 1$ every Banach subspace of L_p is isomorphic to a subspace of L_q.

MSC: 52A21

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1. Introduction

The concept of an intersection body was introduced by Lutwak [Lu] as part of his dual Brunn–Minkowski theory. Let K and L be origin symmetric star bodies in $\mathbb{R}^n$. We say that K is the *intersection body of* L if the radius of K in every direction is equal to the volume of the central hyperplane section of L perpendicular to this direction, i.e. for every $\xi \in S^{n-1}$,

$$\|\xi\|_K^{-1} = vol_{n-1}(L \cap \xi^\perp),$$

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where $\|x\|_K = \min\{a \geqslant 0 : \ x \in aK\}$, $\xi^\perp = \{x \in \mathbb{R}^n : \ (x,\xi) = 0\}$, and vol_{n-1} is the $(n-1)$-dimensional volume. A more general class of *intersection bodies* can be defined as the closure in the radial metric of the class of intersection bodies of star bodies.

Intersection bodies play an important role in the solution of the Busemann–Petty problem posed in [BP] in 1956: suppose that K and L are origin symmetric convex bodies in $\mathbb{R}^n$ so that, for every $\xi \in S^{n-1}$,

$$vol_{n-1}(K \cap \xi^\perp) \leqslant vol_{n-1}(L \cap \xi^\perp).$$

Does it follow that $vol_n(K) \leqslant vol_n(L)$? The problem was completely solved in 1997, and the answer is affirmative if $n \leqslant 4$ and negative if $n \geqslant 5$. The solution has appeared as a result of work of many mathematicians (see [GKS] or [Z] for details). A connection between intersection bodies and the Busemann–Petty problem was established by Lutwak [Lu]: if K is an intersection body then the answer to the Busemann–Petty problem is affirmative for any star body L. On the other hand, if L is a symmetric convex body that is not an intersection body then one can construct K giving together with L a counterexample.

A more general concept of a k-intersection body was introduced in [Ko8,Ko6]. For an integer k, $1 \leqslant k < n$ and star bodies D, L in $\mathbb{R}^n$, we say that D is the k-intersection body of L if for every $(n-k)$-dimensional subspace H of $\mathbb{R}^n$,

$$vol_k(D \cap H^\perp) = vol_{n-k}(L \cap H).$$

Taking the closure in the radial metric of the class of all D's that appear as k-intersection bodies of star bodies, we define the class of *k-intersection bodies*. If $k = 1$ one gets the usual intersection bodies. The class of k-intersection bodies is related to a certain generalization of the Busemann–Petty problem in the same way as intersection bodies are related to the original problem (see [Ko6] for details; this generalization offers a condition on the volume of sections that allows to compare the volumes of two bodies in arbitrary dimensions).

The concept of embedding of finite-dimensional normed spaces in L_p with $p < 0$ was introduced in [Ko5] in relation to some probabilistic problems, as an analytic extension of embedding of normed spaces into L_p with $p > 0$. It is a well-known fact going back to Levy (see for example [BL, p. 189]) that an n-dimensional normed space $(\mathbb{R}^n, \|\cdot\|)$ embeds in L_p, $p > -1$ if and only if there exists a finite Borel measure μ on the sphere S^{n-1} in $\mathbb{R}^n$ so that for every $x \in \mathbb{R}^n$,

$$\|x\|^p = \int_{S^{n-1}} |(x,\xi)|^p \, d\mu(\xi). \tag{1}$$

If we want to extend the latter equality to $p \leqslant -1$ we have to regularize the divergent integral in the right-hand side. The standard way of doing it is by using distributions. Applying both sides of the latter equality to a test function ϕ and using elementary connections between the Fourier and Radon transforms (see [Ko5]), we arrive at the following:

Definition 1. Let X be an n-dimensional normed space, and $-n < p < 0$. We say that X embeds in L_p if there exists a finite Borel measure μ on S^{n-1} so that, for every even Schwartz test function $\phi \in \mathcal{S}(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} \|x\|_X^p \phi(x)\, dx = \int_{S^{n-1}} d\mu(\theta) \int_{\mathbb{R}} |t|^{-p-1} \hat{\phi}(t\theta)\, dt.$$

Expressing the volume of hyperplane sections of a star body in the form

$$vol_{n-1}(L \cap \xi^{\perp}) = \lim_{\varepsilon \to 0} \varepsilon \int_L |(x, \xi)|^{-1+\varepsilon}\, d\xi$$

and considering the definition of intersection bodies and the equality (1), one can suggest that intersection bodies are the unit balls of spaces that embed in L_{-1}. This is indeed true and is a part of a more general connection between intersection bodies and embedding in L_p established in [Ko8]:

Theorem 1.1. *Let $1 \leqslant k < n$. The following are equivalent*:

(i) *An origin symmetric star body D in $\mathbb{R}^n$ is a k-intersection body*;
(ii) *$\|\cdot\|_D^{-k}$ represents a positive definite distribution*;
(iii) *The space $(\mathbb{R}^n, \|\cdot\|_D)$ embeds in L_{-k}.*

The advantage of this connection (and, consequently, of introducing embeddings in negative L_p) is that now one can try to extend to negative values of p different results about usual L_p-spaces. Every such extension gives new information about intersection bodies. Let us give several examples of this approach.

A well-known simple fact is that every two-dimensional normed space embeds in L_1. How does this fact extend to embeddings in L_{-k}? It was proved in [Ko7, Theorem 2] that for every symmetric convex body K in $\mathbb{R}^n$ and every $p \in [n-3, n)$, the function $\|\cdot\|_K^{-p}$ represents a positive definite distribution, so by Theorem 1.1 every n-dimensional Banach space embeds in L_{-n+3}. Putting $n = 2$ we get the property of two-dimensional spaces mentioned above. Putting $n = 4$ we see that every four-dimensional normed space embeds in L_{-1}. By Theorem 1.1, every four-dimensional symmetric convex body is a 1-intersection body, which, by Lutwak's connection, solves in affirmative the critical four-dimensional case of the Busemann–Petty problem.

Another well-known property of L_p-spaces is that, for any $0 < p < q \leqslant 2$, the space L_q embeds isometrically in L_p, so L_p-spaces become larger when p decreases from 2, see for example [BL, p. 189]. This result was extended to negative p in [Ko5, Th 2]: every finite-dimensional subspace of L_q, $0 < q \leqslant 2$ embeds in L_{-p} for every $p \in (0, n)$. Hence, the unit ball of every n-dimensional subspace of L_q, $0 < q \leqslant 2$ is a k-intersection body for every $k = 1, \ldots, n$. This gives plenty of examples of intersection bodies, and in particular, proves that every polar projection body is an intersection body (this was first proved in [Ko3, Theorem 3]).

On the negative side, the solution to Schoenberg's problem (see [Ko1]) shows that for $q > 2$ and $n \geqslant 3$, the spaces l_q^n do not embed in L_p with $0 < p \leqslant 2$. This result was extended to negative p in [Ko4, Lemma 9]: the spaces l_q^n with $q > 2$ embed in L_{-p} only for $p \in [n-3, n)$. Putting $n = 5$ we see that l_q^5 with $q > 2$ does not embed in L_{-1}, so the unit balls of these spaces are not 1-intersection bodies, which provides counterexamples to the Busemann–Petty problem in the critical dimension 5.

All these examples are isometric. In this paper, we give an isomorphic example of this approach by extending to negative values of p the factorization theorem of Maurey and Nikishin [Ma,N,W, p. 264]. This theorem implies that, for $0 < p < q < 1$, every Banach subspace of L_p is isomorphic to a subspace of L_q (see also [Kl] for related results). We prove that, for any $-\infty < p < q < 1,\ p \neq 0,\ q > 0$, every n-dimensional Banach subspace of L_p, $-n < p$, is isomorphic to a subspace of L_q with the Banach–Mazur distance depending only on p and q (see Theorem 4.4 below). In terms of intersection bodies this translates as follows:

Theorem 1.2. *For any $k \in \mathbb{N}$ and $0 < q < 1$, there exists a constant $c(k, q)$ depending on k and q only so that for every $n \in \mathbb{N}$ and every symmetric convex k-intersection body D in $\mathbb{R}^n$, $n > k$ there exists an n-dimensional subspace of $L_q([0, 1])$, whose unit ball L satisfies $L \subset D \subset c(k, q)L$.*

The latter theorem, in conjunction with the fact that the unit ball of any subspace of L_q, $0 < q < 1$ is a k-intersection body for every k (see above), shows that symmetric convex k-intersection bodies are isomorphically equivalent to unit balls of subspaces of $L_q,\ 0 < q < 1$.

2. Moments of stable random variables

We start by observing the following elementary formula:

$$x^z \Gamma(-z) = \int_0^\infty t^{-z} e^{-xt} \frac{dt}{t}, \quad \Re z < 0. \tag{2}$$

Then if f is a non-negative random variable for $\Re z < 0$ we have

$$\mathbb{E}(f^z) = \frac{1}{\Gamma(-z)} \int_0^\infty t^{-z} \mathbb{E}(e^{-tf}) \frac{dt}{t}. \tag{3}$$

Now let us note that by analytic continuation, (2) implies

$$-\frac{1}{z} - x^z \Gamma(z) = \int_0^\infty t^{-z} (\chi_{[0,1]}(t) - e^{-xt}) \frac{dt}{t}$$

whenever $\Re z < 1$ and this implies that if $0 < \Re z < 1$,

$$-x^z\Gamma(-z) = \int_0^\infty t^{-z}(1-e^{-xt})\frac{dt}{t}. \tag{4}$$

These observations lead easily to the following comparison principle:

Lemma 2.1. *If f, g are two positive random variables with $\mathbb{E}(e^{-tf}) \leqslant \mathbb{E}(e^{-tg})$ for $0 \leqslant t < \infty$ then $(\mathbb{E}f^p)^{\frac{1}{p}} \geqslant (\mathbb{E}g^p)^{\frac{1}{p}}$ whenever $-\infty < p \leqslant 1$.*

We shall say that a random-variable γ is normalized Gaussian if for $t \in \mathbb{R}$, $\mathbb{E}(e^{it\gamma}) = e^{-\frac{t^2}{2}}$. We shall say that η is symmetric normalized p-stable where $0 < p < 2$ if $\mathbb{E}(e^{it\eta}) = e^{-|t|^p}$. We shall say that a positive random variable α is normalized positive p-stable for $0 < p < 1$ if $\mathbb{E}(e^{-t\alpha}) = e^{-t^p}$ when $t > 0$.

We shall need some elementary computations of moments for such variables. It is easy to see that if γ is normalized Gaussian

$$\mathbb{E}(|\gamma|^z) = \frac{1}{\sqrt{\pi}}2^{\frac{z}{2}}\Gamma\left(\frac{z+1}{2}\right), \quad \Re z > -1. \tag{5}$$

If α is normalized positive p-stable then (3) and analytic continuation can be used to show that

$$\mathbb{E}(\alpha^z) = \frac{\Gamma(\frac{-z}{p})}{p\Gamma(-z)}, \quad \Re z < p. \tag{6}$$

Next, if η is symmetric p-stable then we note that η has an identical distribution with $\sqrt{2\alpha}\gamma$ where γ, α are independent, γ is normalized Gaussian and α is normalized positive $p/2$-stable. Hence

$$\mathbb{E}(|\eta|^z) = \frac{2^{z+1}\Gamma(\frac{-z}{p})\Gamma(\frac{z+1}{2})}{p\sqrt{\pi}\Gamma(-\frac{z}{2})}, \quad -1 < \Re z < p. \tag{7}$$

In the special case $p = 1$, this can be simplified either by using properties of the Gamma function or by direct calculation

$$\mathbb{E}(|\eta|^z) = \frac{1}{\pi}\int_{-\infty}^\infty \frac{|x|^z}{1+x^2}\,dx = \sec\left(\frac{z\pi}{2}\right), \quad -1 < \Re z < 1. \tag{8}$$

If $0 < p < 1$ and η is normalized symmetric p-stable then η has the same distribution as $\alpha\zeta$ where α, ζ are independent, α is normalized positive p-stable and ζ is normalized symmetric 1-stable. Hence, we can rewrite (7) in the form

$$\mathbb{E}(|\eta|^z) = \mathbb{E}(\alpha^z)\mathbb{E}(|\zeta|^z) = \sec\left(\frac{z\pi}{2}\right)\frac{\Gamma(\frac{-z}{p})}{p\Gamma(-z)}, \quad -1 < \Re z < p. \tag{9}$$

Our next Lemma will be useful later:

Lemma 2.2. *Suppose $-\infty < p < q < 1$ with $p \neq 0$ and $q > 0$. Suppose $(\eta_k)_{k=1}^\infty$ is a sequence of i.i.d. normalized symmetric q-stable random variables. Then for any $a_1, \ldots, a_n \in \mathbb{R}$,*

$$\left(\mathbb{E}\left(\sum_{k=1}^n |a_k||\eta_k|\right)^p\right)^{\frac{1}{p}} \leqslant \left(\sec\left(\frac{q\pi}{2}\right)\right)^{\frac{1}{q}}\left(\frac{\Gamma(\frac{-p}{q})}{q\Gamma(-p)}\right)^{\frac{1}{p}}\left(\sum_{k=1}^n |a_k|^q\right)^{\frac{1}{q}}.$$

Proof. Let $f = \sum_{k=1}^n |a_k||\eta_k|$. Then, for $t \geqslant 0$,

$$\mathbb{E}(e^{-tf}) = \prod_{k=1}^n \mathbb{E}(e^{-t|a_k||\eta_k|}).$$

Now

$$\mathbb{E}(e^{-t|a_k||\eta_k|}) = \mathbb{E}_\eta\mathbb{E}_\zeta(e^{ita_k\eta_k\zeta}),$$

where ζ is a normalized 1-stable random variable independent of η_k. Hence

$$\mathbb{E}(e^{-t|a_k||\eta_k|}) = \mathbb{E}_\zeta(e^{-t^q|a_k|^q|\zeta|^q}) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{e^{-t^q|a_k|^q|x|^q}}{1+x^2}\,dx.$$

Now, we use convexity of the function e^{-x} and the fact that $\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{dx}{1+x^2} = 1$ to get

$$\geqslant \exp\left\{-\frac{|t|^q|a_k|^q}{\pi}\int_{-\infty}^{\infty}\frac{|x|^q}{1+x^2}\,dx\right\} = e^{-\sec(\frac{q\pi}{2})t^q|a_k|^q}.$$

We thus have

$$\mathbb{E}(e^{-tf}) \geqslant e^{-\sec(\frac{q\pi}{2})t^q\sum_{k=1}^n|a_k|^q} = \mathbb{E}e^{-(\sec(\frac{q\pi}{2}))^{\frac{1}{q}}(\sum_{k=1}^n|a_k|^q)^{\frac{1}{q}}t\alpha},$$

where α is a normalized positive q-stable random variable. By Lemma 2.1

$$(\mathbb{E}(f^p))^{\frac{1}{p}} \leqslant \left(\sec\left(\frac{q\pi}{2}\right)\right)^{\frac{1}{q}} \left(\sum_{k=1}^{n} |a_k|^q\right)^{\frac{1}{q}} (\mathbb{E}(\alpha^p))^{\frac{1}{p}}.$$

This implies the conclusion by (6). □

3. Spaces embedding into L_p for $-\infty < p < 0$

Let Ω be a Polish space. We denote by $\mathcal{M}(\Omega)$ the space of all real-valued Borel functions on Ω. A probability measure on Ω is a positive Borel measure of total mass one.

Let μ be any probability measure on Ω. For any $f \in \mathcal{M}(\Omega)$ and any $-\infty < p < \infty$ with $p \neq 0$ we define

$$\|f\|_{p,d\mu} = \left(\int |f|^p \, d\mu\right)^{\frac{1}{p}}.$$

Notice that $0 \leqslant \|f\|_{p,d\mu} < \infty$ for all f and if $p < 0$ then $\|f\|_{p,d\mu} = 0$ does not imply $f = 0$ a.e.

Before proceeding, we will need the following lemma. It is an elementary extension to all p of a well-known property of the standard L_p-norms and we omit the proof.

Lemma 3.1. *Suppose $-\infty < p < q < \infty$. Then for any measurable functions $f_1, \ldots, f_n$ we have*

$$\left\| \left(\sum_{k=1}^{n} |f_k|^p\right)^{\frac{1}{p}} \right\|_q \leqslant \left(\sum_{k=1}^{n} \|f_k\|_q^p\right)^{\frac{1}{p}}$$

and

$$\left\| \left(\sum_{k=1}^{n} |f_k|^q\right)^{\frac{1}{q}} \right\|_{p,d\mu} \geqslant \left(\sum_{k=1}^{n} \|f_k\|_{p,d\mu}^q\right)^{\frac{1}{q}}.$$

Now let Ω be a Polish space and μ some probability measure on Ω. If $p > -1$, a linear map $T : X \to \mathcal{M}(\Omega)$ is a *c-isometric embedding* of X into $L_p(\mu)$ if

$$\|Tx\|_{p,d\mu} = c\|x\|, \quad x \in X. \tag{10}$$

If $c = 1$ we say T is an isometric embedding.

Let $X = (\mathbb{R}^n, \|\cdot\|_X)$ be an n-dimensional normed space and e_k the standard basis in $\mathbb{R}^n$. We construct an embedding $U : X \to \mathcal{M}(S^{n-1}, \mu)$ as follows: for every $x = \sum_{k=1}^n x_i e_i$ put $Ux(u) = (x, u),\ \ u \in S^{n-1}$.

Lemma 3.2. *Suppose that $-n < p < 0$ and $X = (\mathbb{R}^n, \|\cdot\|_X)$ is an n-dimensional normed space embedding in L_p with the corresponding measure μ. If $\eta_1, \ldots, \eta_m$ are independent symmetric q-stable random variables, $0 < q \leqslant 2$ then, for any $f_1, \ldots, f_m \in X$ with the dimension of their linear span not smaller than n,*

$$\mathbb{E}\left\|\sum_{k=1}^m \eta_k f_k\right\|_X^p = c(p,q)\int_{S^{n-1}}\left(\sum_{k=1}^m |Uf_k(u)|^q\right)^{\frac{p}{q}} d\mu(u),$$

where $c(p,q)$ is a positive constant depending only on p and q.

Proof. First consider the case $q = 2$. Then the distribution of $\sum_{k=1}^m \eta_k f_k$ is a Gaussian measure in $\mathbb{R}^n$ with density $\phi \in \mathcal{S}(\mathbb{R}^n)$, whose characteristic function

$$\hat{\phi}(u) = \exp\left(-\frac{1}{2}\sum_{k=1}^m (f_k, u)^2\right).$$

By definition of embedding in L_p (Definition 1 in the Introduction),

$$\begin{aligned}
\mathbb{E}\left\|\sum_{k=1}^m \eta_k f_k\right\|_X^p &= \int_{\mathbb{R}^n} \|x\|_X^p \phi(x)\, dx \\
&= \int_{S^{n-1}} d\mu(u) \int_{\mathbb{R}} |t|^{-1-p} \exp\left(-\frac{t^2}{2}\sum_{k=1}^m (f_k, u)^2\right) dt \\
&= c(p)\int_{S^{n-1}}\left(\sum_{k=1}^m (f_k, u)^2\right)^{\frac{p}{2}} d\mu(u).
\end{aligned}$$

Now, for arbitrary q, use the fact that each symmetric q-stable random variable η_k has the same distribution as $\sqrt{2\alpha_k}\gamma_k$ where α_k, γ_k are independent, γ_k is a normalized Gaussian and α_k is a normalized positive $q/2$-stable random variable. Now we use the case $q = 2$ to see that

$$\mathbb{E}\left\|\sum_{k=1}^{m} \eta_k f_k\right\|_X^p = \mathbb{E}_\alpha \mathbb{E}_\gamma \left\|\sum_{k=1}^{m} \sqrt{2\alpha_k}\gamma_k f_k\right\|_X^p$$
$$= c(p)\mathbb{E}_\alpha \left(\int_{S^{n-1}} \left(\sum_{k=1}^{m} \alpha_k (f_k, u)^2\right)^{\frac{p}{2}} d\mu\right).$$

To finish the proof, note that $\sum_{k=1}^m \alpha_k (f_k, u)^2$ has the same distribution as $\alpha_1 \left(\sum_{k=1}^m |(f_k, u)|^q\right)^{\frac{2}{q}}$, so the latter expectation turns into $\mathbb{E}\alpha_1^p$ which was computed in (6). □

4. Change of density

Our next Lemma is a slight extension of the Komlos theorem ([Kom]).

Lemma 4.1. *Suppose $(f_n)_{n=1}^\infty$ is a sequence of non-negative extended-valued measurable functions. Then there is a sequence (g_n) such that $g_n \in \operatorname{co}\{f_k : k \geqslant n\}$ and such that (g_n) converges a.e. to an extended-valued measurable function g.*

Proof. Let us say a sequence (h'_n) is *subordinate* to a sequence (h_n) if $h'_n \in \operatorname{co}\{h_k : k \geqslant n\}$. Let $\varphi(t) = t(1+t)^{-1}$ with $\varphi(\infty) = 1$. Then for any sequence (h_n) of extended-valued non-negative functions we define

$$\Phi((h_n)) = \sup\left\{\int \liminf_{n\to\infty} \varphi \circ h'_n \, d\mu : \ (h'_n) \text{ is subordinate to } (h_n)\right\}.$$

Now inductively we may construct sequences $(f_n^k)_{n=1}^\infty$ for $k = 1, 2, \ldots$ so that $f_n^1 = f_n$ and for each k, (f_n^{k+1}) is subordinate to (f_n^k) and

$$\int \liminf_{n\to\infty} \varphi \circ f_n^{k+1} \, d\mu > \Phi((f_n^k)) - 2^{-k}.$$

If we then choose $h_n = f_n^n$ then (h_n) is subordinate to (f_n) and

$$\int \liminf_{n\to\infty} \varphi \circ h_n \, d\mu = \Phi((h_n)) := \delta.$$

Let $h = \liminf h_n$. We shall argue that

$$\lim_{n\to\infty} \int |\varphi \circ h_n - \varphi \circ h| \, d\mu = 0. \tag{11}$$

Indeed notice first that

$$\lim_{n\to\infty} \int \varphi \circ h - \varphi \circ \min(h, h_n) \, d\mu = 0. \tag{12}$$

Now suppose we can find a subsequence (u_n) so that

$$\int \varphi \circ \max(h, u_n) - \varphi \circ h \, d\mu > \delta > 0$$

for all n. By passing to a further subsequence and using the Komlos theorem we can suppose that $\frac{1}{n}(\varphi \circ u_1 + \cdots + \varphi \circ u_n)$ converges a.e. to some function $\varphi \circ u$ where u is extended-valued non-negative and measurable. Now by the Bounded Convergence Theorem we have

$$\int \varphi \circ \max(h, u) - \varphi \circ h \, d\mu \geqslant \delta.$$

On the other hand $\frac{1}{n}(u_1 + \cdots + u_n)$ is subordinate to (h_n) and so

$$\int \liminf_{n\to\infty} \varphi \circ \left(\frac{1}{n}(u_1 + \cdots + u_n) \right) d\mu = \int \varphi \circ h \, d\mu.$$

Since $\liminf_{n\to\infty} \frac{1}{n}(u_1 + \cdots + u_n) \geqslant h$ a.e. this implies that

$$\liminf_{n\to\infty} \frac{1}{n}(u_1 + \cdots + u_n) = h, \quad \mu\text{-a.e.}$$

By the concavity of φ this implies that

$$\liminf_{n\to\infty} \frac{1}{n} (\varphi \circ u_1 + \cdots + \varphi \circ u_n) \leqslant \varphi \circ h, \quad \mu\text{-a.e.}$$

i.e. $u \leqslant h$ a.e. which gives a contradiction and establishes

$$\lim_{n\to\infty} \int \varphi \circ \max(h, h_n) - \varphi \circ h \, d\mu = 0. \tag{13}$$

If we combine (12) and (13) we obtain (11).

To complete the argument we pass to further subsequence such that

$$\sum_{n=1}^{\infty} \int |\varphi \circ h_n - \varphi \circ h| \, d\mu < \infty.$$

Then it is clear that h_n converges a.e. to h. □

We can now give an extension to the negative p of the classical change of density result of Maurey (see [Ma,Pi1] or [W, p. 264]).

Theorem 4.2. *Suppose* $-\infty < p < q < \infty$ *with* $q > 0$, $p \neq 0$ *and suppose* $K \subset \mathcal{M}(\Omega)$. *Let*

$$\tilde{K}_q = \left\{ \left(\sum_{j=1}^{m} c_j |f_j|^q \right)^{\frac{1}{q}} : \sum_{j=1}^{m} c_j = 1, c_j \leqslant 1; f_1, \ldots, f_m \in K, m \in \mathbb{N} \right\}.$$

Suppose that there exists $h \in \tilde{K}_q$ *with* $\|h\|_{p,d\mu} > 0$. *Then in order that there exists* $v \in L_1(d\mu)$ *with* $v \geqslant 0$ *a.e. and* $\int v \, d\mu = 1$ *such that*

$$\|f v^{-\frac{1}{p}}\|_{q, v d\mu} \leqslant \sigma \quad \forall f \in K \tag{14}$$

it is necessary and sufficient that

$$\|g\|_{p,d\mu} \leqslant \sigma \quad \forall g \in \tilde{K}_q. \tag{15}$$

Proof. The case $0 < p < q < \infty$ is the standard Maurey theorem and we refer to [Ma] or [W, p. 264] for this. We shall therefore only consider the case when $p < 0$. We remark that it is easy to see that (15) is a necessary condition since if (14) holds then

$$\|g\|_{p,d\mu} = \|g v^{-\frac{1}{p}}\|_{p, v\, d\mu} \leqslant \|g v^{-\frac{1}{p}}\|_{q, v\, d\mu} \leqslant \sigma.$$

We therefore turn to the proof of sufficiency. We can assume that

$$\sigma = \sup\{\|g\|_{p,d\mu} : \ g \in \tilde{K}_q\}.$$

First we prove that if (g_n) is a sequence in $\tilde{K}_q$ such that $g_n \to g$ μ-a.e. where g is an extended-valued measurable function $g : \Omega \to [0, \infty)$ then $\limsup_{n \to \infty} \|g_n\|_{p,d\mu} \leqslant \|g\|_{p,d\mu} \leqslant \sigma$.

Notice that from Fatou's lemma

$$\|g\|_{p,d\mu} \geqslant \limsup_{n\to\infty} \|g_n\|_{p,d\mu},$$

and so we only need to show that $\|g\|_{p,d\mu} \leqslant \sigma$. Suppose $t > 0$ and let $g_n' = ((1-t)g_n^q + th^q)^{\frac{1}{q}}$. Then g_n' converges a.e. to $((1-t)g^q + th^q)^{\frac{1}{q}}$. Now if $p < 0$ it follows easily from the Dominated Convergence Theorem (note that $h^p \in L_1(d\mu)$ because $\|h\|_{p,d\mu} > 0$) that

$$\lim_{n\to\infty} \|g_n'\|_{p,d\mu} = \|((1-t)g^q + th^q)^{\frac{1}{q}}\|_{p,d\mu}. \tag{16}$$

By Lemma 3.1,

$$\|((1-t)g^q + th^q)^{\frac{1}{q}}\|_{p,d\mu} \geqslant ((1-t)\|g\|_{p,d\mu}^q + t\|h\|_{p,d\mu}^q)^{\frac{1}{q}}$$

and

$$((1-t)\|g\|_{p,d\mu}^q + t\|h\|_{p,d\mu}^q)^{\frac{1}{q}} \leqslant \sigma.$$

As $t > 0$ is arbitrary we get $\|g\|_{p,d\mu} \leqslant \sigma$.

Take any sequence (g_n) in $\tilde{K}_q$ such that $\lim_{n\to\infty} \|g_n\|_{p,d\mu} = \sigma$. We can use Lemma 4.1 to find a sequence g_n' so that $(g_n')^q \in \mathrm{co}\{g_k^q : k \geqslant n\}$ and g_n' converges a.e. to some g. Then by Lemma 3.1 we have that $\|g_n'\|_{p,d\mu} \to \sigma$ and the preceding argument shows that $\|g\|_{p,d\mu} = \sigma$.

Now if $f \in K$ we have that $((1-t)g^q + t|f|^q)^{\frac{1}{q}} \in \tilde{K}_q$ for any $t > 0$, so

$$\|((1-t)g^q + t|f|^q)^{\frac{1}{q}}\|_{p,d\mu} \leqslant \sigma = \|g\|_{p,d\mu}.$$

This implies

$$\int ((1-t)g^q + t|f|^q)^{\frac{p}{q}}\, d\mu \geqslant \int g^p\, d\mu.$$

Let $F_M = \min(|f|g^{-1}, M)$. Then

$$\int g^p((1 + t(F_M^q - 1)))^{\frac{p}{q}} - 1)\, d\mu \geqslant 0.$$

Next, we divide both sides of above inequality by t

$$\int g^p \frac{((1+t(F_M^q-1)))^{\frac{p}{q}}-1)}{t}\, d\mu \geqslant 0.$$

It follows from the Dominated Convergence Theorem, since $g^p \in L_1(d\mu)$, that (as $t \to 0$)

$$\int g^p (F_M^q - 1)\, d\mu \leqslant 0.$$

Now, by the Monotone Convergence Theorem (as $M \to \infty$)

$$\int g^{p-q}|f|^q\, d\mu \leqslant \int g^p\, d\mu = \sigma^p.$$

Let $v = \sigma^{-p} g^p$. Then

$$\|v^{-\frac{1}{p}} f\|_{q,v\,d\mu} \leqslant \sigma. \qquad \square$$

Theorem 4.3. *Suppose* $-1 < p < q < 1$ *with* $p \neq 0$ *and* $q > 0$. *Suppose* X *is a Banach subspace of* $L_p(\Omega, \mu)$. *Then there exists a function* $v \in L_1$ *with* $v \geqslant 0$ *a.e. and* $\int v\, d\mu = 1$ *such that for all* $f \in X$ *we have*

$$\|f\|_{p,d\mu} \leqslant \|f v^{-\frac{1}{p}}\|_{q,v\,d\mu} \leqslant \left(\sec\left(\frac{q\pi}{2}\right)\right)^{\frac{1}{q}} \left(\cos\left(\frac{p\pi}{2}\right)\right)^{\frac{1}{p}} \|f\|_{p,d\mu}. \tag{17}$$

In particular, there is a closed subspace Y *of* L_q *such that* $d(X, Y) \leqslant (\sec(\frac{q\pi}{2}))^{\frac{1}{q}} (\cos(\frac{p\pi}{2}))^{\frac{1}{p}}$.

Proof. We first note that it is enough to prove (17) for $f \in X$ with $\|f\|_{p,d\mu} = 1$. The lower bound in (17) follows from monotonicity of L_p-norms.

We use Theorem 4.2 to prove the upper bound. Let

$$K = \{f \in X : \|f\|_{p,d\mu} = 1\}.$$

Consider $g \in \tilde{K}_q$ such that $g = (\sum_{k=1}^n c_k |f_k|^q)^{\frac{1}{q}}$, $\sum_{k=1}^n c_k = 1$.

Let $(\eta_k)_{k=1}^n$ be i.i.d. normalized symmetric q-stable random variables. Then

$$\left(\mathbb{E}\left\|\sum_{k=1}^n \eta_k c_k^{1/q} f_k\right\|_{p,d\mu}^p\right)^{\frac{1}{p}} = \left(\int_\Omega \mathbb{E}\left|\sum_{k=1}^n \eta_k c_k^{1/q} f_k(\omega)\right|^p d\mu(\omega)\right)^{\frac{1}{p}}.$$

Next, we may use that $\sum_{k=1}^n \eta_k c_k^{1/q} f_k(\omega)$ has the same distribution as $(\sum_{k=1}^n c_k|f_k|^q)^{\frac{1}{q}}\eta_1$ and, by (9),

$$= (\mathbb{E}|\eta_1|^p)^{\frac{1}{p}} \left\|\left(\sum_{k=1}^n c_k|f_k|^q\right)^{\frac{1}{q}}\right\|_{p,d\mu}$$

$$= \left(\sec\left(\frac{p\pi}{2}\right)\frac{\Gamma(\frac{-p}{q})}{q\Gamma(-p)}\right)^{\frac{1}{p}} \left\|\left(\sum_{k=1}^n c_k|f_k|^q\right)^{\frac{1}{q}}\right\|_{p,d\mu}.$$

Hence,

$$\|g\|_{p,d\mu} = \left\|\left(\sum_{k=1}^n c_k|f_k|^q\right)^{\frac{1}{q}}\right\|_{p,d\mu}$$

$$= \left(\sec\left(\frac{p\pi}{2}\right)\frac{\Gamma(\frac{-p}{q})}{q\Gamma(-p)}\right)^{-\frac{1}{p}} \left(\mathbb{E}\left\|\sum_{k=1}^n \eta_k c_k^{1/q} f_k\right\|_{p,d\mu}^p\right)^{\frac{1}{p}}.$$

We remind that $f_1,\dots,f_n \in X$, where X is a Banach space, and so by the triangle inequality $\|\sum_{k=1}^n \eta_k c_k^{1/q} f_k\|_{p,d\mu} \leqslant \sum_{k=1}^n |\eta_k| c_k^{1/q}\|f_k\|_{p,d\mu} = \sum_{k=1}^n |\eta_k| c_k^{1/q}$. Finally,

$$\|g\|_{p,d\mu} \leqslant \left(\sec\left(\frac{p\pi}{2}\right)\frac{\Gamma(\frac{-p}{q})}{q\Gamma(-p)}\right)^{-\frac{1}{p}} \left(\mathbb{E}\left(\sum_{k=1}^n |\eta_k| c_k^{1/q}\right)^p\right)^{\frac{1}{p}} \leqslant \frac{(\sec(\frac{q\pi}{2}))^{\frac{1}{q}}}{(\sec(\frac{p\pi}{2}))^{\frac{1}{p}}}$$

by Lemma 2.2, and the fact that $\sum_{k=1}^n c_k = 1$. The result follows from Theorem 4.2. □

Remark. Note that this proof does not work directly in the case $p \leqslant -1$, because $\mathbb{E}|\eta_1|^p$ is no longer a finite number.

Theorem 4.4. *If $-\infty < p < q < 1$, $p \neq 0$ and $q > 0$ then there exists a constant $C = C(p,q)$ so that for any n with $-n < p$ if X is an n-dimensional normed space that embeds into L_p then there is a subspace Y of L_q so that $d(X,Y) \leqslant C$.*

Proof. Let $v = [-p]$. In view of Theorem 4.3, we consider only the case $p \leqslant -1$ and, correspondingly, $v \geqslant 1$.

Recall that the operator $U : X \to \mathcal{M}(S^{n-1}, d\mu)$ is defined by $Uf(u) = (f,u)$, $u \in S^{n-1}$. By Lemma 3.2, for any $f_1, \dots, f_m \in X$

$$\left(\mathbb{E}\left\|\sum_{k=1}^{n} \eta_k f_k\right\|_X^p\right)^{\frac{1}{p}} = \left\|\left(\sum_{k=1}^{n} |Uf_k|^q\right)^{\frac{1}{q}}\right\|_{p,d\mu}, \tag{18}$$

whenever $(\eta_k)_{k=1}^{\infty}$ is a sequence of normalized symmetric q-stable random variables. Hence from Lemma 2.2 we have

$$\left\|\left(\sum_{k=1}^{n} |Uf_k|^q\right)^{\frac{1}{q}}\right\|_{p,d\mu} \leqslant B\left(\sum_{k=1}^{n} \|f_k\|_X^q\right)^{\frac{1}{q}}, \tag{19}$$

where

$$B = B(p,q) = \left(\sec\left(\frac{q\pi}{2}\right)\right)^{\frac{1}{q}} \left(\frac{\Gamma(\frac{-p}{q})}{q\Gamma(-p)}\right)^{\frac{1}{p}}.$$

Let $K = \{Uf : f \in X, \ \|f\|_X = 1\}$. Using the inequality (19), one can immediately verify the conditions of Theorem 4.2. Therefore, using the same argument as in the proof of Theorem 4.3, we get that there exists $v \in L_1$ with $v \geqslant 0$ and $\int v \, d\mu = 1$ so that for all $f \in X$ we have

$$\|Ufv^{-\frac{1}{p}}\|_{q,v\,d\mu} \leqslant C\|f\|_X \tag{20}$$

where $C = C(p,q)$.

Unlike the case $p > -1$, the lower bound $\|fv^{-\frac{1}{p}}\|_{q,v\,d\mu} \geqslant c\|f\|_X$ does not follow from monotonicity of L_p norms, and we need to adjust the embedding to get the lower bound.

We first note that

$$\mathbb{E}\left(\max_{1 \leqslant k \leqslant v+1} |\eta_k|\right)^p = \mathbb{E} \min_{1 \leqslant k \leqslant v+1} |\eta_k|^p \approx c'(p,q,v) = c'(p,q)$$

and we define $c_0 = c(p,q) = (\mathbb{E}(\max_{1\leqslant k\leqslant v+1} |\eta_k|)^p)^{\frac{1}{p}}$. Consider some $c > c_0(v+1)^{-\frac{1}{q}}$. Let us prove that there is a subspace E of X of dimension at most v and a closed subspace F of X of codimension at most v so that if $f \in F$ then

$$\|Ufv^{-\frac{1}{p}}\|_{q,v\,d\mu} \geqslant c\, d(f,E), \tag{21}$$

where $d(f,E) := \inf\{\|f-e\|_X :\ e \in E\}$. To see this, suppose the contrary. Then, by induction we can pick $f_1,\dots,f_{v+1} \in X$ and $\phi_1,\dots,\phi_{v+1} \in X^*$ such that $\|f_k\|_X = \|\phi_k\|_{X^*} = 1$ for $k = 1,2,\dots,v+1$, $\phi_k(f_j) = 0$ if $j \neq k$ and $\phi_k(f_k) = 1$ so that $d(f_k,[f_1,\dots,f_{k-1}]) = 1$ but

$$\|Uf_kv^{-\frac{1}{p}}\|_{q,v\,d\mu} < c.$$

Then

$$\left\|\sum_{k=1}^{v+1} \eta_k f_k\right\|_X \geqslant \max_{1\leqslant i\leqslant v+1} \left|\phi_i\left(\sum_{k=1}^{v+1} \eta_k f_k\right)\right| = \max_{1\leqslant i\leqslant v+1} |\eta_k|$$

and

$$\left(\mathbb{E}\left\|\sum_{k=1}^{v+1} \eta_k f_k\right\|_X^p\right)^{\frac{1}{p}} \geqslant \left(\mathbb{E}\max_{1\leqslant i\leqslant v+1} \eta_i|^p\right)^{\frac{1}{p}} = c_0.$$

But we also have an upper bound

$$\begin{aligned}\left(\mathbb{E}\left\|\sum_{k=1}^{v+1} \eta_k f_k\right\|_X^p\right)^{\frac{1}{p}} &= \left\|\left(\sum_{k=1}^{v+1} |Uf_k|^q\right)^{\frac{1}{q}}\right\|_{p,d\mu} = \left\|\left(\sum_{k=1}^{v+1} |Uf_kv^{-1/p}|^q\right)^{\frac{1}{q}}\right\|_{p,v\,d\mu} \\ &\leqslant \left\|\left(\sum_{k=1}^{v+1} |Uf_kv^{-1/p}|^q\right)^{1/q}\right\|_{q,v\,d\mu} \\ &= \left(\sum_{k=1}^{v+1} \|Uf_kv^{-1/p}\|_{q,v\,d\mu}^q\right)^{1/q} \leqslant c(v+1)^{\frac{1}{q}},\end{aligned}$$

so $c_0 \leqslant c(v+1)^{\frac{1}{q}}$ which gives a contradiction.

Next, we use John's theorem saying that the Banach–Mazur distance from any m-dimensional normed space E to the m-dimensional Euclidean space is at most

$\sqrt{m}$. There exists a bounded linear operator $S_0 : E \to \ell_2^{\nu+1}$, $m = \dim E \leqslant \nu$ with $\|S_0 e\|_2 \geqslant \|e\|_X$ for $e \in E$ and $\pi_2(S_0) \leqslant \sqrt{\nu}$, where $\pi_2(S_0)$ denotes the 2-summing norm of the operator S_0 (see [Pi2, p. 8] for definition and properties). Moreover S_0 can be extended to a bounded operator $S : X \to \ell_2^m$ with $\|S\| \leqslant \pi_2(S) \leqslant \sqrt{\nu}$ [Pi2, Corollary 3.9].

There is also a linear operator $T : X \to \ell_2^m$ with $\|T\| \leqslant \sqrt{\nu}$ and $\|Tf\|_2 \geqslant d(f, F)$ for $f \in X$. In fact, again by John's theorem, there exists an operator $T' : X/F \to l_2^m$ with $\|T'\| \leqslant \sqrt{m}$ and $\|T'\| \geqslant 1$. Then we can put $T = T'Q$ where $Q : X \to X/F$ is the quotient map.

Consider the space $Y = (\ell_2^m \oplus \ell_2^m \oplus L_q(v^{-1}\, d\mu))_q$ which embeds isometrically into L_q. We define an operator $V : X \to Y$ by $Vf = (Sf, Tf, Jf)$ where $Jf = Ufv^{-\frac{1}{p}}$. Then $\|V\| \leqslant C_1 = C_1(p, q)$, and our goal is to find a lower bound. Assume $\|f\|_X = 1$; we show that $\|Vf\|_q \geqslant \delta$ where $\delta > 0$ is chosen so that

$$\left[1 + (\sqrt{\nu} + 1)((2C + 1)^{\frac{1}{q}} c^{-1} + 2)\right] < \delta^{-1},$$

when C is defined by inequality (20). We may pick $g \in F$ with $\|f - g\|_X < 2d(f, F)$. Then we can find $h \in E$ so that $\|g - h\|_X < 2d(g, E)$. Assume $\|Vf\|_q < \delta$. Then $\|Tf\|_2 < \delta$ (and $\|Sf\|_2 < \delta$) so that $d(f, F) < \delta$. Hence $\|g\|_X > 1 - 2\delta$. Now, from inequality (20), we get

$$\|Jg\|_{q,v\,d\mu}^q < \|Jf\|_{q,v\,d\mu}^q + \|Jf - Jg\|_{q,v\,d\mu}^q \leqslant \|Jf\|_{q,v\,d\mu}^q + 2C\delta^q < (2C + 1)\delta^q$$

and Eq. (21) gives $d(g, E) < (2C + 1)^{\frac{1}{q}} c^{-1}\delta$. Hence

$$\|f - h\|_X < ((2C + 1)^{\frac{1}{q}} c^{-1} + 2)\delta.$$

Thus,

$$\|Sh\|_2 \leqslant \|Sf\|_2 + \|S(f - h)\|_2 \leqslant \left[1 + \sqrt{\nu}((2C + 1)^{\frac{1}{q}} c^{-1} + 2)\right]\delta.$$

However,

$$\|Sh\|_2 \geqslant \|h\|_X \geqslant \|f\|_X - \|f - h\|_X \geqslant 1 - ((2C + 1)^{\frac{1}{q}} c^{-1} + 2)\delta.$$

This gives a contradiction and completes the proof. □

Theorem 1.2 now follows from Theorems 4.4 and 1.1.

Remarks. (i) If $k = 1$ and D is the 1-intersection body of a convex symmetric body one can choose L in Theorem 1.2 as an ellipsoid. This follows from a result of Hensley

[He] that there exist absolute constants c_1, c_2 so that every symmetric convex body D in $\mathbb{R}^n$ admits a linear transformation TD satisfying

$$c_1 \leqslant \frac{vol_{n-1}(TD \cap \xi^{\perp})}{vol_{n-1}(TD \cap \eta^{\perp})} \leqslant c_2$$

for any $\xi, \eta \in S^{n-1}$. However, the most interesting and useful examples of 1-intersection bodies are not the ones generated by symmetric convex bodies, but rather by star bodies or measures on the sphere (in the latter case they belong to the closure of intersection bodies of star bodies). For example, the cube in $\mathbb{R}^4$ and l_1^n-balls for all n are 1-intersection bodies generated by measures on the sphere with unbounded densities.

(ii) The constant $c(k, q)$ in Theorem 1.2 tends to infinity as $q \to 1$, so the method of this paper does not work in the case $q = 1$. Since the unit balls of subspaces of L_1 are polar projection bodies (see for example [Ga, p. 134]), the statement of Theorem 1.2 in the case $q = 1$ would mean, if true, that intersection bodies are isomorphically equivalent to polar projection bodies. This would represent a big step towards resolving one of the mysteries of convex geometry—the duality between sections and projections. The corresponding result in the theory of L_p-spaces is also open and is the matter of Kwapien's problem [Kw] posed in 1970: is every Banach subspace of L_p with $0 < p < 1$ isomorphic to a subspace of L_1? For partial results on Kwapien's problem see [Kl]. Note that the isometric version of Kwapien's problem has a negative answer, see [Ko2,KK]. This means, in particular, that not every intersection body is a polar projection body. The result of [KK] also implies that not every intersection body is the unit ball of a subspace of L_q (isometrically), for every $q \in (0, 1)$.

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