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A Unified Theory of Commutator Estimates for a Class of Interpolation Methods[1]

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PÜHENDATUD MEIE ÕPETAJALE, MEIE KOLLEEGILE JA VÄGA KALLILE SÕBRALE JAAK PEETRELE, KÕIGE SOOJEMATE ÕNNITLUSTEGA JA KÕIGE PAREMATE SOOVIDEGA TEMA 65. SÜNNIPÄEVA PUHUL.

DEDICATED TO OUR TEACHER, OUR COLLEAGUE, AND OUR VERY DEAR FRIEND, JAAK PEETRE, WITH WARMEST GREETINGS AND BEST WISHES, ON THE OCCASION OF HIS 65TH BIRTHDAY.

A general family of interpolation methods is introduced which includes, as special cases, the real and complex methods and also the so-called $\pm$ or G_1 and G_2 methods defined by Peetre and Gustavsson–Peetre. Derivation operators Ω and translation operators $\mathcal{R}$ are introduced for all methods of this family. A theorem is proved about

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the boundedness of the commutators $[T, \Omega]$ and $[T, \mathscr{R}]$ for operators T which are bounded on the spaces of the pair to which the interpolation method is applied. This extends and unifies results previously known for derivation and translation operators in the contexts of the real and complex methods. Other results deal with higher order commutators and also include an "equivalence theorem," i.e. it is shown that, as previously known only for real interpolation spaces, all these interpolation spaces have two different equivalent definitions in the style of the "J method" and "K method." Auxiliary results which may also be of independent interest include the equivalence of Lions–Schechter complex interpolation spaces defined using an annulus with the same spaces defined in the usual way, using a strip.

Key Words: derivation operator; commutator estimate; pseudolattice; real interpolation; complex interpolation; plus–minus interpolation method; Lions–Schechter interpolation space.

1. INTRODUCTION

The purpose of interpolation theory is to study the properties of interpolation functors and their applications in analysis. In particular the well-known real and complex methods of interpolation play a central role in the theory. However, the study of these concrete interpolation methods has traditionally emphasized their differences, which in turn has often led to the development of disjoint sets of techniques to attack essentially similar problems. While some of these differences are unavoidable, and indeed part of the richness of the subject, it seems to us that a good deal of the basic underlying theory behind the real and complex method can and should be given a unified treatment. Indeed this has already been done in the impressive work of Janson [29] where the real, complex and most other known interpolation functors are revealed to be special cases of either the minimal or the maximal functors of Aronszajn–Gagliardo. Another general construction of interpolation spaces, which includes the real and complex methods as special cases, has been presented by Williams [61].

The purpose of this paper is to introduce and study a different approach to a unified treatment of some of the basic theoretical properties of the real and complex methods of interpolation. This approach, initially suggested by some definitions and remarks of Jaak Peetre ([48, pp. 174–177]), is motivated here by the wish to better understand the so-called theory of "commutators," and we emphasize that theory here. We hope to treat other aspects of our general approach elsewhere.

To be more specific, in this paper we introduce new interpolation functors which provide a general method for constructing scales of interpolation spaces. We then define certain (possibly nonlinear) "derivation" mappings Ω and "translation" mappings $\mathscr{R}$ with respect to each such scale and prove that their commutators with bounded linear operators on the interpolation scale are bounded operators. Our general method includes the real and complex and other interpolation methods as special cases. Thus our results unify, simplify and generalize known commutator theorems for the real and complex methods in [32, 54], cf. also [27, 42] and also give new analogous results for other methods such as the $\pm$ methods of Peetre and Gustavsson–Peetre.

Ours is not the first paper to present commutator theorems in a general abstract setting. This has been done in a different way by Carro, Cerdà and Soria, using the framework of [61] as their point of departure. See e.g. [8–10], and, in particular, [11] for a summary and general survey of their work on this topic. Their initial paper [8] also includes answers to some questions raised in [20]. At various stages in this paper we shall make a number of comparisons between this material, particularly in [8, 9], and our own approach.

Our scale of interpolation spaces is indexed by a parameter $\theta \in (0,1)$ (or by e^{θ} or by $e^{\theta+i\tau}$). We are not aware of any way in which we could extend our approach, including the construction of the mappings Ω and $\mathcal{R}$, to the case of "function parameters," i.e. positive concave functions $\rho : (0,\infty) \to (0,\infty)$ which generalize the role of the numerical parameter θ (which corresponds to the function $\rho(t) = t^{\theta}$). By contrast, Janson's approach is well adapted to function parameters, but to date there is apparently no known way of obtaining a version of the theory of commutators in his setting. In this direction we mention that generalized commutator estimates for the real method have been considered in [2]. There are also a number of other approaches to interpolation which do not fit into the format we present here (or, at least, we do not see how they do). These include methods based on convexity and envelopes [16, 52, 57], based on differential equations and geometric considerations [56] (see also [31, Section 4]), and based on harmonic functions of several variables [31]. Yet another approach [49] extends the ideas in Marcel Riesz's original proof of the Riesz(-Thorin) theorem. Cf. also the method of quadratic means referred to in [49] for still other methods see [45]. We feel that the problem of bringing more unity to these diverse viewpoints is an interesting one.

In their study of H^p spaces on $\mathbb{R}^n$, Coifman *et al.* (cf. [15]) proved, among many other things, that if $b \in BMO(\mathbb{R}^n)$ and T is a Calderón–Zygmund operator, then the commutator defined by

$$[T,b](f) = T(bf) - bT(f)$$

is bounded on $L^p(\mathbb{R}^n)$ for $1<p<\infty$. Note that both of the operators $f \mapsto T(bf)$ and $f \mapsto bT(f)$ which appear in the definition of $[T,b]$ are not bounded on $L^p(\mathbb{R}^n)$. The remarkable feature here is the subtle cancellation that occurs when we subtract these two unbounded operators and make their difference $[T, b]$ bounded. From our point of view here, the second proof of this L^p boundedness, given on [15, p. 621], is particularly interesting. It can be seen as the forerunner of various arguments which appear later in [54] and several other papers including this one.

Commutator theorems have a long history in harmonic analysis (cf. [14] and the references therein to the fundamental work of Calderón and Zygmund and others). Motivated by the classical theorems on commutators of singular integral operators, Rochberg and Weiss, in the early 1980 [54], initiated the study of commutator estimates in interpolation theory. Since then the theory has developed in various directions and applications have been found to pde's, harmonic analysis, and functional analysis. For some of these developments we refer the reader to [23, 25, 26, 28, 34, 35, 44, 50, 51], where more references and further applications and examples can be found. For a survey of the earlier work we refer also to [20].

Remark 1.1. Let us briefly recall two, by now rather standard examples of commutator theorems in the context of L^p spaces: For the first example, we choose some fixed $p \in (1, \infty)$ and consider the couple $(L^p(w_0), L^p(w_1))$ of weighted L^p spaces on some underlying measure space (X, Σ, μ). If the linear operator T maps $L^p(w_j)$ boundedly onto itself for $j = 0, 1$, then of course T is also bounded on the space $L^p(w_\theta)$ where $w_\theta = w_0^{1-\theta} w_1^\theta$ and $\theta \in (0, 1)$. But it is also known (see, e.g. [54, pp. 335–337; 32, p. 203]) that the commutator $[T, \Omega] = T\Omega - \Omega T$ is a bounded map on $L^p(w_\theta)$ where the map Ω is defined by $\Omega f = f \log (w_1/w_0)$. As explained in [54, pp. 335–336], this result is very closely related to the result from [15] mentioned just above.

For the second example we consider a linear operator T which is bounded on L^1 and L^∞. Then it is of course bounded on L^p for every $p \in (1, \infty)$. But it is also known (see [54, pp. 315–318]) that $[T, \Omega]$ is bounded on L^p, where this time Ω is a nonlinear operator and is defined by $\Omega f = f \log |f|$.

The theory that has evolved around commutator estimates has largely followed the pattern referred to above, i.e. it was essentially developed separately for the real and complex methods. But it was also asked long ago, e.g. in [20], whether a general method could be given to unify the approaches to the real and complex methods, and of course the above-mentioned work [8] subsequently showed one way in which this is possible.

One notable feature of our approach is the systematic use of analytic functions and holomorphic structure for our general method and thus by implication for the real and $\pm$ methods, not just the complex one. The idea of using analytic functions in the framework of the real method may seem a little exotic, but it is certainly not new. For example, it can be seen in a setting closely related to this paper in [63, Sect. 2; 20, pp. 180–181], and it also plays a limited role in [8, p. 209]). Perhaps its first at least implicit appearance was in [41, Sect. 1.4, pp. 29–31]. There the analytic functions serve to show (cf. also [47, pp. 22–23; 17, p. 1008]), that the real and complex methods are in some sense "Fourier transforms" of each other.

"Traditionally," since the work of Thorin [60], the analytic functions in interpolation theory are defined on a strip, and we could have developed our theory here using functions on Thorin's strip. However, we have chosen to replace the strip by an annulus. This simplifies certain steps and corresponds to "discretising" (as in the real method). It is also more convenient for observing the connection with the $\pm$ method. The price we pay for this convenience is that we have to show later that, in the case of the real and complex methods, our constructions using the annulus are equivalent to previously used constructions using the strip. Intuitively this seems obvious, in the light of the well known and easily proved equivalences between the "discrete/annular" and "continuous/strip" versions of the real method and the complex method, respectively. But our proofs have turned out to be longer than might have been expected.

While the earlier work on commutators emphasized the specific role of certain "derivation" mappings, more recently the emphasis has been on trying to understand the role that cancellations play in the theory (cf. [7, 44]). This has been particularly fruitful in obtaining "higher order" commutator estimates and has led to some simplification of the theory. The method which we develop here enables us now to express the cancellation conditions for the real, complex, $\pm$ and other methods in the same unified way: certain derivatives of the analytic functions representing elements of the interpolation space have to vanish at the point corresponding to the parameter of the interpolation space. This approach leads efficiently to higher order estimates and characterization of domain spaces unifying and generalizing methods developed in [7, 44].

The paper is organized as follows: In Section 2, we define a general method to construct interpolation spaces using analytic functions on an annulus, and observe that the real and complex and also the $\pm$ interpolation methods all arise as particular cases of this method. In Section 3, we construct the derivation and translation mappings and prove a general commutator theorem.

Section 4 provides the above-mentioned proofs, in the case of the real and complex methods, that the modification of our constructions with the annulus replaced by a strip, gives essentially the same derivation and translation operators. This shows that our general commutator theorem contains the commutator theorems obtained in earlier papers as special cases.

In Section 5, we indicate some connections and some differences between our approach and that of Carro, Cerdà and Soria.

In Section 6, we extend the results of Section 3 emphasizing the role of cancellations in the computation of the norms in interpolation spaces and use these observations to prove higher order commutator theorems. In Section 7, we obtain a general characterization of the domain spaces associated with derivation operators. Finally, in Section 8 we obtain an "equivalence theorem" for our interpolation spaces, i.e. a result which generalizes the equivalence of the two "standard" ways (i.e. the J and K methods) for defining real interpolation spaces.

There are many natural questions which can be asked concerning the new interpolation spaces introduced in this paper. To give just one example: do these space satisfy some version of Wolff's Theorem [62]? Can this be proved using the methods of [62] and/or [33] and/or [30]?

2. INTERPOLATION SPACES DEFINED VIA ANALYTIC FUNCTIONS ON AN ANNULUS

In this section we introduce a class of interpolation methods defined using analytic functions on an annulus. As we shall see, these methods include as special examples the complex and real methods of interpolation as well as the $\pm$ methods of Peetre and Gustavsson–Peetre. The idea of looking at these particular methods in a unified way goes back to Peetre, and our construction here has its origins in his definitions and remarks on pp. 174–177 of [48].

We start with some general definitions.

DEFINITION 2.1. Let **Ban** be the class of all Banach spaces over the complex numbers. A mapping $\mathscr{X}$: **Ban** $\to$ **Ban** will be called a *pseudolattice*, or a *pseudo-$\mathbb{Z}$-lattice*, if

(i) for each $B \in$ **Ban** the space $\mathscr{X}(B)$ consists of B valued sequences $\{b_n\}_{n\in\mathbb{Z}}$ and if

(ii) whenever A is a closed subspace of B it follows that $\mathscr{X}(A)$ is a closed subspace of $\mathscr{X}(B)$ and if

(iii) there exists a positive constant $C = C(\mathscr{X})$ such that, for all $A, B \in$ **Ban** and all bounded linear operators $T : A \to B$ and every sequence $\{a_n\}_{n\in\mathbb{Z}} \in \mathscr{X}(A)$, the sequence $\{Ta_n\}_{n\in\mathbb{Z}} \in \mathscr{X}(B)$ and satisfies the estimate

$$\|\{Ta_n\}_{n\in\mathbb{Z}}\|_{\mathscr{X}(B)} \leqslant C(\mathscr{X})\|T\|_{A\to B}\|\{a_n\}\|_{\mathscr{X}(A)}.$$

Let us now present a number of examples of pseudolattices. For each of them we have $C(\mathscr{X}) = 1$.

EXAMPLE 2.2. Let X be a Banach lattice of real valued functions defined on $\mathbb{Z}$. We will use the notation $\mathscr{X} = X$ to mean that, for each $B \in$ **Ban**,

$\mathscr{X}(B)$ is the space, usually denoted by $X(B)$, consisting of all B valued sequences $\{b_n\}$ such that $\{\|b_n\|_B\}_{n\in\mathbb{Z}} \in X$. It is normed by $\|\{b_n\}_{n\in\mathbb{Z}}\|_{X(B)} = \|\{\|b_n\|_B\}_{n\in\mathbb{Z}}\|_X$.

EXAMPLE 2.3. For each $B \in \mathbf{Ban}$ let $FC(B)$ be the space of all B valued sequences $\{b_n\}_{n\in\mathbb{Z}}$ such that $b_n = \frac{1}{2\pi}\int_0^{2\pi} e^{-int} f(e^{it})\,dt$ for all n and some continuous function $f : \mathbb{T} \to B$. $FC(B)$ is normed by $\|\{b_n\}\|_{FC(B)} = \sup_{\tau\in\mathbb{T}} \|f(\tau)\|_B$. The notation $\mathscr{X} = FC$ will mean that $\mathscr{X}(B) = FC(B)$ for each B. Where necessary, we may use the more explicit notations $FC_{\mathbb{T}}$ and $FC_{\mathbb{T}}(B)$.

("FC" indicates that here we are dealing with the Fourier transform $FC(\mathbb{T}, B)$ of the space $C = C(\mathbb{T}, B)$ of continuous (B-valued) functions on $\mathbb{T}$. Cf. [29] where analogous sequence spaces FL^1 and FL^∞ also appear and play important roles. There are of course obvious possible variants of this example where the space of continuous B valued functions on $\mathbb{T}$ is replaced by some other suitable space of B valued functions on $\mathbb{T}$.)

EXAMPLE 2.4. We shall use the notation $\mathscr{X} = UC$, when $\mathscr{X}(B) = UC(B)$ for all $B \in \mathbf{Ban}$, where $UC(B)$ denotes the Banach space of all B valued sequences $\{b_n\}_{n\in\mathbb{Z}}$ such that the series $\sum_{n\in\mathbb{Z}} b_n$ is unconditionally convergent in B. As norm we take $\|\{b_n\}\|_{\mathscr{X}(B)} = \sup\|\sum_{n\in F} \varepsilon_n b_n\|_B$ where the supremum is taken over all finite subsets $F \subset \mathbb{Z}$ and all sequences ε_n taking only the values ± 1.

EXAMPLE 2.5. We shall also consider a variant of the preceding example where b_n is required only to be *weakly* unconditionally convergent in B. The norm is as above, and we shall use the notation $WUC(B)$ for the corresponding Banach space and $\mathscr{X} = WUC$ for the pseudolattice.

Remark 2.6. Each of the pseudolattices $\mathscr{X}$ in the preceding Examples 2.3–2.5 has the property that

$$\|b_m\|_B \leqslant \|\{b_n\}\|_{\mathscr{X}(B)} \tag{2.1}$$

for all $m \in \mathbb{Z}$, all $\{b_n\}_{n\in\mathbb{Z}} \in \mathscr{X}(B)$ and all Banach spaces B. The same also holds for Example 2.2 provided the lattice X has the property that

$$\|\{\delta_{m,n}\}_{n\in\mathbb{Z}}\|_X \leqslant 1 \qquad \text{for each } m \in \mathbb{Z} \tag{2.2}$$

(Here $\delta_{m,n}$ denotes the usual Kronecker delta.)

We shall use the usual notation $\vec{B} = (B_0, B_1)$ for *Banach pairs* (also often referred to as "Banach couples" in the literature) of Banach spaces B_0 and B_1 (cf. [4, Chap. 2] or [5, p. 91]). Also the notation $T : \vec{A} \to \vec{B}$ will have the usual meaning, that T is a linear operator $T : A_0 + A_1 \to B_0 + B_1$ such that T maps A_j to B_j continuously for $j = 0, 1$. We set $\|T\|_{\vec{A}\to\vec{B}} = \max\{\|T\|_{A_0\to B_0}, \|T\|_{A_1\to B_1}\}$.

Let $\mathscr{X}_0$ and $\mathscr{X}_1$ be any two pseudolattices. We consider them as a pair, which we denote by $\mathbf{X} = \{\mathscr{X}_0, \mathscr{X}_1\}$. (Note that a pseudolattice pair is an essentially different object from a Banach pair.)

Definition 2.7. For each Banach pair $\vec{B}$ and pseudolattice pair $\mathbf{X}$ we define $\mathscr{J}(\mathbf{X}, \vec{B})$ to be the space of all $B_0 \cap B_1$ valued sequences $\{b_n\}_{n\in\mathbb{Z}}$ for which the sequence $\{e^{jn}b_n\}_{n\in\mathbb{Z}}$ is in $\mathscr{X}_j(B_j)$ for $j = 0, 1$. This space is normed by

$$\|\{b_n\}_{n\in\mathbb{Z}}\|_{\mathscr{J}(\mathbf{X},\vec{B})} = \max_{j=0,1} \|\{e^{jn}b_n\}_{n\in\mathbb{Z}}\|_{\mathscr{X}_j(B_j)}.$$

It will be convenient to exclude some "pathological" phenomena by requiring the space $\mathscr{J}(\mathbf{X}, \vec{B})$ to be "not too small." One way of doing this is to impose the following condition. As we shall see later, it is also equivalent to various other seemingly stronger conditions.

Definition 2.8. Let $\mathbb{A}$ be the open annulus $\{z \in \mathbb{C} : 1 < |z| < e\}$. We shall say that the pseudolattice pair $\mathbf{X}$ is *nontrivial* if, for the special one-dimensional Banach pair $\vec{B} = (\mathbb{C}, \mathbb{C})$ and each $s \in \mathbb{A}$, there exists $\{b_n\}_{n\in\mathbb{Z}} \in \mathscr{J}(\mathbf{X}, \vec{B})$ such that the series $\sum_{n\in\mathbb{Z}} s^n b_n$ converges to a nonzero number.

In all concrete examples to be considered in this paper it will be immediately evident that this condition is fulfilled because the sequence $\{b_n\}_{n\in\mathbb{Z}}$ defined by $b_0 = 1$ and $b_n = 0$ for all $n \neq 0$ will be an element of $\mathscr{J}(\mathbf{X}, (\mathbb{C}, \mathbb{C}))$.

Definition 2.9. We shall say that the pseudolattice pair $\mathbf{X}$ is *Laurent compatible* if it is nontrivial and if for every Banach pair $\vec{B}$, every vector valued sequence $\{b_n\}_{n\in\mathbb{Z}}$ in $\mathscr{J}(\mathbf{X}, \vec{B})$ and every fixed z in the open annulus $\mathbb{A}$ the Laurent series $\sum_{n\in\mathbb{Z}} z^n b_n$ converges in $B_0 + B_1$ and $\|\sum_{n\in\mathbb{Z}} z^n b_n\|_{B_0+B_1} \leqslant C\|\{b_n\}_{n\in\mathbb{Z}}\|_{\mathscr{J}(\mathbf{X},\vec{B})}$ for some constant $C = C(z)$ independent of the choice of $\{b_n\}_{n\in\mathbb{Z}}$.

Remark 2.10. This convergence of $\sum_{n\in\mathbb{Z}} z^n b_n$ implies of course that

$$\lim_{n\to\pm\infty} \rho^n \|b_n\|_{B_0+B_1} = 0 \qquad \text{for all } \rho \in (1, e). \tag{2.3}$$

It follows from (2.3) that $\sum_{n\in\mathbb{Z}} z^n b_n$ converges absolutely (with respect to the norm of $B_0 + B_1$) and uniformly on every compact subset of $\mathbb{A}$. Consequently the sum of this series is an analytic function of z in $\mathbb{A}$ and can be differentiated term-by-term. The series for its derivative $f'(z) = \sum_{n\in\mathbb{Z}} nz^n b_n$ must also converge absolutely in $B_0 + B_1$.

It is easy to check that any pair $\mathbf{X} = \{\mathscr{X}_0, \mathscr{X}_1\}$ of pseudolattices both of which satisfy (2.1) must be Laurent compatible. Thus, from Remark 2.6 we have several examples of such pairs. It is also clear that considerably weaker conditions on each of the pseudolattices $\mathscr{X}_j$ would suffice to give Laurent compatibility, for example

$$\|b_m\|_B \leqslant C(1+|m|)^{\lambda}\|\{b_n\}\|_{\mathscr{X}_j(B)} \qquad \text{for all } m \in \mathbb{Z} \text{ and } j = 0, 1, \tag{2.4}$$

where λ is any positive constant.

Definition 2.11. For each Banach pair $\vec{B}$, each Laurent compatible pair $\mathbf{X}$ and each fixed $s \in \mathbb{A}$ we define the space $\vec{B}_{\mathbf{X},s}$ to consist of all elements of the form $b = \sum_{n\in\mathbb{Z}} s^n b_n$ where $\{b_n\}_{n\in\mathbb{Z}} \in \mathscr{J}(\mathbf{X}, \vec{B})$, with the natural quotient norm

$$\|b\|_{\vec{B}_{\mathbf{X},s}} = \inf\left\{\|\{b_n\}_{n\in\mathbb{Z}}\|_{\mathscr{J}(\mathbf{X},\vec{B})} : b = \sum_{n\in\mathbb{Z}} s^n b_n\right\}. \tag{2.5}$$

The Laurent compatibility of $\mathbf{X}$ and the completeness of $\mathscr{J}(\mathbf{X}, \vec{B})$ imply, respectively, that $\|\cdot\|_{\vec{B}_{\mathbf{X},s}}$ is indeed a norm, rather than merely a seminorm and that $\vec{B}_{\mathbf{X},s}$ is a Banach space. The nontriviality of $\mathbf{X}$ implies that $(\mathbb{C}, \mathbb{C})_{\mathbf{X},s} = \mathbb{C}$ for all $s \in \mathbb{A}$. In fact these two conditions are equivalent.

Under additional conditions on $\mathbf{X}$ we can replace (2.5) by a useful "convexity" estimate. (Cf. [41, Lemme (3.1), p. 12]). These conditions are conveniently formulated in terms of shift operators, which we will also need for some other purposes later.

Definition 2.12. Let S denote the left-shift operator on two-sided (vector valued) sequences defined by $S(\{b_n\}_{n\in\mathbb{Z}}) = \{b_{n+1}\}_{n\in\mathbb{Z}}$. Then of course S^{-1} is the right-shift operator $S^{-1}(\{b_n\}_{n\in\mathbb{Z}}) = \{b_{n-1}\}_{n\in\mathbb{Z}}$.

Lemma 2.13. *Suppose that S maps $\mathscr{X}_j(B_j)$ isometrically onto itself for $j = 0, 1$. Then, for each $s \in \mathbf{A}$ and each $b \in \vec{B}_{\mathbf{X},s}$,*

$$\|b\|_{\vec{B}_{\mathbf{X},s}} \leqslant e \inf\Bigg\{\|\{b_n\}_{n\in\mathbb{Z}}\|^{1-\theta}_{\mathscr{X}_0(B_0)}\|\{e^n b_n\}_{n\in\mathbb{Z}}\|^{\theta}_{\mathscr{X}_1(B_1)} : \{b_n\}_{n\in\mathbb{Z}} \in \mathscr{J}(\mathbf{X}, \vec{B}), \\ b = \sum_{n\in\mathbb{Z}} s^n b_n\Bigg\}, \tag{2.6}$$

where $\theta = \log|s|$.

Proof. This is very similar to the argument on [41, p. 13]. Given any $\{b_n\}_{n\in\mathbb{Z}} \in \mathscr{J}(\mathbf{X}, \vec{B})$ with $b = \sum_{n\in\mathbb{Z}} s^n b_n$ we see that, for each $k \in \mathbb{Z}$, we have that $\{b_{n+k}\}_{n\in\mathbb{Z}} \in \mathscr{J}(\mathbf{X}, \vec{B})$. More specifically, by the isometry of S, we have $\|\{b_{n+k}\}_{n\in\mathbb{Z}}\|_{\mathscr{X}_0(B_0)} = \|\{b_n\}_{n\in\mathbb{Z}}\|_{\mathscr{X}_0(B_0)}$ and $\|\{e^n b_{n+k}\}_{n\in\mathbb{Z}}\|_{\mathscr{X}_1(B_1)} = e^{-k}\|\{e^{n+k} b_{n+k}\}_{n\in\mathbb{Z}}\|_{\mathscr{X}_1(B_1)} = e^{-k}\|\{e^n b_n\}_{n\in\mathbb{Z}}\|_{\mathscr{X}_1(B_1)}$. So

$$\|\{b_{n+k}\}_{n\in\mathbb{Z}}\|_{\mathscr{J}(\mathbf{X},\vec{B})} = \max\{\|\{b_n\}_{n\in\mathbb{Z}}\|_{\mathscr{X}_0(B_0)}, e^{-k}\|\{e^n b_n\}_{n\in\mathbb{Z}}\|_{\mathscr{X}_1(B_1)}\}.$$

Let us now choose k so that

$$\|\{b_n\}_{n\in\mathbb{Z}}\|_{\mathscr{X}_0(B_0)} \leqslant e^{-k}\|\{e^n b_n\}_{n\in\mathbb{Z}}\|_{\mathscr{X}_1(B_1)} \leqslant e\|\{b_n\}_{n\in\mathbb{Z}}\|_{\mathscr{X}_0(B_0)}.$$

Then $\sum_{n\in\mathbb{Z}} s^n b_{n+k} = \sum_{n\in\mathbb{Z}} s^{n-k} b_n = s^{-k} b$ with

$$\|s^{-k} b\|_{\vec{B}_{\mathbf{X},s}} \leqslant e\|\{b_n\}_{n\in\mathbb{Z}}\|_{\mathscr{X}_0(B_0)}. \tag{2.7}$$

Note that

$$|s^k| = e^{\theta k} \leqslant \left(\frac{\|\{e^n b_n\}_{n\in\mathbb{Z}}\|_{\mathscr{X}_1(B_1)}}{\|\{b_n\}_{n\in\mathbb{Z}}\|_{\mathscr{X}_0(B_0)}}\right)^{\theta}$$

so that, after multiplying (2.7) by $e^{\theta k}$, we obtain

$$\|b\|_{\vec{B}_{\mathbf{X},s}} \leqslant e\|\{b_n\}_{n\in\mathbb{Z}}\|^{1-\theta}_{\mathscr{X}_0(B_0)}\|\{e^n b_n\}_{n\in\mathbb{Z}}\|^{\theta}_{\mathscr{X}_1(B_1)}.$$

To complete the proof we simply take the infimum over all $\{b_n\}_{n\in\mathbb{Z}} \in \mathscr{J}(\mathbf{X}, \vec{B})$ with $b = \sum_{n\in\mathbb{Z}} s^n b_n$. ∎

THEOREM 2.14. *Let $\vec{B} = (B_0, B_1)$ be a Banach pair, let $\mathbf{X}$ be a Laurent compatible pair and let s be any point in $\mathbb{A}$. Then*

(i) *the space $\vec{B}_{\mathbf{X},s}$ is* intermediate, *i.e. it satisfies the continuous inclusions $B_0 \cap B_1 \subset \vec{B}_{\mathbf{X},s}$ and $\vec{B}_{\mathbf{X},s} \subset B_0 + B_1$.*

(ii) *Let $\vec{A} = (A_0, A_1)$ be another Banach pair and suppose that $T : A_0 + A_1 \to B_0 + B_1$ is a linear operator which maps A_j boundedly to B_j with norm M_j for $j = 0, 1$. Then T maps $\vec{A}_{\mathbf{X},s}$ boundedly to $\vec{B}_{\mathbf{X},s}$ for each $s \in \mathbb{A}$, with norm*

$$\|T\|_{\vec{A}_{\mathbf{X},s}\to\vec{B}_{\mathbf{X},s}} \leqslant \max_{j=0,1} M_j C(\mathscr{X}_j).$$

(ii′) *If, furthermore, S maps $\mathscr{X}_j(B_j)$ isometrically onto itself for $j = 0, 1$ then the norm $\|T\|_{\vec{A}_{\mathbf{X},s}\to\vec{B}_{\mathbf{X},s}}$ also satisfies*

$$\|T\|_{\vec{A}_{\mathbf{X},s}\to\vec{B}_{\mathbf{X},s}} \leqslant e(C(\mathscr{X}_0)M_0)^{1-\theta}(C(\mathscr{X}_1)M_1)^{\theta},$$

where $\theta = \log|s|$.

Proof. We start with (ii) which has an obvious proof which can be left to the reader. A small modification of that proof using Lemma 2.13 gives (ii′). The second inclusion in (i) is an immediate consequence of the Laurent compatibility of **X**. It remains to prove the first inclusion of (i). We choose $A_0 = A_1 = \mathbb{C}$ and use the nontriviality of **X** to ensure that $\vec{A}_{\mathbf{X},s} = \mathbb{C}$. Then we apply (ii) to the operator $T : \vec{A} \to \vec{B}$ defined by $T\zeta = \zeta b$ for all $\zeta \in \mathbb{C}$, where b is an arbitrary fixed element of $B_0 \cap B_1$. This gives that $b = T1 \in \vec{B}_{\mathbf{X},s}$ with $\|b\|_{\vec{B}_{\mathbf{X},s}} \leqslant \max_{j=0,1} M_j C(\mathscr{X}_j)\|1\|_{\vec{A}_{\mathbf{X},s}} \leqslant \|b\|_{B_0 \cap B_1} \times \max_{j=0,1} C(\mathscr{X}_j)\|1\|_{\vec{A}_{\mathbf{X},s}}$. ∎

The conclusions of Theorem 2.14 can be re-expressed more formally by stating that, for each **X** and s, the map $\vec{B} \mapsto \vec{B}_{\mathbf{X},s}$ is an *interpolation functor* (cf. [4, p. 28] or [5, p. 140]). It is easy to check (cf. also [48, pp. 174–177]) that this new general interpolation functor coincides with various "classical" interpolation methods for suitable choices of **X**: More specifically, let us set $s = e^{\theta}$ for some $\theta \in (0, 1)$. Then, if $\mathscr{X}_0 = \mathscr{X}_1 = \ell^p$, the space $\vec{B}_{\mathbf{X},s}$ coincides with the Lions–Peetre real method space $\vec{B}_{\theta,p} = (B_0, B_1)_{\theta,p}$ using the equivalent "discrete definition." (See e.g. [41, p. 17] where this space is denoted by $s(p, \theta, B_0; p, \theta - 1, B_1)$ or [4, Chap. 3].) If $\mathscr{X}_0 = \mathscr{X}_1 = FC$, then $\vec{B}_{\mathbf{X},s}$ coincides, to within equivalence of norm, with the Calderón complex method space $\vec{B}_{[\theta]} = [B_0, B_1]_{\theta} = [\vec{B}]_{\theta}$. (See [17]. This is also discussed below in more detail in the course of the proof of Theorem 4.2.) If $\mathscr{X}_0 = \mathscr{X}_1 = UC$, then $\vec{B}_{\mathbf{X},s}$ is the Peetre $\pm$ method space $\vec{B}_{\langle\theta\rangle} = \langle B_0, B_1\rangle_{\theta}$, [48, p. 176]. If we replace UC by WUC, we obtain the Gustavsson–Peetre variant of $\langle B_0, B_1\rangle_{\theta}$ which is denoted by $\langle \vec{B}, \rho_{\theta}\rangle$. (See [24, p. 45; 29].)

Remark 2.15. The previous identifications of the space $\vec{B}_{\mathbf{X},s}$ also hold for any other $s \in \mathbb{A}$ on the circle $|s| = e^{\theta}$ since each of the pseudolattices $\mathscr{X}$ used to define them has the property that the "rotation map" $\{b_n\}_{n\in\mathbb{Z}} \mapsto \{e^{in\tau} b_n\}_{n\in\mathbb{Z}}$ is an isometry of $\mathscr{X}(B)$ onto itself for every real τ and every Banach space B.

One can of course obtain many other (often more exotic) interpolation spaces by making other choices of $\{\mathscr{X}_0, \mathscr{X}_1\}$. We need not, as we have done so far, always require that $\mathscr{X}_0 = \mathscr{X}_1$. For example, in [48], Peetre also briefly considers the space $\langle B_0, B_1\rangle_{\theta,p_0,p_1}$. This is a generalized version of $\langle B_0, B_1\rangle_{\theta}$ which corresponds to defining $\mathscr{X}_j$ in terms of "p_j-unconditionally summable sequences" for $j = 0, 1$. Certain spaces corresponding to the case where $\mathscr{X}_0$ and $\mathscr{X}_1$ are both (possibly different) lattices have been studied by some authors. See e.g. the work of Dmitriev [22]. We should mention one case with $\mathscr{X}_0 \neq \mathscr{X}_1$ for which the description of $\vec{B}_{\mathbf{X},s}$ is well known. If we take $\mathscr{X}_0 = \ell^{p_0}$ and $\mathscr{X}_1 = \ell^{p_1}$, then it is clear that $\vec{B}_{\mathbf{X},s}$ is exactly the space denoted by $s(p_0, \theta, B_0; p_1, \theta - 1, B_1)$ in [41, p. 17] which is the same, to within equivalence of norm, as the space $S(\vec{B}, p_0, p_1, \theta)$ in [4, Sect. 3.12] and the space $(B_0, B_1)_{\theta,p_0,p_1}$ in [46]. But then, by Théorème 1 of [46, p. 252] (cf. also [4, pp. 70–72; pp. 85–86]) this space coincides, to within equivalence of norms, with $(B_0, B_1)_{\theta,p}$ where $1/p = (1-\theta)/p_0 + \theta/p_1$. As a contrasting example, it is tantalizing to wonder what "hybrid" choices like $\mathbf{X} = \{\ell^1, FC\}$ might give. It is easy to give concrete descriptions of $(L^1, L^\infty)_{\{\ell^{p_0},\ell^{p_1}\},s}$ and $(L^1, L^\infty)_{\{FC,FC\},s}$, but what can be said about the space $(L^1, L^\infty)_{\{\ell^1,FC\},s}$?

3. DERIVATION MAPS, TRANSLATION MAPS, AND COMMUTATOR THEOREMS

We are now ready to define the "derivation" mappings Ω and "translation" mappings $\mathscr{R}$ associated with the interpolation spaces $\vec{B}_{\mathbf{X},s}$. Both Ω and $\mathscr{R}$ map $\vec{B}_{\mathbf{X},s}$ into $B_0 + B_1$. In general they are nonlinear, and may also be taken to be multiple valued. As one particular case of our derivation mappings we shall obtain mappings which are equivalent (in view of the results to be presented in Section 4) to the derivation mappings which are defined and studied in [32] explicitly for the real method (in [19] they are referred to as "quasilogarithmic operators"). In another particular case our mappings will be equivalent (cf. Section 4) to the derivation mappings defined in [54] explicitly for the complex method. We shall establish a "commutator" theorem for our derivation mappings, which includes as special cases the commutator theorems developed in the preceding references and thus also the two results mentioned in Remark 1.1. We recall once more that an alternative method of putting these kinds of results into a more general abstract framework has been developed in [8].

Our translation mappings essentially generalize previously studied maps which are used in several contexts. For example, in the case of the real method, such mappings appear explicitly or implicitly in [32, Sect. 5; 63, Sect. 2; 20, pp. 179–182]. Let us also mention various versions of such maps which are known in the case of the complex method. For example in [12, pp. 276–277; 13, pp. 142–146] a related (one-valued) map is introduced and used in the context of finite dimensional spaces, (infinite families rather than just pairs). Daher [21] and Kalton (unpublished) have (independently) used one-valued versions of (complex method) translation mappings to provide homeomorphisms between unit balls of certain uniformly convex Banach spaces. Another treatment of this material, including further details concerning the moduli of continuity of such homeomorphisms, is given in [3, pp. 204–206]. In these preceding examples the "optimality constant" (as defined below) is chosen to equal 1 (cf. Remark 3.3). Translation maps also appear, at least implicitly, in the work of Shneiberg [58] (cf. also [59]). Translation maps are also considered from the point of view of Banach space theory in [37].

We obtain a commutator theorem for our general translation mappings which can be considered as a sort of generalization of results first obtained in [27] (for L^p spaces using the complex method) and later in [43] (for the real method).

Although in previous papers it has usually been customary to work with a derivation mapping Ω which is single valued, it is perhaps a little more convenient and natural to instead consider a certain set-valued mapping $\tilde{\Omega}$. The mapping Ω can then be taken to be any single valued "selection" of $\tilde{\Omega}$. (There is of course a great deal of arbitrariness in the definition of Ω.) Similarly, we can consider set-valued and single-valued versions of the translation mappings, analogously denoted by $\tilde{\mathscr{R}}$ and $\mathscr{R}$.

Let us now explicitly describe these various mappings:

DEFINITION 3.1. Let us fix a positive constant C_{opt} (an "optimality" constant) usually satisfying $C_{\text{opt}} > 1$, a Laurent compatible pair of pseudolattices $\mathbf{X}$, and a point $s \in \mathbb{A}$. For each Banach pair $\vec{B}$ and each element $b \in \vec{B}_{\mathbf{X},s}$ let $E(b)$ denote the set of all sequences $\{b_n\}_{n\in\mathbb{Z}}$ in $\mathscr{J}(\mathbf{X}, \vec{B})$ such that $\sum_{n\in\mathbb{Z}} s^n b_n = b$ and $\|\{b_n\}_{n\in\mathbb{Z}}\|_{\mathscr{J}(\mathbf{X},\vec{B})} \leqslant C_{\text{opt}}\|b\|_{\vec{B}_{\mathbf{X},s}}$.

(i) Let $\tilde{\Omega}(b)$ denote the set of all elements $b' \in B_0 + B_1$ of the form $b' = \sum_{n\in\mathbb{Z}} n s^{n-1} b_n$ for all choices of the sequence $\{b_n\}_{n\in\mathbb{Z}}$ in $E(b)$.

(ii) For each element $b \in \vec{B}_{\mathbf{X},s}$ we choose some element $\Omega(b) \in \tilde{\Omega}(b)$. (Assume $C_{\text{opt}} > 1$.)

(iii) Fix a second point $s' \in \mathbb{A}$ and for each $\vec{B}$ and each element $b \in \vec{B}_{\mathbf{X},s}$ as above, let $\tilde{\mathscr{R}}(b)$ denote the set of all elements $b' \in B_0 + B_1$ of the form $b' = \sum_{n\in\mathbb{Z}} (s')^n b_n$ for all choices of the sequence $\{b_n\}_{n\in\mathbb{Z}}$ in $E(b)$.

(iv) For each element $b \in \vec{B}_{\mathbf{X},s}$ we choose some element $\mathscr{R}(b) \in \tilde{\mathscr{R}}(b)$. (Assume $C_{\text{opt}} > 1$.)

Where necessary we shall use the notation $\tilde{\Omega}_{\vec{B}}$ and $\Omega_{\vec{B}}$ or $\tilde{\mathscr{R}}_{\vec{B}}$ and $\mathscr{R}_{\vec{B}}$ or $E_{\vec{B}}$ to indicate the underlying pair $\vec{B}$ with respect to which these mappings or sets are defined. (Later, in Section 4, we shall use still more elaborate notation indicating the dependence on other parameters also.)

Remark 3.2. The convergence in $B_0 + B_1$ of the series $\sum_{n\in\mathbb{Z}} n s^{n-1} b_n$ is a consequence of our hypotheses on $\{b_n\}$ (cf. Remark 2.10) and its sum is of course $f'(s)$ where $f(z) = \sum_{n\in\mathbb{Z}} z^n b_n$.

Remark 3.3. It may be interesting in some cases, e.g. when $\vec{B}$ is a Banach pair of finite dimensional spaces, to choose C_{opt} to equal 1 in the above definition. This corresponds to what is done in certain papers mentioned above ([3, 12,13, 21]). But, for general Banach pairs, such a choice could cause the set $E(b)$ and therefore also $\tilde{\Omega}(b)$ and $\tilde{\mathscr{R}}(b)$ to be empty for some elements b. Of course if $C_{\text{opt}} > 1$, as will usually be assumed, then these sets are always nonempty. and so $\Omega(b)$ and $\mathscr{R}(b)$ are also defined. On the other hand, if $C_{\text{opt}} < 1$ then these sets will be empty for each $b \neq 0$.

The preceding definitions of Ω and $\mathscr{R}$ are rather abstract, so perhaps it is useful to briefly recall a simple and relatively concrete example. It arises in the framework of the complex interpolation method. Here the relevant complex variables will range over a strip rather than the annulus $\mathbb{A}$. (But, as already mentioned, we will see in Section 4 that the strip and annulus give essentially the same operators.)

We will be quite informal, both with precise definitions and specific hypotheses. Suppose W is a (possibly unbounded) positive linear operator on a Banach space X and that W admits enough of a functional calculus so that we can make good sense of the semigroup $W^s, s \geqslant 0$. In this case the family of spaces X_s defined by $\|x\|_{X_s} = \|W^s x\|_X$, $0 \leqslant s \leqslant 1$ will be a (complex) interpolation scale. For this family the operator Ω will be the infinitesimal generator of the semigroup and the translation operators $\mathscr{R}$ will be given by the powers of W; in particular $W^{s_1 - s_0}$ is the translation operator which maps X_{s_1} boundedly (in this case, in fact, isometrically) to X_{s_0}. This pair of viewpoints, one case focusing on $W^s x$ as a varying family of vectors residing in the fixed space X or, alternatively, focusing on x as a fixed vector seen as living in a family of spaces, the X_s, is reminiscent of the duality in quantum mechanics between the Schrödinger picture and the Heisenberg picture.

In the case where $X = L^p$ and W is given by pointwise multiplication by a positive, possibly unbounded, function w, then Ω is multiplication by $\log w$, i.e. we recapture the first example mentioned in Remark 1.1. (We have to choose $w_0 = w^{-\theta}$ and $w_1 = w_0 w = w^{1-\theta}$ for some $\theta \in (0,1)$.)

In the context of such semigroup considerations it is interesting to compare the results here with some of the basic facts about semigroups of operators such as the Hille–Yosida–Phillips theorem. In general, as we noted, the operators Ω and $\mathscr{R}$ are not linear. The previous discussion suggests that in this case there may be some relation with nonlinear semigroup theory or, more generally, nonlinear evolution equations. In fact it was already noted in [12, 13] that the translation operators, there called $A(z, z_0; \cdot)$ satisfy the propagator equation ((2.13) on [12, p. 276] or (4.7) on [13, p. 143]) which characterizes evolution equations.

Definition 3.4. Let $\mathbf{X} = \{\mathscr{X}_0 . \mathscr{X}_1\}$ be a pair of pseudolattices. We shall say that $\mathbf{X}$ *admits differentiation* if it is Laurent compatible and, for each complex Banach space B,

(i) for every $r \in (0,1)$ each element $\{b_n\}_{n\in\mathbb{Z}} \in \mathscr{X}_0(B)$ satisfies $\lim_{k\to-\infty} r^{-k}\|b_k\|_B = 0$ and each element $\{b_n\}_{n\in\mathbb{Z}} \in \mathscr{X}_1(B)$ satisfies $\lim_{k\to\infty} r^k\|b_k\|_B = 0$, and

(ii) for every complex number ρ satisfying $0<|\rho|<1$, for $j=0,1$ and for every sequence $\{b_n\}_{n\in\mathbb{Z}} \in \mathscr{X}_j(B)$, the new sequence $\{b_n^j\}_{n\in\mathbb{Z}}$ is also in $\mathscr{X}_j(B)$, where $\{b_n^0\}_{n\in\mathbb{Z}}$ and $\{b_n^1\}_{n\in\mathbb{Z}}$ are defined by setting

$$b_n^0 = \sum_{k<0} \rho^{-k} b_{n+k+1} \quad \text{and} \quad b_n^1 = \sum_{k\geqslant 0} \rho^k b_{n+k+1},$$

(where the convergence of all these sums is of course guaranteed by (i) and is in fact equivalent to (i),) and if also

(iii) for $j=0,1$ and each ρ as above, the linear map $D_{j,\rho}$ defined on $\mathscr{X}_j(B)$ by setting $D_{j,\rho}(\{b_n\}_{n\in\mathbb{Z}}) = \{b_n^j\}_{n\in\mathbb{Z}}$ maps $\mathscr{X}_j(B)$ boundedly into itself.

(iv) If, furthermore, for $j=0,1$, both the norms $\|D_{j,\rho}\|_{\mathscr{X}_j(B)\to\mathscr{X}_j(B)}$ are bounded functions of ρ on each compact subset of the punctured open unit disk, then we shall say that $\mathbf{X}$ admits differentiation *uniformly*.

Remark 3.5. It can be shown that conditions (i)–(iii) in the preceding definition in fact imply an apparently stronger condition, namely that there exists a (finite) constant $C_*(\mathbf{X},\rho)$, depending only on $\mathbf{X}$ and ρ, such that $\|D_{j,\rho}\|_{\mathscr{X}_j(B)\to\mathscr{X}_j(B)} \leqslant C_*(\mathbf{X},\rho)$ for all complex Banach spaces B and $j=0,1$. We defer the proof of this to an appendix: See Subsection A.1 (Corollary A.3(i)). Analogously, condition (iv) turns out to be equivalent to a stronger condition, where the upper bound for $\|D_{j,\rho}\|_{\mathscr{X}_j(B)\to\mathscr{X}_j(B)}$ as ρ ranges over any give compact set is independent of the particular choice of B. For the proof we refer again to Subsection A.1 (Corollary A.3(ii)).

The following lemma gives a simple sufficient condition in terms of the shift operator S (Definition 2.12) for a pair $\mathbf{X}$ to admit differentiation uniformly:

LEMMA 3.6. *Let $\mathbf{X} = \{\mathscr{X}_0, \mathscr{X}_1\}$ be a Laurent compatible pair and suppose that for each $B \in$ **Ban**, S is bounded on $\mathscr{X}_1(B)$ and S^{-1} is bounded on $\mathscr{X}_0(B)$ and furthermore that*

$$\sum_{k>0} r^k \|S^{-k}\|_{\mathscr{X}_0(B)\to\mathscr{X}_0(B)} < \infty \quad \text{and} \quad \sum_{k>0} r^k \|S^k\|_{\mathscr{X}_1(B)\to\mathscr{X}_1(B)} < \infty \tag{3.1}$$

for each $r \in (0,1)$. Then the pair $\mathbf{X} = \{\mathscr{X}_0, \mathscr{X}_1\}$ admits differentiation uniformly.

Proof. Obvious. ∎

Remark 3.7. Conditions (3.1) are in fact equivalent to stronger conditions which hold uniformly for all Banach spaces, i.e. with each term $\|S^m\|_{\mathscr{X}_j(B)\to\mathscr{X}_j(B)}$ replaced by $\sup_{B\in\mathbf{Ban}}\|S^m\|_{\mathscr{X}_j(B)\to\mathscr{X}_j(B)}$. For a proof, see Subsection A.1, Corollary A.4.

It is clear from the preceding lemma that the pair $\mathbf{X} = \{\mathscr{X}_0, \mathscr{X}_1\}$ admits differentiation uniformly whenever $\mathscr{X}_0$ and $\mathscr{X}_1$ are each chosen to be any of ℓ^p, FC, UC or WUC because for all of these $\mathscr{X}_j(B)$ is isometrically invariant under shifts for every $B \in \mathbf{Ban}$. Furthermore, pairs of pseudolattices satisfying a rather weaker form of shift invariance also admit differentiation uniformly. For example one could take both $\mathscr{X}_j$'s to be weighted ℓ^p spaces X of scalar sequences, where the weight varies comparatively slowly, such as

$$\|\{\alpha_n\}_{n\in\mathbb{Z}}\|_X = \left(\sum_{n\in\mathbb{Z}} |\alpha_n|^p(1+|n|)^\lambda\right)^{1/p},$$

where λ is any positive constant.

We can now present a generalized version of the "commutator theorem" for derivation mappings and also for translation mappings:

THEOREM 3.8. *Let* $\mathbf{X}$ *be a pair of pseudolattices which admits differentiation. Let* $\vec{A}$ *and* $\vec{B}$ *be arbitrary Banach pairs. Fix a point* $s \in \mathbb{A}$ *and a constant* $C_{\text{opt}} > 1$.

(i) *Let* $\tilde{\Omega}_{\vec{A}}$, $\Omega_{\vec{A}}$, $\tilde{\Omega}_{\vec{B}}$ *and* $\Omega_{\vec{B}}$ *denote derivation mappings corresponding to these choices of* s, C_{opt} *and* $\mathbf{X}$ *for the pairs* $\vec{A}$ *and* $\vec{B}$, *respectively. Let* $T : \vec{A} \to \vec{B}$ *be a bounded linear operator. Then the commutator* $[T, \Omega]$ *maps* $\vec{A}_{\mathbf{X},s}$ *boundedly into* $\vec{B}_{\mathbf{X},s}$. *More explicitly,* $T(\Omega_{\vec{A}}(a)) - \Omega_{\vec{B}}(Ta) \in \vec{B}_{\mathbf{X},s}$ *for each* $a \in \vec{A}_{\mathbf{X},s}$ *and*

$$\|T(\Omega_{\vec{A}}(a)) - \Omega_{\vec{B}}(Ta)\|_{\vec{B}_{\mathbf{X},s}} \leqslant \tilde{C}\|T\|_{\vec{A}\to\vec{B}}\|a\|_{\vec{A}_{\mathbf{X},s}} \tag{3.2}$$

for some constant $\tilde{C}$ *not depending on* a. (*See Remark* 3.10 *for various estimates for* $\tilde{C}$.)

(ii) *Fix a second point* $s' \in \mathbb{A}$ *and let* $\tilde{\mathscr{R}}_{\vec{A}}$, $\mathscr{R}_{\vec{A}}$, $\tilde{\mathscr{R}}_{\vec{B}}$ *and* $\mathscr{R}_{\vec{B}}$ *denote translation mappings corresponding to* s', s, C_{opt} *and* $\mathbf{X}$ *for the pairs* $\vec{A}$ *and* $\vec{B}$, *respectively. Then, for* T *as above, the commutator* $[T, \mathscr{R}]$ *maps* $\vec{A}_{\mathbf{X},s}$ *boundedly into* $\vec{B}_{\mathbf{X},s'}$ *and satisfies the estimate*

$$\|T(\mathscr{R}_{\vec{A}}(a)) - \mathscr{R}_{\vec{B}}(Ta)\|_{\vec{B}_{\mathbf{X},s'}} \leqslant |s-s'|\tilde{C}\|T\|_{\vec{A}\to\vec{B}}\|a\|_{\vec{A}_{\mathbf{X},s}} \tag{3.3}$$

for the same constant $\tilde{C}$.

Remark 3.9. An equivalent and perhaps slightly better formulation of (3.2) (cf. the preamble to Definition 3.1) would be that for each $a' \in \tilde{\Omega}_{\vec{A}}(a)$ and each $b' \in \tilde{\Omega}_{\vec{B}}(Ta)$, the element $Ta' - b' \in \vec{B}_{\mathbf{X},s}$ with the same norm estimate. Analogously one could reformulate (3.3) as the condition $Ta' - b' \in \vec{B}_{\mathbf{X},s'}$ with a corresponding norm estimate, for each $a' \in \tilde{\mathscr{R}}_{\vec{A}}(a)$ and each $b' \in \tilde{\mathscr{R}}_{\vec{B}}(Ta)$.

Remark 3.10. The constant $\tilde{C}$ can be taken to be

$$\tilde{C} = 2C_{\mathrm{opt}} \cdot C(\mathbf{X}) \max \left\{ \|D_{0,1/s}\|_{\mathscr{X}_0(B_0)\to\mathscr{X}_0(B_0)}, \frac{1}{e}\|D_{1,s/e}\|_{\mathscr{X}_1(B_1)\to\mathscr{X}_1(B_1)} \right\}, \quad (3.4)$$

where C_{opt} is the optimality constant chosen in the definition of the derivation mappings and $C(\mathbf{X}) = \max_{j=0,1} C(\mathscr{X}_j)$. Furthermore, (cf. Remark 3.5 and Corollary A.3) the last factor in (3.4) is bounded above by a constant which is independent of the particular Banach spaces B_0 and B_1.

In the case where the shift operator S is an isometry of $\mathscr{X}_j(B)$ onto itself for $j = 0, 1$ and each $B \in \mathbf{Ban}$, then we can use Lemma 2.13 and part (ii′) of Theorem 2.14 in a slightly modified version of the proof of Theorem 3.8. This gives alternative versions of estimates (3.2) and (3.3) where $\tilde{C}\|T\|_{\vec{A}\to\vec{B}}$ is replaced by an expression depending more explicitly on each of the norms $M_j = \|T\|_{A_j\to B_j}$ and on $\theta = \log|s|$ and $\theta' = \log|s'|$. By analogy with part (ii′) of Theorem 2.14 one might initially expect the expression $e(C(\mathscr{X}_0)M_0)^{1-\theta}(C(\mathscr{X}_1)M_1)^{\theta}$ to appear as a multiplicative factor in such estimates. But we obtain more complicated expressions. In (3.2) $\tilde{C}\|T\|_{\vec{A}\to\vec{B}}$ can be replaced by

$$C_{\mathrm{opt}}(\|D_{0,1/s}\|_{\mathscr{X}_0(B_0)\to\mathscr{X}_0(B_0)}(C(\mathscr{X}_0)M_0 + e(C(\mathscr{X}_0)M_0)^{1-\theta}(C(\mathscr{X}_1)M_1)^{\theta}))^{1-\theta}$$
$$\times (e^{-1}\|D_{1,s/e}\|_{\mathscr{X}_1(B_1)\to\mathscr{X}_1(B_1)}(C(\mathscr{X}_1)M_1 + e(C(\mathscr{X}_0)M_0)^{1-\theta}(C(\mathscr{X}_1)M_1)^{\theta}))^{\theta}$$

and in (3.3) $\tilde{C}\|T\|_{\vec{A}\to\vec{B}}$ can be replaced by

$$C_{\mathrm{opt}}(\|D_{0,1/s}\|_{\mathscr{X}_0(B_0)\to\mathscr{X}_0(B_0)}(C(\mathscr{X}_0)M_0 + e(C(\mathscr{X}_0)M_0)^{1-\theta}(C(\mathscr{X}_1)M_1)^{\theta}))^{1-\theta'}$$
$$\times (e^{-1}\|D_{1,s/e}\|_{\mathscr{X}_1(B_1)\to\mathscr{X}_1(B_1)}(C(\mathscr{X}_1)M_1 + e(C(\mathscr{X}_0)M_0)^{1-\theta}(C(\mathscr{X}_1)M_1)^{\theta}))^{\theta'}.$$

We leave the details to the reader.

Proof. When dealing with various sequences in $\mathscr{J}(\mathbf{X}, \vec{A})$ or $\mathscr{J}(\mathbf{X}, \vec{B})$ we shall (cf. earlier proofs for the complex method) tend to work more with the corresponding vector valued analytic functions on $\mathbb{A}$ which have those sequences as their Laurent coefficients.

It will be convenient to present part of the proof in the format of a preliminary lemma which we will also refer to later in the paper for other purposes:

LEMMA 3.11. *Let* $\mathbf{X}$ *be a pair of pseudolattices which admits differentiation and let* $\vec{B}$ *be a Banach pair. Let the sequence* $\{f_n\}_{n\in\mathbb{Z}}$ *be an element of* $\mathscr{J}(\mathbf{X},\vec{B})$ *and let* $f:\mathbb{A}\to B_0+B_1$ *be the analytic function defined by* $f(z)=\sum_{n\in\mathbb{Z}} z^n f_n$. *Suppose that* $f(s)=0$ *for some point* $s\in\mathbb{A}$ *and let* $g:\mathbb{A}\to B_0+B_1$ *be the analytic function obtained by setting* $g(s)=f'(s)$ *and* $g(z)=f(z)/(z-s)$ *for all* $z\in\mathbb{A}\backslash\{s\}$. *Let* $\{g_n\}_{n\in\mathbb{Z}}$ *be the sequence of coefficients in the Laurent expansion* $g(z)=\sum_{n\in\mathbb{Z}} z^n g_n$ *of* g *in* $\mathbb{A}$. *Then* $\{g_n\}_{n\in\mathbb{Z}}$ *is also an element of* $\mathscr{J}(\mathbf{X},\vec{B})$. *More specifically,*

$$\|\{g_n\}_{n\in\mathbb{Z}}\|_{\mathscr{J}(\mathbf{X},\vec{B})}\leqslant C\|\{f_n\}_{n\in\mathbb{Z}}\|_{\mathscr{J}(\mathbf{X},\vec{B})} \tag{3.5}$$

for some constant C *which can be bounded above by*

$$\max\{\|D_{0,1/s}\|_{\mathscr{X}_0(B_0)\to\mathscr{X}_0(B_0)}, e^{-1}\|D_{1,s/e}\|_{\mathscr{X}_1(B_1)\to\mathscr{X}_1(B_1)}\}.$$

To prove this lemma we first note that for each $z\in\mathbb{A}$ with $|z|>|s|$ we have

$$\begin{aligned} g(z) &= \frac{1}{z-s}\sum_{n\in\mathbb{Z}} z^n f_n = \frac{1}{z}\sum_{k\geqslant 0}\left(\frac{s}{z}\right)^k \sum_{n\in\mathbb{Z}} z^n f_n \\ &= \sum_{k\geqslant 0}\sum_{n\in\mathbb{Z}} z^{n-k-1}s^k f_n = \sum_{k\geqslant 0}\sum_{m\in\mathbb{Z}} z^m s^k f_{m+k+1}. \end{aligned}$$

Because of absolute convergence we can interchange the order of summation to obtain that the preceding expression equals $\sum_{m\in\mathbb{Z}} z^m \sum_{k\geqslant 0} s^k f_{m+k+1}$. Since the Laurent expansion of g is unique we must have $g_n=\sum_{k\geqslant 0} s^k f_{n+k+1}$. We shall need to deduce a second formula for g_n:

$$\begin{aligned} g_n &= s^{-n-1}\sum_{k\geqslant 0} s^{n+k+1} f_{n+k+1} = -s^{-n-1}\sum_{k<0} s^{n+k+1} f_{n+k+1} \quad (\text{since } f(s)=0) \\ &= -\sum_{k<0} s^k f_{n+k+1}, \end{aligned}$$

i.e.

$$\{g_n\}_{n\in\mathbb{Z}} = -D_{0,1/s}(\{f_n\}_{n\in\mathbb{Z}}). \tag{3.7}$$

Also $e^n g_n = e^{-1} \sum_{k \geqslant 0} (s/e)^k (e^{n+k+1} f_{n+k+1})$, i.e.

$$\{e^n g_n\}_{n\in\mathbb{Z}} = e^{-1} D_{1,s/e}(\{e^n f_n\}_{n\in\mathbb{Z}}). \tag{3.8}$$

Since $\mathbf{X}$ admits differentiation we obtain that $\{e^{jn} g_n\}_{n\in\mathbb{Z}} \in \mathscr{X}_j(B_j)$ for $j = 0, 1$, and furthermore

$$\{g_n\}_{n\in\mathbb{Z}} \in \mathscr{J}(\mathbf{X}, \vec{B}) \tag{3.9}$$

with norm not exceeding

$$\max\{\|D_{0,1/s}\|_{\mathscr{X}_0(B_0)\to\mathscr{X}_0(B_0)}, e^{-1}\|D_{1,s/e}\|_{\mathscr{X}_1(B_1)\to\mathscr{X}_1(B_1)}\}\|\{f_n\}_{n\in\mathbb{Z}}\|_{\mathscr{J}(\mathbf{X},\vec{B})}.$$

This completes the proof of the lemma.

We can now complete the proof of Theorem 3.8. We may suppose, without loss of generality, that $\|T\|_{\vec{A}\to\vec{B}} = 1$. Fix $a \in \vec{A}_{\mathbf{X},s}$, which we may also take to have norm 1. Then (cf. Theorem 2.14) the element $b = Ta$ is in $\vec{B}_{\mathbf{X},s}$ with norm not exceeding $C(\mathbf{X}) = \max_{j=0,1} C(\mathscr{X}_j)$. Let $\{a_n\}_{n\in\mathbb{Z}}$ be any sequence in $E_{\vec{A}}(a)$ and let $\{b_n\}_{n\in\mathbb{Z}}$ be any sequence in $E_{\vec{B}}(Ta)$. Since, by definition, $\|\{e^{jn}a_n\}_{n\in\mathbb{Z}}\|_{\mathscr{X}_j(A_j)} \leqslant C_{\text{opt}}$ for $j = 0, 1$, we deduce that $\|\{e^{jn}Ta_n\}_{n\in\mathbb{Z}}\|_{\mathscr{X}_j(B_j)} \leqslant C_{\text{opt}}C(\mathbf{X})$ for $j = 0, 1$. We also have by definition that $\|\{e^{jn}b_n\}_{n\in\mathbb{Z}}\|_{\mathscr{X}_j(B_j)} \leqslant C_{\text{opt}}C(\mathbf{X})$. Consequently, the sequence $\{f_n\}_{n\in\mathbb{Z}} :=$ $\{Ta_n - b_n\}_{n\in\mathbb{Z}}$ is in $\mathscr{J}(\mathbf{X}, \vec{B})$, with norm not exceeding $2C_{\text{opt}}C(\mathbf{X})$. Our hypotheses (cf. Remark 2.10) ensure that

$$\sum_{n\in\mathbb{Z}} s^n f_n = Ta - Ta = 0. \tag{3.10}$$

Thus we can apply Lemma 3.11 to the sequence $\{f_n\}_{n\in\mathbb{Z}}$. For f and g and $\{g_n\}_{n\in\mathbb{Z}}$ defined as in the statement of the lemma, this gives that $\{g_n\}_{n\in\mathbb{Z}} \in \mathscr{J}(\mathbf{X}, \vec{B})$ with norm not exceeding

$$C_1 = 2C_{\text{opt}}C(\mathbf{X})\max\{\|D_{0,1/s}\|_{\mathscr{X}_0(B_0)\to\mathscr{X}_0(B_0)}, e^{-1}\|D_{1,s/e}\|_{\mathscr{X}_1(B_1)\to\mathscr{X}_1(B_1)}\}.$$

Consequently, $f'(s) = g(s) = \sum_{n\in\mathbb{Z}} s^n g_n$ is in $\vec{B}_{\mathbf{X},s}$ also with norm not exceeding C_1. Also $g(s')$ must be in $\vec{B}_{\mathbf{X},s'}$, again with norm not exceeding C_1. So if we choose the above sequences $\{a_n\}_{n\in\mathbb{Z}}$ and $\{b_n\}_{n\in\mathbb{Z}}$ so that $a' = \Omega_{\vec{A}}(a) = \sum_{n\in\mathbb{Z}} ns^{n-1}a_n$ and $b' = \Omega_{\vec{B}}(Ta) = \sum_{n\in\mathbb{Z}} ns^{n-1}b_n$, then we shall have the required estimate for $T\Omega_{\vec{A}}(a) - \Omega_{\vec{B}}(Ta) = f'(s)$. Alternatively, if we choose $\{a_n\}_{n\in\mathbb{Z}}$ and $\{b_n\}_{n\in\mathbb{Z}}$ so that $a' = \mathscr{R}_{\vec{A}}(a) = \sum_{n\in\mathbb{Z}}(s')^n a_n$ and $b' = \mathscr{R}_{\vec{B}}(Ta) = \sum_{n\in\mathbb{Z}} (s')^n b_n$, then we shall have the required estimate for $T\mathscr{R}_{\vec{A}}(a) - \mathscr{R}_{\vec{B}}(Ta) = f(s') = (s' - s)g(s')$. ∎

4. COMPARISONS WITH EARLIER RESULTS

The purpose of this section is to show that, in the special cases of the complex and real interpolation methods, the operators Ω and $\mathscr{R}$, which we introduced and studied in Section 3, essentially coincide with analogous operators which appear in [27, 32, 34, 35, 42, 54] and elsewhere. This means, among other things, that a number of theorems in these papers, which are in the style of our Theorem 3.8, can be viewed as consequences of Theorem 3.8 (modulo possible changes in the constants appearing in the norm estimates). Conversely, the special cases of Theorem 3.8 for the real and complex methods could also be easily deduced from those theorems.

The essential difference between our definitions of Ω and $\mathscr{R}$ and those in the previous papers is that we have found it convenient to use interpolation spaces defined using a "discrete" definition (i.e. functions defined on an annulus) whereas previous papers used a "continuous" method (i.e. functions defined on a strip in the complex plane). Thus our proofs here amount to obtaining more elaborate versions of known results (see [17] for the complex method, and [41] or [4] for the real method) which show that "discrete" and "continuous" definitions give the same interpolation spaces, in each of these cases.

Let us first deal with the complex method. Let $\mathbb{S}$ be the "unit strip" $\mathbb{S} = \{z \in \mathbb{C} : 0 < \operatorname{Re} z < 1\}$. For any Banach pair $\vec{B}$ we let $\mathscr{F}_\infty(\vec{B})$ denote the space of bounded continuous functions $f : \bar{\mathbb{S}} \to B_0 + B_1$ which are analytic in $\mathbb{S}$ and such that, for $j = 0, 1$, the function $t \mapsto f(j + it)$ is continuous and bounded from $\mathbb{R}$ into B_j. We let $\mathscr{F}(\vec{B})$ be the space introduced in [6] which is the subspace of $\mathscr{F}_\infty(\vec{B})$ consisting of those functions for which $\lim_{|t| \to \infty} \|f(j + it)\|_{B_j} = 0$ for $j = 0, 1$. Both $\mathscr{F}_\infty(\vec{B})$ and $\mathscr{F}(\vec{B})$ are normed by

$$\|f\|_{\mathscr{F}(\vec{B})} = \sup_{j=0,1, t \in \mathbb{R}} \|f(j + it)\|_{B_j}.$$

For each $\theta \in [0, 1]$, Calderón's complex interpolation space $[\vec{B}]_\theta$ is defined by

$$[\vec{B}]_\theta = \{f(\theta) : f \in \mathscr{F}(\vec{B})\}$$

with norm

$$\|b\|_{[\vec{B}]_\theta} = \inf\{\|f\|_{\mathscr{F}(\vec{B})} : f \in \mathscr{F}(\vec{B}),\ f(\theta) = b\}.$$

It is well known and easy to show (e.g. with the help of scalar analytic functions $e^{\delta z^2}$ for small $\delta > 0$) that replacing $\mathscr{F}$ by $\mathscr{F}_\infty$ in the preceding definition gives the same interpolation space and the same norm.

Let us fix a point $\sigma = \theta + i\tau \in \mathbb{S}$ and some constant $C_{\rm opt} > 1$. Then, for each $b \in [\vec{B}]_\theta$, let $\mathscr{F}_\infty(b, C_{\rm opt}, \sigma, \vec{B})$ be the subset of $\mathscr{F}_\infty(\vec{B})$ consisting of those functions f which satisfy $f(\sigma) = b$ and $\|f\|_{\mathscr{F}(\vec{B})} \leqslant C_{\rm opt}\|b\|_{[\vec{B}]_\theta}$. We also set $\mathscr{F}(b, C_{\rm opt}, \sigma, \vec{B}) = \mathscr{F}_\infty(b, C_{\rm opt}, \sigma, \vec{B}) \cap \mathscr{F}(\vec{B})$.

The versions of the operators Ω and $\mathscr{R}$ which are encountered in previous papers dealing with the complex method correspond to (particular values of) the multivalued operators acting on $[\vec{B}]_\theta$ defined by

$$\tilde{\Omega}_*(b, C_{\rm opt}, \sigma, \vec{B}) = \{f'(\sigma) : f \in \mathscr{F}(b, C_{\rm opt}, \sigma, \vec{B})\} \tag{4.1}$$

and, for some fixed $\sigma' = \theta' + i\tau' \in \mathbb{S}$,

$$\tilde{\mathscr{R}}_*(b, C_{\rm opt}, \sigma, \sigma', \vec{B}) = \{f(\sigma') : f \in \mathscr{F}(b, C_{\rm opt}, \sigma, \vec{B})\}. \tag{4.2}$$

It will be a little more convenient to work with variants of these operators, which are obtained by replacing $\mathscr{F}$ by $\mathscr{F}_\infty$ in these last two definitions, i.e.

$$\tilde{\Omega}_{*,\infty}(b, C_{\rm opt}, \sigma, \vec{B}) = \{f'(\sigma) : f \in \mathscr{F}_\infty(b, C_{\rm opt}, \sigma, \vec{B})\} \tag{4.3}$$

and

$$\tilde{\mathscr{R}}_{*,\infty}(b, C_{\rm opt}, \sigma, \sigma', \vec{B}) = \{f(\sigma') : f \in \mathscr{F}_\infty(b, C_{\rm opt}, \sigma, \vec{B})\}. \tag{4.4}$$

Let us now observe that these variants are in fact almost the same as the original operators:

THEOREM 4.1. *For each* $\varepsilon > 0$,

$$\tilde{\Omega}_*(b, C_{\rm opt}, \sigma, \vec{B}) \subset \tilde{\Omega}_{*,\infty}(b, C_{\rm opt}, \sigma, \vec{B}) \subset \tilde{\Omega}_*(b, (1+\varepsilon)C_{\rm opt}, \sigma, \vec{B}) \tag{4.5}$$

and

$$\begin{aligned}\tilde{\mathscr{R}}_*(b, C_{\rm opt}, \sigma, \sigma', \vec{B}) &\subset \tilde{\mathscr{R}}_{*,\infty}(b, C_{\rm opt}, \sigma, \sigma', \vec{B}) \\ &\subset \tilde{\mathscr{R}}_*(b, (1+\varepsilon)C_{\rm opt}, \sigma, \sigma', \vec{B})\end{aligned} \tag{4.6}$$

Proof. Suppose that $a \in \tilde{\Omega}_{*,\infty}(b, C_{\rm opt}, \sigma, \vec{B})$ so that $a = f'(\sigma)$ for some $f \in \mathscr{F}_\infty(b, C_{\rm opt}, \sigma, \vec{B})$. For each $\delta > 0$ let $f_\delta(z) = e^{\delta(z-\sigma)^2} f(z)$ for all $z \in \bar{\mathbb{S}}$. Then $f_\delta \in \mathscr{F}(b, (1+\varepsilon)C_{\rm opt}, \sigma, \vec{B})$ for all sufficiently small δ and $f_\delta'(\sigma) = f'(\sigma) = a$. This establishes the second inclusion in (4.5), and the first inclusion is obvious. The proof of (4.6) is almost the same, except that this time we define $f_\delta(z) = e^{\delta((z-\sigma)^2 + (z-\sigma')^2 - (\sigma'-\sigma)^2)} f(z)$. ∎

In this section it will be convenient to use more detailed notation for the various sets or multivalued operators which we introduced in Definition 3.1: We shall use $E(b, C_{\rm opt}, s, \mathbf{X}, \vec{B})$ to denote the set of sequences $E(b)$.

Correspondingly, $\tilde{\Omega}(b, C_{\mathrm{opt}}, s, \mathbf{X}, \vec{B})$ will denote the set $\tilde{\Omega}(b)$ and $\tilde{\mathcal{R}}(b, C_{\mathrm{opt}}, s, s', \mathbf{X}, \vec{B})$ will denote $\tilde{\mathcal{R}}(b)$. Since we are currently dealing with the complex method we now consider the case where the pair of pseudolattices $\mathbf{X} = \{\mathcal{X}_0, \mathcal{X}_1\}$ is $\mathbf{FC} = \{FC, FC\}$.

We are now ready to compare $\tilde{\Omega}_{*,\infty}(b, C_{\mathrm{opt}}, \sigma, \vec{B})$ and $\tilde{\mathcal{R}}_{*,\infty}(b, C_{\mathrm{opt}}, \sigma, \sigma', \vec{B})$ with the operators $\tilde{\Omega}(b, C_{\mathrm{opt}}, s, \mathbf{FC}, \vec{B})$ and $\tilde{\mathcal{R}}(b, C_{\mathrm{opt}}, s, s', \mathbf{FC}, \vec{B})$ defined in Section 3, Definition 3.1.

THEOREM 4.2. *There exist absolute positive constants $C_\#$ and C_1, and, for each pair of points σ and σ' in $\mathbb{S}$, there exists a positive constant $C_{\sigma,\sigma'}$, depending only on those points, such that, for each Banach pair $\vec{B}$, each $b \in [\vec{B}]_{\mathrm{Re}\,\sigma}$ and each $C_{\mathrm{opt}} > 1$,*

$$\tilde{\Omega}(b, C_\#^{-1} C_{\mathrm{opt}}, e^{\sigma}, \mathbf{FC}, \vec{B}) \subset e^{-\sigma} \tilde{\Omega}_{*,\infty}(b, C_{\mathrm{opt}}, \sigma, \vec{B}) \subset \tilde{\Omega}(b, C_1 C_{\mathrm{opt}}, e^{\sigma}, \mathbf{FC}, \vec{B}). \tag{4.7}$$

Furthermore, provided $e^{\sigma} \neq e^{\sigma'}$,

$$\tilde{\mathcal{R}}(b, C_\#^{-1} C_{\mathrm{opt}}, e^{\sigma}, e^{\sigma'}, \mathbf{FC}, \vec{B}) \subset \tilde{\mathcal{R}}_{*,\infty}(b, C_{\mathrm{opt}}, \sigma, \sigma', \vec{B}) \subset \tilde{\mathcal{R}}(b, C_{\sigma,\sigma'} C_{\mathrm{opt}}, e^{\sigma}, e^{\sigma'}, \mathbf{FC}, \vec{B}). \tag{4.8}$$

For each $\delta \in (0, 2\pi)$, the constants $C_{\sigma,\sigma'}$ satisfy

$$\sup\{C_{\sigma,\sigma'} : \sigma, \sigma' \in \mathbb{S}, |\sigma - \sigma'| \leqslant \delta\} < \infty. \tag{4.9}$$

Remark 4.3. We cannot in general dispense with the condition $e^{\sigma} \neq e^{\sigma'}$ in (4.8). In the trivial case where $\sigma = \sigma'$, all three sets in (4.8) are either the singleton $\{b\}$ or the empty set and (4.8) in fact does hold. But if $e^{\sigma} = e^{\sigma'}$ with $\sigma \neq \sigma'$, then $\tilde{\mathcal{R}}(b, C_{\mathrm{opt}}, e^{\sigma}, e^{\sigma'}, \mathbf{FC}, \vec{B}) = \{b\}$ for all choices of $C_{\mathrm{opt}} > 1$. Then (4.8) does not hold because, again for all choices of $C_{\mathrm{opt}} > 1$, the set $\tilde{\mathcal{R}}_{*,\infty}(b, C_{\mathrm{opt}}, \sigma, \sigma', \vec{B})$ will contain all elements of the form $b + a$ where $\|a\|_{B_0 \cap B_1}$ is smaller than some positive number depending on C_{opt}, σ and σ'. An additional argument, which we defer to an appendix (Subsection A.3), shows that, in general, $C_{\sigma,\sigma'}$ cannot remain bounded as $e^{\sigma'}$ becomes arbitrarily close to e^{σ}.

Remark 4.4. The boundedness condition (4.9) is needed if one wishes to show that results in the style of (3.3) in the settings of earlier papers imply (3.3) in the case of the complex method. We defer further discussion of the behaviour of $C_{\sigma,\sigma'}$ to Remark 4.6.

Proof. Let $\mathscr{F}_{2\pi}(\vec{B})$ be the subspace of $\mathscr{F}(\vec{B})$ consisting of those functions f which satisfy $f(z+2\pi i)=f(z)$ for all $z\in\bar{\mathbb{S}}$. The absolute constant $C_\#$ in the statement of the theorem will be the same as appears in the result in [17] that the space $[\vec{B}]_\theta^{2\pi}=\{f(\theta): f\in\mathscr{F}_{2\pi}(\vec{B})\}$ coincides with $[\vec{B}]_\theta$ for each $\theta\in(0,1)$ and the norm $\|b\|_{[\vec{B}]_\theta^{2\pi}}=\inf\{\|f\|_{\mathscr{F}(\vec{B})}: f\in\mathscr{F}_{2\pi}(\vec{B}), f(\theta)=b\}$ satisfies

$$\|b\|_{[\vec{B}]_\theta}\leqslant\|b\|_{[\vec{B}]_\theta^{2\pi}}\leqslant C_\#\|b\|_{[\vec{B}]_\theta}\qquad\text{for each } b\in[\vec{B}]_\theta. \tag{4.10}$$

Our proof here will in fact include and extend that result (cf. Remark 4.5).

We observe that each $f\in\mathscr{F}_{2\pi}(\vec{B})$ corresponds to a unique function $G_f: \bar{\mathbb{A}}\to B_0+B_1$ which is analytic in $\mathbb{A}$ and satisfies $f(z)=G_f(e^z)$ for all $z\in\bar{\mathbb{S}}$. We consider the sequence of Fourier coefficients $\{\hat{f}_n\}_{n\in\mathbb{Z}}$ of f, or equivalently, Laurent coefficients of G_f, defined, for each $\lambda\in[0,1]$, by

$$\hat{f}_n=\frac{1}{2\pi}\int_0^{2\pi}e^{-(\lambda+it)n}f(\lambda+it)\,dt=\frac{1}{2\pi i}\oint_{|\zeta|=e^\lambda}\frac{G_f(\zeta)}{\zeta^{n+1}}\,d\zeta. \tag{4.11}$$

Of course, by Cauchy's theorem, their values are independent of λ. Furthermore, $\{\hat{f}_n\}_{n\in\mathbb{Z}}\in\mathscr{J}(\mathbf{FC},\vec{B})$ with

$$\|\{\hat{f}_n\}_{n\in\mathbb{Z}}\|_{\mathscr{J}(\mathbf{FC},\vec{B})}=\|f\|_{\mathscr{F}(\vec{B})}. \tag{4.12}$$

Conversely, given any sequence $\{b_n\}_{n\in\mathbb{Z}}\in\mathscr{J}(\mathbf{FC},\vec{B})$, let $G(\zeta)=\sum_{n\in\mathbb{Z}}\zeta^n b_n$. Clearly the $(C,1)$ means of the partial sums of this series converge uniformly in B_j on $\{|\zeta|=e^j\}$ for $j=0,1$ and therefore uniformly in B_0+B_1 on $\bar{\mathbb{A}}$. Thus the function $h(z)=G(e^z)$ is an element of $\mathscr{F}_{2\pi}(\vec{B})$ with $\|h\|_{\mathscr{F}(\vec{B})}=\|\{b_n\}_{n\in\mathbb{Z}}\|_{\mathscr{J}(\mathbf{FC},\vec{B})}$.

The preceding remarks show that, for any $\sigma\in\mathbb{S}$, we have $\vec{B}_{\mathbf{FC},s}=[\vec{B}]_\theta^{2\pi}=[\vec{B}]_\theta$, where $s=e^\sigma$ and $\theta=\ln|s|=\operatorname{Re}\sigma$ and

$$\|b\|_{[\vec{B}]_\theta}\leqslant\|b\|_{[\vec{B}]_\theta^{2\pi}}=\|b\|_{\vec{B}_{\mathbf{FC},s}}\leqslant C_\#\|b\|_{[\vec{B}]_\theta}\qquad\text{for each } b\in[\vec{B}]_\theta. \tag{4.13}$$

We can now easily obtain the first inclusions in (4.47) and (4.48). Each $b'\in\tilde{\Omega}(b,C_\#^{-1}C_{\text{opt}},e^\sigma,\mathbf{FC},\vec{B})$ and each $b''\in\tilde{\mathscr{R}}(b,C_\#^{-1}C_{\text{opt}},e^\sigma,e^{\sigma'},\mathbf{FC},\vec{B})$ are of the forms $b'=\sum_{n\in\mathbb{Z}}n(e^\sigma)^{n-1}b_n$ and $b''=\sum_{n\in\mathbb{Z}}(e^{\sigma'})^n b_n$ respectively, for some choices of sequences $\{b_n\}_{n\in\mathbb{Z}}\in E(b,C_\#^{-1}C_{\text{opt}},e^\sigma,\mathbf{FC},\vec{B})$. For each such sequence $\{b_n\}_{n\in\mathbb{Z}}$, the function $f(z)=\sum_{n\in\mathbb{Z}}e^{nz}b_n$ is an element of $\mathscr{F}_{2\pi}(\vec{B})$ with $\|f\|_{\mathscr{F}(\vec{B})}=\|\{b_n\}_{n\in\mathbb{Z}}\|_{\mathscr{J}(\mathbf{FC},\vec{B})}\leqslant C_\#^{-1}C_{\text{opt}}\|b\|_{\vec{B}_{\mathbf{FC},e^\sigma}}\leqslant C_{\text{opt}}\|b\|_{[\vec{B}]_\theta}$. Thus $b'=\sum_{n\in\mathbb{Z}}n(e^\sigma)^{n-1}b_n=e^{-\sigma}f'(\sigma)\in e^{-\sigma}\tilde{\Omega}_{*,\infty}(b,C_{\text{opt}},\sigma,\vec{B})$ and $f(\sigma')=\sum_{n\in\mathbb{Z}}e^{n\sigma'}b_n\in\tilde{\mathscr{R}}_{*,\infty}(b,C_{\text{opt}},\sigma,\sigma',\vec{B})$, which proves these inclusions. Of course in the preceding argument we assumed that the set $E(b,C_\#^{-1}C_{\text{opt}},e^\sigma,\mathbf{FC},\vec{B})$ is nonempty. But if it is empty, as happens for example if $b\neq 0$ and $C_{\text{opt}}<C_\#$, then there is nothing to prove.

The proofs of the second inclusions in (4.47) and(4.48) are where we elaborate upon the methods of [17]. We shall need to use some scalar valued analytic functions with some special properties. Let H_0 be the set of all entire functions $\psi : \mathbb{C} \to \mathbb{C}$ which satisfy

$$\psi(0) = 1 \tag{4.14}$$

and

$$\psi(2\pi ni) = 0 \qquad \text{for all } n \in \mathbb{Z} \text{ with } n \neq 0 \tag{4.15}$$

and, for each $\rho > 0$, there exists a constant $C(\psi, \rho)$ depending only on ψ and ρ such that

$$|\psi(s + it)| \leqslant C(\psi, \rho)e^{-t^2} \qquad \text{for all } s \in [-\rho, \rho] \text{ and all } t \in \mathbb{R}. \tag{4.16}$$

For example, the function $\psi_0 : \mathbb{C} \to \mathbb{C}$ defined by

$$\psi_0(0) = 1 \quad \text{and} \quad \psi_0(z) = e^{z^2} \frac{e^z - 1}{z} \qquad \text{for all } z \neq 0 \tag{4.17}$$

is an element of H_0. For our purposes we shall need several more functions in H_0. The first of these, ψ_1, is defined by

$$\psi_1(0) = 1 \quad \text{and} \quad \psi_1(z) = e^{z^2} \frac{e^{z/2} - e^{-z/2}}{z} \qquad \text{for all } z \neq 0. \tag{4.18}$$

Apart from being in H_0, ψ_1 is an even function and so also satisfies $\psi_1'(0) = 0$. Our next function ψ_2, defined by $\psi_2(z) = (\psi_1(z))^2$ is also in H_0 and satisfies

$$\psi_2'(2\pi ni) = 0 \qquad \text{for all } n \in \mathbb{Z}. \tag{4.19}$$

Now, given any $b' \in e^{-\sigma}\tilde{\Omega}_{*,\infty}(b, C_{\text{opt}}, \sigma, \vec{B})$, let f be an element of $\mathscr{F}_{\infty}(b, C_{\text{opt}}, \sigma, \vec{B})$ such that $b' = e^{-\sigma} f'(\sigma)$. We define a new function $F : \mathbb{S} \to B_0 + B_1$

$$F(z) = \sum_{n \in \mathbb{Z}} \psi_2(z - \sigma + 2\pi ni) f(z + 2\pi ni). \tag{4.20}$$

We introduce the finite constants

$$\gamma = \sup_{t \in \mathbb{R}} \sum_{n \in \mathbb{Z}} e^{-(2\pi n - t)^2} = \sup_{t \in [0, 2\pi]} \sum_{n \in \mathbb{Z}} e^{-(2\pi n - t)^2} \quad \text{and}$$
$$C_1 = \gamma C(\psi_2, 1). \tag{4.21}$$

Since ψ_2 satisfies condition (4.16), it follows that series (4.20) converges uniformly in $B_0 + B_1$ norm on every compact subset of $\bar{\mathbb{S}}$ and satisfies $\sup_{z\in\bar{\mathbb{S}}} \|F(z)\|_{B_0+B_1} \leqslant C_1 \sup_{z\in\bar{\mathbb{S}}} \|f(z)\|_{B_0+B_1}$. For similar reasons, the same series also converges uniformly in B_j norm on every compact subset of the line $\{j + it: t \in \mathbb{R}\}$ and satisfies $\sup_{t\in\mathbb{R}} \|F(j + it)\|_{B_j} \leqslant C_1 \sup_{t\in\mathbb{R}} \|f(j + it)\|_{B_j}$, for $j = 0, 1$. This means that $F \in \mathscr{F}_\infty(\vec{B})$. We also see that $F \in \mathscr{F}_{2\pi}(\vec{B})$ and $F(\sigma) = f(\sigma) = b$. Furthermore, since series (4.20) can be differentiated term-by-term for all $z \in \mathbb{S}$, we have, using (4.19) and the fact that $\psi_2(0) = 1$, that $F'(\sigma) = f'(\sigma) = e^\sigma b'$. Let $G_F: \bar{\mathbb{A}} \to B_0 + B_1$ be the continuous function which is analytic on $\mathbb{A}$ such that $F(z) = G_F(e^z)$ and let $\{b_n\}_{n\in\mathbb{Z}}$ be the sequence of Fourier coefficients of F, i.e., Laurent coefficients of G_F defined as in (4.11). The arguments given above (cf. (4.11), (4.12), etc.) show that $\{b_n\}_{n\in\mathbb{Z}} \in \mathscr{J}(\mathbf{FC}, \vec{B})$ with $\|\{b_n\}_{n\in\mathbb{Z}}\|_{\mathscr{J}(\mathbf{FC},\vec{B})} = \|F\|_{\mathscr{F}(\vec{B})} \leqslant C_1 \|f\|_{\mathscr{F}(\vec{B})} \leqslant C_1 C_{\mathrm{opt}} \|b\|_{[\vec{B}]_{\mathrm{Re}\,\sigma}} \leqslant C_1 C_{\mathrm{opt}} \|b\|_{\vec{B}_{\mathbf{FC},e^\sigma}}$. This implies that the series $\sum_{n\in\mathbb{Z}} \zeta^n b_n$ and $\sum_{n\in\mathbb{Z}} n\zeta^{n-1} b_n$ both converge in $B_0 + B_1$ for all $\zeta \in \mathbb{A}$ and their sums are necessarily $G_F(\zeta)$, and its derivative $G_F'(\zeta)$, respectively. In particular, when $\zeta = e^\sigma$ we get $G_F(e^\sigma) = F(\sigma) = b$ and $e^\sigma G_F'(e^\sigma) = F'(\sigma) = e^\sigma b'$. This shows that $b' \in \tilde{\Omega}(b, C_1 C_{\mathrm{opt}}, e^\sigma, \mathbf{FC}, \vec{B})$ and so we have established the second inclusion of (4.7).

The proof of the second inclusion of (4.8) is similar. We will need to use yet another entire function ψ_3. Let us first note that the imposed condition $e^\sigma \neq e^{\sigma'}$, i.e. $\frac{\sigma-\sigma'}{2\pi i} \notin \mathbb{Z}$ ensures that $\psi_1(\sigma - \sigma') \neq 0$. So we can define

$$\psi_3(z) = \frac{\psi_1(z-\sigma)\psi_1(z-\sigma')}{\psi_1(\sigma-\sigma')} \qquad \text{for all } z \in \mathbb{C}. \tag{4.22}$$

Then $\psi_3(\sigma) = \psi_3(\sigma') = 1$ and $\psi_3(\sigma + 2\pi ni) = \psi_3(\sigma' + 2\pi ni) = 0$ for every nonzero integer n. We also have $|\psi_3(s + it)| \leqslant C(\sigma, \sigma') e^{-(t-\mathrm{Re}\,\sigma)^2}$ for all $s \in [-1, 1]$ and $t \in \mathbb{R}$, where

$$C(\sigma, \sigma') = C(\psi_1, 1)^2 / |\psi(\sigma - \sigma')|. \tag{4.23}$$

This gives, for γ as in (4.21), that

$$\sup_{z\in\mathbb{S}} \sum_{n\in\mathbb{Z}} |\psi_3(z + 2\pi ni)| \leqslant \gamma C(\sigma, \sigma'). \tag{4.24}$$

Now, given any $b'' \in \tilde{\Omega}_{*,\infty}(b, C_{\mathrm{opt}}, \sigma, \sigma', \vec{B})$, let f be an element of $\mathscr{F}_\infty(b, C_{\mathrm{opt}}, \sigma, \vec{B})$ such that $b'' = f(\sigma')$. This time we define a new function $F : \bar{\mathbb{S}} \to B_0 + B_1$ by

$$F(z) = \sum_{n\in\mathbb{Z}} \psi_3(z + 2\pi ni) f(z + 2\pi ni). \tag{4.25}$$

Analogous to the previous step, we obtain from (4.24) and the other properties of ψ_3 that $F \in \mathscr{F}_{2\pi}(\vec{B})$ with $\|F\|_{\mathscr{F}(\vec{B})} \leqslant C_{\sigma,\sigma'}\|f\|_{\mathscr{F}(\vec{B})} \leqslant C_{\sigma,\sigma'}C_{\text{opt}}\|b\|_{[\vec{B}]_{\text{Re}\,\sigma}} \leqslant C_{\sigma,\sigma'}C_{\text{opt}}\|b\|_{\vec{B}_{\mathbf{FC},e^\sigma}}$ where $C_{\sigma,\sigma'} = \gamma C(\sigma,\sigma')$ depends only on σ and σ'. We also have $F(\sigma) = f(\sigma) = b$ and $F(\sigma') = f(\sigma') = b''$. We obtain the function G_F and the sequence $\{b_n\}_{n\in\mathbb{Z}}$ from F exactly as in the previous step. Thus we have $G_F(e^\sigma) = b$ and $G_F(e^{\sigma'}) = b''$ and so $b'' \in \tilde{\mathscr{R}}(b, C_{\sigma,\sigma'}C_{\text{opt}}, e^\sigma, e^{\sigma'}, \mathbf{FC}, \vec{B})$. This completes the proof of (4.8). Finally we deduce (4.9) from the fact that $\inf\{|\psi_1(\sigma-\sigma')| : \sigma,\sigma' \in \mathbb{S}, |\sigma-\sigma'| \leqslant \delta\} > 0$. ∎

Remark 4.5. We have not sought to find the optimal value of $C_\#$ and this problem is not addressed in [17] either. To obtain a crude estimate, we can observe that the proof in [17] of (4.10) uses a formula just like (4.20) but there ψ_2 can also be replaced, for example, by ψ_0. Since $\sup_{s\in[0,1]}|\psi_0(s+it)| \leqslant e^{1-t^2}\max\{e+1, \sup_{|z|}|\frac{e^z-1}{z}|\}$ we obtain, for γ as in (4.21), that

$$1 \leqslant C_\# \leqslant \gamma e \max\left\{e+1, \sup_{|z|\leqslant 1}\left|\frac{e^z-1}{z}\right|\right\}.$$

Remark 4.6. It seems that the most interesting aspects of the behaviour of the operators $\tilde{\mathscr{R}}$ are when σ and σ' are close to each other. But we can also describe the behaviour of the constant $C_{\sigma,\sigma'}$ appearing in (4.8) when $|\sigma-\sigma'|$ is large. The upper bound which we obtained in the proof of Theorem 4.2 for $C_{\sigma,\sigma'}$ becomes arbitrarily large as $|\sigma-\sigma'|$ tends to ∞, even if we restrict ourselves to a subset of σ and σ' where $|e^\sigma - e^{\sigma'}|$ is bounded from below by some positive number. However, there is an alternative approach which gives a uniform estimate for $C_{\sigma,\sigma'}$. More precisely, for each $\delta > 0$, we can show that

$$\sup\{C_{\sigma,\sigma'} : \sigma,\sigma' \in \mathbb{S}, |e^\sigma - e^{\sigma'}| \geqslant \delta\} < \infty. \tag{4.26}$$

We defer the proof of this to an appendix (Subsection A.2).

We now turn our attention to analogous results for the real method. This means that we must now take the pair of pseudolattices $\mathbf{X}$ to be $\{\ell^p, \ell^p\}$, for which we will use the abbreviated notation $\mathbf{l}^p$. We shall consider p in the "usual" range $1 \leqslant p \leqslant \infty$ (although of course there is a great deal that can be done in other contexts of real interpolation for p beyond this range).

For any Banach pair $\vec{B}$ we define $\mathscr{J}_r(\vec{B},p)$ to be the space of strongly measurable functions $v : \mathbb{R} \to B_0 \cap B_1$ such that

$$\|v\|_{\mathscr{J}_r(\vec{B},p)} = \max_{j=0,1}\left(\int_{-\infty}^{\infty}\|e^{jx}v(x)\|_{B_j}^p\,dx\right)^{1/p} < \infty.$$

As usual, if $p = \infty$, the integrals are replaced by essential suprema. For each $\theta \in (0,1)$ it is clear (cf. [4, 41] and some remarks below) that the real method interpolation space $\vec{B}_{\theta,p}$ is the set of all elements of the form $b = \int_{-\infty}^{\infty} e^{\theta x} v(x)\,dx$ for some $v \in \mathscr{J}_r(\vec{B}, p)$, and that any norm for $\vec{B}_{\theta,p}$ is equivalent to

$$\|b\|^{\mathbb{R}}_{\vec{B}_{\theta,p}} = \inf\left\{ \|v\|_{\mathscr{J}_r(\vec{B},p)} : v \in \mathscr{J}_r(\vec{B},p),\ b = \int_{-\infty}^{\infty} e^{\theta x} v(x)\,dx \right\}.$$

In particular, the norms $\|\cdot\|^{\mathbb{R}}_{\vec{B}_{\theta,p}}$ and $\|\cdot\|^{\mathbb{Z}}_{\vec{B}_{\theta,p}}$ are equivalent, where (cf. Definition 2.11 with $s = e^{\theta}$ and $\mathbf{X} = \mathbf{l}^p$)

$$\|b\|^{\mathbb{Z}}_{\vec{B}_{\theta,p}} = \|b\|_{\vec{B}_{\mathbf{l}^p, e^{\theta}}} = \inf\left\{ \|\{b_n\}_{n\in\mathbb{Z}}\|_{\mathscr{J}(\mathbf{l}^p,\vec{B})} : b = \sum_{n\in\mathbb{Z}} e^{\theta n} b_n \right\}.$$

We remark that if θ is replaced by an arbitrary point $\sigma \in \mathbb{S}$, then all the above definitions still make sense, and the various norms obtained are equal to those that would be obtained with $\operatorname{Re}\sigma$ in place of σ.

Given any $\sigma = \theta + i\tau \in \mathbb{S}$, and any $b \in \vec{B}_{\theta,p}$ and some positive constant C_{opt}, let $\mathscr{J}_r(b, C_{\text{opt}}, \sigma, \vec{B}, p)$ be the subset of $\mathscr{J}_r(\vec{B}, p)$ consisting of those functions v which satisfy $\int_{-\infty}^{\infty} e^{\sigma x} v(x) dx = b$ and $\|v\|_{\mathscr{J}_r(\vec{B},p)} \leqslant C_{\text{opt}} \|b\|^{\mathbb{R}}_{\vec{B}_{\theta,p}}$. The operators Ω and $\mathscr{R}$ encountered in previous papers dealing with the real method correspond to (particular values of) the multivalued operators acting on $\vec{B}_{\theta,p}$ defined by

$$\tilde{\Omega}_r(b, C_{\text{opt}}, \sigma, \vec{B}, p) = \left\{ \int_{-\infty}^{\infty} x e^{\sigma x} v(x)\,dx : v \in \mathscr{J}_r(b, C_{\text{opt}}, \sigma, \vec{B}, p) \right\} \tag{4.27}$$

and, for some fixed $\sigma' = \theta' + i\tau' \in \mathbb{S}$,

$$\tilde{\mathscr{R}}_r(b, C_{\text{opt}}, \sigma, \sigma', \vec{B}, p) = \left\{ \int_{-\infty}^{\infty} e^{\sigma' x} v(x)\,dx : v \in \mathscr{J}_r(b, C_{\text{opt}}, \sigma, \vec{B}, p) \right\}. \tag{4.28}$$

We have obtained the preceding definitions from corresponding ones appearing variously in [4, 20, 32] and implicitly in [63] by two trivial transformations. In "standard" definitions, the space $\vec{B}_{\theta,p}$ is defined to be the set of all elements of the form $b = \int_0^{\infty} u(t)\frac{dt}{t}$ for a suitable class of functions $u : (0,\infty) \to B_0 \cap B_1$. Here we have replaced each such function u by a function $v : \mathbb{R} \to B_0 \cap B_1$ which satisfies $u(t) = t^{\sigma} v(\ln t)$, and then we have used the change of variables $x = \ln t$ to move from integrals on $((0,\infty), \frac{dt}{t})$ to integrals on $(\mathbb{R}, dx)$. We have thus returned to a notation which is closer to that appearing in the seminal Lions–Peetre paper [41] and also more convenient for our purposes here. Definition (4.27) corresponds to the J-functional definition of an operator Ω given in [32]. We recall that the

K-functional variant of this definition given in that paper was shown in [19, Theorem 2.8 (p. 602)] to give the essentially the same operator.

THEOREM 4.7. *There exists an absolute positive constant* C^* *and, for each pair of points* σ *and* σ' *in* $\mathbb{S}$, *there exist positive constants* $C^{\#}_{\sigma,\sigma'}$ *and* $C^*_{\sigma,\sigma'}$, *depending only on those points, such that, for each Banach pair* $\vec{B}$, *each* $p \in [1,\infty]$, *each* $b \in \vec{B}_{\mathrm{Re}\,\sigma,p}$ *and each* $C_{\mathrm{opt}} > 1$,

$$\tilde{\Omega}(b, e^{-1/2}C_{\mathrm{opt}}, e^{\sigma}, \ell^p, \vec{B}) \subset \tilde{\Omega}_r(b, C_{\mathrm{opt}}, \sigma, \vec{B}, p) \subset \tilde{\Omega}(b, C^* C_{\mathrm{opt}}, e^{\sigma}, \ell^p, \vec{B}) \quad (4.29)$$

and, provided $e^{\sigma} \neq e^{\sigma'}$,

$$\tilde{\mathscr{R}}(b, (C^{\#}_{\sigma,\sigma'})^{-1} C_{\mathrm{opt}}, e^{\sigma}, e^{\sigma'}, \ell^p, \vec{B}) \subset \tilde{\mathscr{R}}_r(b, C_{\mathrm{opt}}, \sigma, \sigma', \vec{B}, p)$$
$$\subset \tilde{\mathscr{R}}(b, C^*_{\sigma,\sigma'} C_{\mathrm{opt}}, e^{\sigma}, e^{\sigma'}, \ell^p, \vec{B}). \quad (4.30)$$

For each $\delta \in (0, 2\pi)$ *the constants* $C^{\#}_{\sigma,\sigma'}$ *and* $C^*_{\sigma,\sigma'}$ *satisfy*

$$\sup\{C^{\#}_{\sigma,\sigma'} : \sigma, \sigma' \in \mathbb{S}, |\sigma - \sigma'| \leqslant \delta\} < \infty \quad (4.31)$$

and

$$\sup\{C^*_{\sigma,\sigma'} : \sigma, \sigma' \in \mathbb{S}, |\sigma - \sigma'| \leqslant \delta\} < \infty. \quad (4.32)$$

Remark 4.8. For similar reasons to those given in Remark 4.3, we cannot dispense in general with the condition $e^{\sigma} \neq e^{\sigma'}$. We have not sought to find the sharpest estimates for the constants appearing in (4.29) and (4.30). We have preferred instead to find constants which do not depend on some of the parameters, even if this means they are larger. Although we shall not pursue this here, it seems likely that the argument in Subsection A.2 can be adapted to show that, when $|\sigma - \sigma'|$ is not too small, the set $\tilde{\mathscr{R}}_r(b, C_{\mathrm{opt}}, \sigma, \sigma', \vec{B}, p)$ is uniformly comparable with the ball $C_{\mathrm{opt}} \|b\|^{\mathbb{R}}_{\vec{B}_{\mathrm{Re}\,\sigma,p}} \mathscr{B}_{\vec{B}_{\mathrm{Re}\,\sigma',p}}$. This in turn should make it possible to show that the constants $C^*_{\sigma,\sigma'}$ and $C^{\#}_{\sigma,\sigma'}$ are bounded above on the set $\{(\sigma,\sigma') \in \mathbb{S} \times \mathbb{S} : |e^{\sigma} - e^{\sigma'}| \geqslant \delta\}$ for each $\delta > 0$, and also (cf. Subsection A.3) that $C^*_{\sigma,\sigma'}$ is unbounded as $|e^{\sigma} - e^{\sigma'}|$ tends to zero.

Proof. Let α and β be any two complex numbers. Given any $\{b_n\}_{n\in\mathbb{Z}} \in \mathscr{J}(\ell^p, \vec{B})$ we define a strongly measurable function $v_\alpha : \mathbb{R} \to B_0 \cap B_1$ by setting

$$v_\alpha(x) = \sum_{n\in\mathbb{Z}} e^{\alpha(n-x)}\chi_{(n-1/2,n+1/2]}(x)b_n. \tag{4.33}$$

Then

$$\begin{aligned}\int_{-\infty}^{\infty} e^{\beta x}v_\alpha(x)\,dx &= \sum_{n\in\mathbb{Z}} e^{\alpha n}\left(\int_{n-1/2}^{n+1/2} e^{(\beta-\alpha)x}\,dx\right)b_n\\ &= \left(\frac{e^{(\beta-\alpha)/2}-e^{(\alpha-\beta)/2}}{\beta-\alpha}\right)\sum_{n\in\mathbb{Z}} e^{\beta n}b_n,\end{aligned} \tag{4.34}$$

where of course the last expression in parentheses is replaced by 1 if $\alpha=\beta$.

If $\alpha\in\mathbb{S}$ then, for $j=0,1$ and each $p\in[1,\infty)$,

$$\begin{aligned}\left(\int_{n-1/2}^{n+1/2}\|e^{jx}v_\alpha(x)\|_{B_j}^p\,dx\right)^{1/p} &= e^{\mathrm{Re}\,\alpha n}\|b_n\|_{B_j}\left(\int_{n-1/2}^{n+/2} e^{p(j-\mathrm{Re}\,\alpha)x}\,dx\right)^{1/p}\\ &\leqslant e^{\mathrm{Re}\,\alpha n}\|b_n\|_{B_j}\sup_{x\in[n-1/2,n+1/2]} e^{(j-\mathrm{Re}\,\alpha)x}\\ &= e^{\mathrm{Re}\,\alpha n}\|b_n\|_{B_j}e^{(j-\mathrm{Re}\,\alpha)n+|j-\mathrm{Re}\,\alpha|/2}\\ &= e^{jn}\|b_n\|_{B_j}e^{|j-\mathrm{Re}\,\alpha|/2}\leqslant e^{jn}\|b_n\|_{B_j}\sqrt{e},\end{aligned}$$

and the same estimate holds for $p=\infty$. Consequently,

$$\left(\int_{-\infty}^{\infty}\|e^{jx}v_\alpha(x)\|_{B_j}^p\,dx\right)^{1/p}\leqslant\sqrt{e}\|\{e^{jn}b_n\}_{n\in\mathbb{Z}}\|_{\ell^p(B_j)} \tag{4.35}$$

for $j=0,1$ and $p\in[1,\infty]$.

We can now prove the first inclusion of (4.29). Given any $b'\in\tilde{\Omega}(b,e^{-1/2}C_{\mathrm{opt}},e^\sigma,\mathbf{l}^p,\vec{\boldsymbol{B}})$ we choose a sequence $\{b_n\}_{n\in\mathbb{Z}}\in E(b,e^{-1/2}C_{\mathrm{opt}},e^\sigma,\mathbf{l}^p,\vec{\boldsymbol{B}})$ satisfying $b'=\sum_{n\in\mathbb{Z}} ne^{\sigma n}b_n$. We set $\alpha=\sigma$ in (4.33) and let $v=v_\sigma$. Then (cf. (4.34)) we have $\int_{-\infty}^{\infty} e^{\sigma x}v(x)=\sum_{n\in\mathbb{Z}} e^{\sigma n}b_n=b$ and also

$$\int_{-\infty}^{\infty} xe^{\sigma x}v(x)\,dx=\sum_{n\in\mathbb{Z}} ne^{\sigma n}b_n=b'. \tag{4.36}$$

Also, it follows from (4.35) that $v\in\mathscr{J}_r(b,C_{\mathrm{opt}},\sigma,\vec{\boldsymbol{B}},p)$. We deduce that $b'\in\tilde{\Omega}_r(b,C_{\mathrm{opt}},\sigma,\vec{\boldsymbol{B}},p)$, establishing the required inclusion. We remark that this step also shows that

$$\|b\|^{\mathbb{R}}_{\vec{\boldsymbol{B}}_{\theta,p}}\leqslant\sqrt{e}\|b\|^{\mathbb{Z}}_{\vec{\boldsymbol{B}}_{\theta,p}}, \tag{4.37}$$

where $\theta = \operatorname{Re}\sigma$, and is adapted from the proof in Lemma 3.2.3 of [41, pp. 18–19] or [4, p. 43] that the norms $\|\cdot\|^{\mathbb{Z}}_{\vec{B}_{\theta,p}}$ and $\|\cdot\|^{\mathbb{R}}_{\vec{B}_{\theta,p}}$ satisfy $\|b\|^{\mathbb{R}}_{\vec{B}_{\theta,p}} \leqslant$ const. $\|b\|^{\mathbb{Z}}_{\vec{B}_{\theta,p}}$. In fact a stronger version of (4.37) holds, namely

$$\|b\|^{\mathbb{R}}_{\vec{B}_{\theta,p}} \leqslant e^{\theta(1-\theta)}\|b\|^{\mathbb{Z}}_{\vec{B}_{\theta,p}} \tag{4.38}$$

as is shown in [41] essentially by taking $v(x)$ to be $e^{\sigma\tau}v_\sigma(x+\tau)$ instead of $v_\sigma(x)$ for $\tau = \frac{1}{2} - \theta$. (But this v does not satisfy (4.36).)

We next give an analogous argument to prove the first inclusion of (4.30). First, note that the condition $e^\sigma \neq e^{\sigma'}$ implies (and is in fact equivalent to) $\sinh(\frac{\sigma-\sigma'}{2}) \neq 0$. Given any $b' \in \tilde{\mathscr{R}}(b, (C^{\#}_{\sigma,\sigma'})^{-1}C_{\text{opt}}, e^\sigma, e^{\sigma'}, \mathbf{l}^p, \vec{B})$ and an associated sequence $\{b_n\}_{n\in\mathbb{Z}} \in E(b, C^{-1}_{\sigma\sigma',p}C_{\text{opt}}, e^\sigma, \mathbf{l}^p, \vec{B})$ we choose $\alpha = (\sigma+\sigma')/2$ in (4.33) and let $v = \frac{(\sigma-\sigma')/2}{\sinh((\sigma-\sigma')/2)}v_\alpha$. These choices are made because then, for $\beta = \sigma$ and also for $\beta = \sigma'$, the expression in parentheses in (4.34) has the same value $\frac{\sinh((\sigma-\sigma')/2)}{(\sigma-\sigma')/2}$. So (4.34) gives

$$\int_{-\infty}^{\infty} e^{\sigma x}v(x)\,dx = \sum_{n\in\mathbb{Z}} e^{\sigma n}b_n = b$$

and also

$$\int_{-\infty}^{\infty} e^{\sigma' x}v(x)\,dx = \sum_{n\in\mathbb{Z}} e^{\sigma' n}b_n = b'.$$

By (4.35) we have

$$\begin{aligned}\|v\|_{\mathscr{J}_r(\vec{B},p)} &\leqslant \sqrt{e}\left|\frac{(\sigma-\sigma')/2}{\sinh((\sigma-\sigma')/2)}\right| \|\{b_n\}_{n\in\mathbb{Z}}\|_{\mathscr{J}(\mathbf{l}^p,\vec{B})} \\ &\leqslant \sqrt{e}\left|\frac{(\sigma-\sigma')/2}{\sinh((\sigma-\sigma')/2)}\right| (C^{\#}_{\sigma,\sigma'})^{-1}C_{\text{opt}}\|b\|_{\vec{B}_{\mathbf{l}^p,e^\sigma}}.\end{aligned}$$

If we choose $C^{\#}_{\sigma,\sigma'} = \sqrt{e}\left|\frac{(\sigma-\sigma')/2}{\sinh((\sigma-\sigma')/2)}\right|$ then the preceding estimate proves the first inclusion of (4.30) and we also obtain (4.31).

We now turn to proving the second inclusions in (4.29) and (4.30). We are given either an element $b' \in \tilde{\Omega}_r(b, C_{\text{opt}}, \sigma, \vec{B}, p)$ or an element $b'' \in \tilde{\mathscr{R}}_r(b, C_{\text{opt}}, \sigma, \sigma', \vec{B}, p)$. In both cases we have to deal with a function $v \in \mathscr{J}_r(b, C_{\text{opt}}, \sigma, \vec{B}, p)$ for which either $\int_{-\infty}^{\infty} xe^{\sigma x}v(x)\,dx = b'$ or $\int_{-\infty}^{\infty} e^{\sigma' x}v(x)\,dx = b''$. We want to use this function to construct a sequence $\{b_n\}_{n\in\mathbb{Z}}$ in $E(b, C^*C_{\text{opt}}, e^\sigma, \mathbf{l}^p, \vec{B})$ or in $E(b, C^*_{\sigma,\sigma'}C_{\text{opt}}, e^\sigma, \mathbf{l}^p, \vec{B})$ such that $\sum_{n\in\mathbb{Z}} ne^{\sigma n}b_n = b'$ or $\sum_{n\in\mathbb{Z}} e^{\sigma' n}b_n = b''$. It would seem at first that the natural thing to do is to define $b_n = e^{-\sigma n}\int_{n-1/2}^{n+1/2} e^{\sigma x}v(x)\,dx$ for each $n \in \mathbb{Z}$. (This corresponds to what works in the analogous stage of the simpler proof in [41, p. 18] or [4, p. 44] that the norms $\|\cdot\|^{\mathbb{Z}}_{\vec{B}_{\theta,p}}$ and $\|\cdot\|^{\mathbb{R}}_{\vec{B}_{\theta,p}}$ satisfy $\|b\|^{\mathbb{Z}}_{\vec{B}_{\theta,p}} \leqslant \text{const.}\|b\|^{\mathbb{R}}_{\vec{B}_{\theta,p}}$.) This will certainly give $\sum_{n\in\mathbb{Z}} e^{\sigma n} b_n = b$ and $\|\{b_n\}_{n\in\mathbb{Z}}\|_{\mathscr{J}(\mathbf{l}^p,\vec{B})} \leqslant \sqrt{e}\|v\|_{\mathscr{J}_r(\vec{B},p)}$ so that $\{b_n\}_{n\in\mathbb{Z}} \in E(b, \sqrt{e}C_{\text{opt}}, e^{\sigma}, \mathbf{l}^p, \vec{B})$. But it will not give that $\sum_{n\in\mathbb{Z}} n e^{\sigma n} b_n = b'$. Instead, we will only be able to show that $\sum_{n\in\mathbb{Z}} n e^{\sigma n} b_n - b' \in \vec{B}_{\theta,p}$ and that $\|\sum_{n\in\mathbb{Z}} n e^{\sigma n} b_n - b'\|^{\mathbb{R}}_{\vec{B}_{\theta,p}} \leqslant \frac{1}{2}\|v\|_{\mathscr{J}_r(\vec{B},p)} \leqslant \frac{1}{2} C_{\text{opt}} \|b\|^{\mathbb{R}}_{\vec{B}_{\theta,p}}$. This would be sufficient for some purposes, such as exhibiting a result like (3.2) in Theorem 3.8 in the case of the real method, as a consequence of similar results in [32]. To get the stronger result of (4.29) we shall need an alternative more elaborate way of defining $\{b_n\}_{n\in\mathbb{Z}}$, which comes, perhaps surprisingly, from the complex method (cf. also [41, pp. 29–31]). It corresponds to what we did at the analogous step in the proof of Theorem 4.2.

Since $v \in \mathscr{J}_r(\vec{B}, p)$, we have by Hölder's inequality, that $\int_{-\infty}^{0} \|e^{\xi x} v(x)\|_{B_0}\, dx < \infty$ for all $\xi > 0$ and $\int_0^{\infty} \|e^{\xi x} v(x)\|_{B_1}\, dx < \infty$ for all $\xi < 1$. Thus

$$\int_{-\infty}^{\infty} \|e^{\xi x} v(x)\|_{B_0+B_1}\, dx < \infty \qquad \text{for all } \xi \in (0,1) \tag{4.39}$$

and so we can define an analytic function $f : \mathbb{S} \to B_0 + B_1$ by setting

$$f(\zeta) = \int_{-\infty}^{\infty} e^{\zeta x} v(x)\, dx. \tag{4.40}$$

Note that, for each compact subinterval $[\alpha, \beta] \subset (0,1)$ we have

$$\sup\{\|f(\zeta)\|_{B_0+B_1} : \operatorname{Re}\zeta \in [\alpha,\beta]\} < \infty. \tag{4.41}$$

Furthermore $f(\sigma) = b$ and $f'(\sigma) = \int_{-\infty}^{\infty} x e^{\sigma x} v(x)\, dx = b'$.

We now use the function $\psi_1(z) = e^{z^2} \frac{e^{z/2} - e^{-z/2}}{z}$ and $\psi_2(z) = (\psi_1(z))^2$ which appeared in the proof of Theorem 4.2 and, as in that proof, we obtain a new function $F : \mathbb{S} \to B_0 + B_1$ from our f by formula (4.20). It follows from (4.41) and arguments similar to those of the proof of Theorem 4.2 that F is analytic. It also satisfies $F(\sigma) = b$ and $F'(\sigma) = b'$ and $F(\zeta + 2\pi i) = F(\zeta)$ for all $\zeta \in \mathbb{S}$. We now define the sequence $\{b_n\}_{n\in\mathbb{Z}}$ to be the Fourier coefficients of F (or the Laurent coefficients of $G_F : \mathbb{A} \to B_0 + B_1$ defined as before by $G_F(e^z) = F(z)$). By standard properties of Laurent expansions we have that $F(\sigma) = \sum_{n\in\mathbb{Z}} e^{\sigma n} b_n$ and $F'(\sigma) = \sum_{n\in\mathbb{Z}} n e^{\sigma n} b_n$, where both of these series converge in $B_0 + B_1$. So, to establish the second inclusion of (4.29) it remains to show that $b_n \in B_0 \cap B_1$ and $\{b_n\}_{n\in\mathbb{Z}} \in \mathscr{J}(\mathbf{l}^p, \vec{B})$ with $\|\{b_n\}_{n\in\mathbb{Z}}\|_{\mathscr{J}(\mathbf{l}^p,\vec{B})} \leqslant C^* C_{\text{opt}} \|b\|_{\vec{B}_{\mathbf{l}^p, e^{\sigma}}}$.

The coefficients b_n are given by

$$b_n = \frac{1}{2\pi} \int_0^{2\pi} e^{-n(\lambda + it)} F(\lambda + it)\, dt,$$

where, by Cauchy's theorem, the value of the integral is the same for all choices of $\lambda \in (0,1)$. Since the $B_0 + B_1$ valued series in (4.20) converges uniformly on each compact subset of $\mathbb{S}$ we have, for each $\lambda \in (0,1)$, that

$$\begin{aligned} b_n &= \frac{1}{2\pi} \int_0^{2\pi} e^{-n(\lambda+it)} \left(\sum_{k\in\mathbb{Z}} \psi_2(\lambda + it - \sigma + 2\pi ki) f(\lambda + it + 2\pi ki) \right) dt \\ &= \frac{1}{2\pi} \sum_{k\in\mathbb{Z}} \int_0^{2\pi} e^{-n(\lambda+it)} \psi_2(\lambda - \sigma + i(t + 2\pi k)) f(\lambda + i(t + 2\pi k))\, dt \\ &= \frac{1}{2\pi} \sum_{k\in\mathbb{Z}} \int_0^{2\pi} e^{-n(\lambda+it)} \psi_2(\lambda - \sigma + i(t + 2\pi k)) \left(\int_{-\infty}^{\infty} e^{(\lambda+i(t+2\pi k))x} v(x)\, dx \right) dt. \end{aligned}$$

In view of (4.39) we can apply an obvious generalization of Fubini's theorem for Bochner integrable $B_0 + B_1$ valued functions to each term of the preceding series, so that

$$\begin{aligned} b_n &= \frac{1}{2\pi} \sum_{k\in\mathbb{Z}} \int_{-\infty}^{\infty} \left(\int_0^{2\pi} \psi_2(\lambda - \sigma + i(t + 2\pi k)) e^{(\lambda+it)(x-n)+2\pi ikx}\, dt \right) v(x)\, dx \\ &= \frac{1}{2\pi} \sum_{k\in\mathbb{Z}} \int_{-\infty}^{\infty} \left(\int_0^{2\pi} \psi_2(\lambda + i(t + 2\pi k) - \sigma) e^{(\lambda+i(t+2\pi k)(x-n)}\, dt \right) v(x)\, dx \\ &= \frac{1}{2\pi} \sum_{k\in\mathbb{Z}} \int_{-\infty}^{\infty} \left(\int_{2\pi k}^{2\pi(k+1)} \psi_2(\lambda + it - \sigma) e^{(\lambda+it)(x-n)}\, dt \right) v(x)\, dx. \end{aligned}$$

Since ψ_2 satisfies an estimate of the form (4.16), the series

$$\sum_{k\in\mathbb{Z}} \int_{2\pi k}^{2\pi(k+1)} \psi_2(\lambda + it - \sigma) e^{(\lambda+it)(x-n)}\, dt$$

converges absolutely and its sum $\int_{-\infty}^{\infty} \psi_2(\lambda + it - \sigma) e^{(\lambda+it)(x-n)}\, dt$ has absolute value bounded by a constant multiple of $e^{\lambda x}$. So using (4.39) and an obvious generalization of the dominated convergence theorem, we can interchange the order of summation and integration with respect to x in the preceding formula for b_n and obtain that

$$b_n = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \psi_2(\lambda + it - \sigma) e^{(\lambda+it)(x-n)}\, dt \right) v(x)\, dx.$$

We rewrite this as

$$b_n = \int_{-\infty}^{\infty} \Xi_\sigma(n - x) v(x)\, dx,$$

where $\Xi_\sigma : \mathbb{R} \to \mathbb{C}$ is the function

$$\Xi_\sigma(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \psi_2(\lambda + it - \sigma) e^{-(\lambda+it)x}\, dt.$$

Since ψ_2 is in the class H_0 of entire functions defined above in the proof of Theorem 4.2 it is clear, using condition (4.16) and Cauchy's theorem, that the above formula for Ξ_σ is valid and independent of λ for all choices of λ, not just in the restricted range $\lambda \in (0,1)$. In particular, a simple calculation shows that

$$\Xi_\sigma(x) = \Xi_0(x) e^{-\sigma x} \qquad \text{for each } \sigma \in \mathbb{C}. \tag{4.42}$$

For each $m \in \mathbb{N}$, the mth derivative $\frac{d^m}{dz^m}\psi_1(z)$ is a finite sum of functions of the form $\psi(z) = P(z)e^{z^2}(e^{z/2} + \varepsilon e^{-z/2})z^{-k}$ where $P(z)$ is a polynomial in z, ε is either 1 or -1, and k is a nonnegative integer. For each constant $\alpha \in \mathbb{C}$, each such function ψ satisfies $\sup\{|y|^n |\psi(\alpha + iy)| : y \in \mathbb{R},\ |y| \geqslant 1\} < \infty$ for every positive integer n. This means that the function $t \mapsto \psi_1(\alpha + it)$ is in the Schwartz class $\mathscr{S}(\mathbb{R})$. The square of this function, namely $t \mapsto \psi_2(\alpha + it)$ must also be in $\mathscr{S}(\mathbb{R})$, and therefore the same is true of its Fourier transform. In particular, for every constant λ, the function $x \mapsto e^{\lambda x}\Xi_\sigma(x)$ is in $\mathscr{S}(\mathbb{R})$. Since $e^{-\lambda|x|}v(x)$ is an integrable $B_0 \cap B_1$ valued function for all $\lambda > 1$ we deduce that $b_n \in B_0 \cap B_1$.

For $j = 0$ or 1 and $p \in [1, \infty)$ we have

$$\begin{aligned}
\left(\sum_{n\in\mathbb{Z}} \|e^{jn} b_n\|_{B_j}^p\right)^{1/p} &= \left(\sum_{n\in\mathbb{Z}} \left\| e^{jn} \int_{-\infty}^{\infty} \Xi_\sigma(n-x) v(x)\, dx \right\|_{B_j}^p\right)^{1/p} \\
&= \left(\sum_{n\in\mathbb{Z}} \left\| \int_{-\infty}^{\infty} e^{j(n-x)} \Xi_\sigma(n-x) e^{jx} v(x)\, dx \right\|_{B_j}^p\right)^{1/p} \\
&\leqslant \left(\sum_{n\in\mathbb{Z}} \left(\int_{-\infty}^{\infty} e^{j(n-x)} |\Xi_\sigma(n-x)| \|e^{jx} v(x)\|_{B_j}\, dx\right)^p\right)^{1/p} \\
&= \left(\sum_{n\in\mathbb{Z}} \int_{n-1/2}^{n+1/2} \left(\int_{-\infty}^{\infty} e^{j(n-x)} |\Xi_\sigma(n-x)| \right.\right. \\
&\qquad \left.\left. \times \|e^{jx} v(x)\|_{B_j}\, dx\right)^p dt\right)^{1/p}.
\end{aligned} \tag{4.43}$$

Let us define the function $\Theta_j : \mathbb{R} \to \mathbb{R}$ by $\Theta_j(t) = \sup\{e^{jy}|\Xi_\sigma(y)| : t - \frac{1}{2} \leqslant y \leqslant t + \frac{1}{2}\}$. Then, for each $n \in \mathbb{Z}$ and all $t \in [n - \frac{1}{2}, n + \frac{1}{2})$, we have $e^{j(n-x)}|\Xi_\sigma(n-x)| \leqslant \Theta_j(t-x)$. Substituting this in each term of (4.43), we obtain that

$$\left(\sum_{n\in\mathbb{Z}}\|e^{jn}b_n\|_{B_j}^p\right)^{1/p}\leqslant\left(\sum_{n\in\mathbb{Z}}\int_{n-1/2}^{n+1/2}\left(\int_{-\infty}^{\infty}\Theta_j(t-x)\|e^{jx}v(x)\|_{B_j}\,dx\right)^p dt\right)^{1/p}$$
$$=\left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\Theta_j(x)\|e^{j(t-x)}v(t-x)\|_{B_j}\,dx\right)^p dt\right)^{1/p}. \quad (4.44)$$

By the integral form of Minkowski's inequality, this last expression is dominated by

$$\int_{-\infty}^{\infty}\Theta_j(x)\left(\int_{-\infty}^{\infty}\|e^{j(t-x)}v(t-x)\|_{B_j}^p\,dt\right)^{1/p}dx$$
$$\leqslant\left(\max_{j=0,1}\int_{-\infty}^{\infty}\Theta_j(x)\,dx\right)\|v\|_{\mathscr{J}_r(\vec{B},p)}. \quad (4.45)$$

Since $y\mapsto e^{jy}\Xi_\sigma(y)$ is in $\mathscr{S}(\mathbb{R})$ it follows readily that $\int_{-\infty}^{\infty}\Theta_j(t)\,dt<\infty$ for $j=0$ and 1. Combining (4.44) and (4.45) and (4.38) gives that

$$\|\{b_n\}_{n\in\mathbb{Z}}\|_{\mathscr{J}(l^p,\vec{B})}\leqslant\left(\max_{j=0,1}\int_{-\infty}^{\infty}\Theta_j(x)\,dx\right)C_{\rm opt}e^{\theta(1-\theta)}\|b\|_{\vec{B}_{\theta,p}}^{\mathbb{Z}}, \quad (4.46)$$

where $\theta=\operatorname{Re}\sigma$. As the reader can easily check, easier variants of the same arguments establish this same inequality also when $p=\infty$.

If we choose $C^*=e^{\theta(1-\theta)}(\max_{j=0,1}\int_{-\infty}^{\infty}\Theta_j(x)\,dx)$ then (4.46) completes the proof of the second inclusion of (4.29). However this choice of C^* depends on σ. To obtain the same inclusion for a constant independent of σ we simply observe, using (4.42), that for all $\sigma\in\mathbb{S}$ the functions Θ_j in the above proof are dominated by $\Theta_j^+(t)=\sup\{e^{jy}|\Xi_0(y)|e^{|y|}:t-\frac{1}{2}\leqslant y\leqslant t+\frac{1}{2}\}$. Since $e^{2y}\Xi_0(y)$ and $e^{-2y}\Xi_0(y)$ are both in $\mathscr{S}(\mathbb{R})$ the functions Θ_j^+ are obviously also integrable. Also $\theta(1-\theta)\leqslant\frac{1}{4}$. So we obtain (4.29) for the absolute constant $C^*=4\sqrt{e}(\max_{j=0,1}\int_{-\infty}^{\infty}\Theta_j^+(x)\,dx)$.

We still have to prove the second inclusion of (4.30). Most of the steps for this are rather obvious variants of steps in the preceding argument (again motivated by the analogous part of the proof of Theorem 4.2). We are given, as mentioned earlier, an element $b''\in\tilde{\mathscr{R}}_r(b,C_{\rm opt},\sigma,\sigma',\vec{B},p)$ and some function $v\in\mathscr{J}_r(b,C_{\rm opt},\sigma,\vec{B},p)$ for which $\int_{-\infty}^{\infty}e^{\sigma'x}v(x)\,dx=b''$. We define the analytic function $f:\mathbb{S}\to B_0+B_1$ exactly as in (4.40). As would be expected, we then replace ψ_2 by the function ψ_3 as defined in (4.22) so that this time the function $F:\mathbb{S}\to B_0+B_1$ is given by (4.25). It satisfies $F(\sigma)=f(\sigma)=b$, $F(\sigma')=f(\sigma')=b''$ and $F(\zeta+2\pi i)=F(\zeta)$ for all $\zeta\in\mathbb{S}$. Its Fourier coefficients b_n satisfy $\sum_{n\in\mathbb{Z}}e^{\sigma}b_n=b$ and $\sum_{n\in\mathbb{Z}}e^{\sigma'}b_n=b''$. They are given by the formula

$$b_n = \int_{-\infty}^{\infty} \Xi_{\sigma,\sigma'}(n-x)v(x)\,dx,$$

where $\Xi_{\sigma,\sigma'} : \mathbb{R} \to \mathbb{C}$ is the function

$$\Xi_{\sigma,\sigma'}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \psi_3(\lambda+it)e^{-(\lambda+it)x}\,dt. \tag{4.47}$$

As before, the value of the integral here is independent of the choice of the constant λ. For all constants $\alpha, \beta \in \mathbb{C}$, the function $t \mapsto \psi_1(\alpha+it)\psi_1(\beta+it)$ is a product of functions in $\mathscr{S}(\mathbb{R})$ and so is itself in $\mathscr{S}(\mathbb{R})$. Since ψ_3 is a function of this form, divided by the constant $\psi_1(\sigma-\sigma')$, we obtain that, for each constant λ, the function $x \mapsto e^{\lambda x}\Xi_{\sigma,\sigma'}(x)$ is in $\mathscr{S}(\mathbb{R})$. This means that the rest of the proof of (4.30) can proceed exactly as in the proof of (4.29), except that Ξ_σ has to be replaced throughout by $\Xi_{\sigma,\sigma'}$. Thus the constant $C^*_{\sigma,\sigma'}$ can be chosen to be

$$C^*_{\sigma,\sigma'} = 4\sqrt{e}\left(\max_{j=0,1}\int_{-\infty}^{\infty}\Theta_j(x)\,dx\right), \tag{4.48}$$

where now, however, the functions Θ_j must be defined by

$$\Theta_j(t) = \sup\{e^{jy}|\Xi_{\sigma,\sigma'}(y)| : t-\tfrac{1}{2} \leqslant y \leqslant t+\tfrac{1}{2}\}. \tag{4.49}$$

Finally, to show that $C^*_{\sigma,\sigma'}$ satisfies estimates (4.32), we have to estimate our new functions Θ_j from above by other integrable functions which depend on σ and σ' in a suitable way. To start this calculation, we choose $\lambda = 0$ in (4.47) and substitute from (4.22) to get

$$\Xi_{\sigma,\sigma'}(x) = \frac{1}{2\pi\psi_1(\sigma-\sigma')}\int_{-\infty}^{\infty}\psi_1(it-\sigma)\psi_1(it-\sigma')e^{-itx}\,dt. \tag{4.50}$$

We use auxiliary functions defined by $\Phi(x,\alpha) = \int_{-\infty}^{\infty} e^{-itx}\psi_1(\alpha+it)\,dt$ for each constant $\alpha \in \mathbb{C}$. By the same reasoning as before, we see that $x \mapsto e^{\lambda x}\Phi(x,\alpha)$ is in $\mathscr{S}(\mathbb{R})$ for each constant λ and α. Also $\psi_1(\alpha+it) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{itx}\Phi(x,\alpha)\,dx$ and so

$$\begin{aligned}\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{itx}\left[\int_{-\infty}^{\infty}\Phi(x-y,-\sigma)\Phi(y,-\sigma')\,dy\right]dx \\ = 2\pi\psi_1(-\sigma+it)\psi_1(-\sigma'+it).\end{aligned}$$

This shows that

$$2\pi \int_{-\infty}^{\infty} \psi_1(-\sigma + it)\psi_1(-\sigma' + it)e^{-itx}\, dt$$
$$= \int_{-\infty}^{\infty} \Phi(x - y, -\sigma)\Phi(y, -\sigma')\, dy$$

and (4.50) becomes

$$\Xi_{\sigma,\sigma'}(x) = \frac{1}{4\pi^2\psi_1(\sigma - \sigma')} \int_{-\infty}^{\infty} \Phi(x - y, -\sigma)\Phi(y, -\sigma')\, dy. \qquad (4.51)$$

We observe that $\Phi(x, \alpha) = e^{\alpha x} \int_{-\infty}^{\infty} e^{-(\alpha+it)x}\psi_1(\alpha + it)\, dt = e^{\alpha x}\Phi(x, 0)$. So

$$|\Phi(x, -\sigma)| \leqslant e^{|x|}|\Phi(x, 0)| \qquad \text{for all } x \in \mathbb{R}.$$

Since $e^{3x}\Phi(x, 0)$ and $e^{-3x}\Phi(x, 0)$ are both in $\mathscr{S}(\mathbb{R})$, this shows that for some absolute constant C we have

$$|\Phi(x, -\sigma)| \leqslant Ce^{-2|x|} \qquad \text{for all } x \in \mathbb{R}$$

and of course $|\Phi(x, -\sigma')|$ satisfies the same estimate. So

$$\left|\int_{-\infty}^{\infty} \Phi(x - y, -\sigma)\Phi(y, -\sigma')\, dy\right| \leqslant C^2 \int_{-\infty}^{\infty} e^{-2|x-y|-2|y|}\, dy$$
$$= C^2e^{-2|x|}\left(\frac{1}{2} + |x|\right).$$

Substituting this in (4.51) and then, in turn, in (4.49) gives that

$$\Theta_j(t) \leqslant \frac{C^2\sqrt{e}}{4\pi^2|\psi_1(\sigma - \sigma')|} e^{-|t|}(1 + |t|) \qquad \text{for all } t \in \mathbb{R} \text{ and } j = 0, 1.$$

This enables us to obtain (4.32) from (4.48) and so completes the proof of the theorem. ∎

5. A COMPARISON WITH THE APPROACH OF CARRO, CERDÀ, AND SORIA

Let us here try to clarify the similarities and differences between our approach to commutator theorems and the one presented in [8].

As we shall see, the approach in [8] is more abstract, and it is more general than ours, when it comes to constructing and studying "derivation" mappings Ω.

On the other hand, our construction has more "built in structure." This is helpful when it comes to constructing particular examples associated with specific interpolation functors. It also means that the verifications that required hypotheses are met can be simpler and more systematic. Our construction also enables the construction and study of "translation" mappings $\mathscr{R}$ which apparently cannot be treated at this stage by the method of [8].

DEFINITION 5.1. Given an arbitrary Laurent compatible pair $\mathbf{X} = \{\mathscr{X}_0, \mathscr{X}_1\}$ of pseudolattices and a point $s \in \mathbb{A}$, let us define a functor H and an *interpolator* Φ over H in the sense of Definition 2.1 of [8, p. 200] as follows:

(i) For each Banach pair $\vec{A}$, let $H(\vec{A})$ be $\mathscr{J}(\mathbf{X}, \vec{A})$ and then let $\Phi_{\vec{A}} : H(\vec{A}) \to A_0 + A_1$ be the map $\Phi_{\vec{A}}(\{a_n\}_{n\in\mathbb{Z}}) = \sum_{n\in\mathbb{Z}} s^n a_n$.

(ii) For every other Banach pair $\vec{B}$ and each bounded linear operator $T : \vec{A} \to \vec{B}$, the operator $H(T) : H(\vec{A}) \to H(\vec{B})$ is defined by $H(T)(\{a_n\}_{n\in\mathbb{Z}}) = \{Ta_n\}_{n\in\mathbb{Z}}$.

In this case we have that $H(\vec{A})$ is a Banach space, and condition (1) of [8, p. 200] is clearly satisfied. Furthermore the space $\vec{A}_\Phi$, as defined in [8], coincides with $\vec{A}_{\mathbf{X},s}$ with equality of norms.

In order to ensure that $A_0 \cap A_1$ is continuously embedded in $\vec{A}_\Phi$, Carro, Cerdà and Soria require the following condition to hold:

(*) *For each interpolator Φ and associated functor H and each Banach pair $\vec{A}$, there exists an operator $\varphi : A_0 \cap A_1 \to H(\vec{A})$ such that $\Phi_{\vec{A}} \circ \varphi$ is the identity map on $A_0 \cap A_1$.*

In all concrete examples of pseudolattice pairs $\mathbf{X}$ which we have considered in this paper, such a map does indeed exist for each Φ and H arising as above from $\mathbf{X}$ and $s \in \mathbb{A}$, and it can be defined by setting $\varphi(a) = \{a_n\}_{n\in\mathbb{Z}}$ where $a_0 = a$ and $a_n = 0$ for all $n \neq 0$. It is difficult to think of a "natural" example of $\mathbf{X}$ for which the above particular choice for the map φ would not have the required properties. But, conceivably, in some exotic examples one might need to replace it, e.g. by setting $\varphi(a) = \{\phi_n a\}$ for some suitable sequence of scalars ϕ_n satisfying $\sum_{n\in\mathbb{Z}} s^n \phi_n = 1$.

In any case, it turns out that our requirement that $\mathbf{X}$ is nontrivial (Definition 2.8) is equivalent to condition (*). As observed above, nontriviality is equivalent to the condition $(\mathbb{C}, \mathbb{C})_{\mathbf{X},s} = \mathbb{C}$, i.e.

$$(\mathbb{C}, \mathbb{C})_\Phi = \mathbb{C} \tag{5.1}$$

and, even in the more general context of any interpolator Φ and functor H in the sense of [8], this is equivalent to (*). It is obvious that (*) implies (5.1). Conversely, if (5.1) holds, then there exists $h \in H(\mathbb{C}, \mathbb{C})$ such that $\Phi_{(\mathbb{C},\mathbb{C})}(h) = 1$. Now, for an arbitrary Banach pair $\vec{A}$ and each $a \in A_0 \cap A_1$, let $T_a : (\mathbb{C}, \mathbb{C}) \to \vec{A}$ be the operator given by $T_a \zeta = \zeta a$ for all $\zeta \in \mathbb{C}$. Then the map φ defined by $\varphi(a) = H(T_a)h$ has the properties required to establish (*).

Having defined our particular H and Φ, we next define a second interpolator Ψ on the same spaces $H(\vec{A})$. Here we are motivated by Definition 3.1 on [8, p. 203]. The natural choice is to set

$$\Psi_{\vec{A}}(\{a_n\}_{n\in\mathbb{Z}}) = \sum_{n\in\mathbb{Z}} ns^{n-1}a_n \tag{5.2}$$

for each Banach pair $\vec{A}$ and each $\{a_n\}_{n\in\mathbb{Z}} \in H(\vec{A})$. This will ensure that the definition of the mapping Ω given in Definition 3.3 on [8, p. 204] will coincide with ours (Definition 3.1). We know from part (i) of our Theorem 3.8 that the conclusions of Theorem 3.4 and Corollary 3.5 of [8] must hold for these particular choices of Φ and Ψ whenever $\mathbf{X}$ admits differentiation. To obtain these same conclusions by the methods of [8] we would need to know that the pair (Φ, Ψ) is *almost compatible*, i.e. that it satisfies condition (3a) of Definition 3.1 of [8]. (Note that this condition in fact already implies the inclusion $\psi_{\vec{A}}(\mathrm{Ker}\, \Phi_{\vec{A}}) \subset \mathrm{Im}\, \Phi_{\vec{A}}$ mentioned in Remark 3.2 of [8].) In fact the proof of Theorem 3.8 contains exactly what is needed for showing this, namely the step presented separately as Lemma 3.11. Thus, the latter part of the proof of part (i) of Theorem 3.8 can be seen as a special case of the proof of Theorem 3.4 of [8].

On the other hand, it is not at all clear to us at this stage how one could obtain a result like part (ii) of Theorem 3.8 in the abstract setting of [8].

We mention that, if our pseudolattice pair $\mathbf{X}$ has the additional property that the left-shift operator S^{-1} maps $\mathscr{J}(\mathbf{X}, \vec{B})$ boundedly into itself, then the second part of Lemma 6.2 (cf. also Remark 6.3) is exactly what is needed to show that the above pair of interpolators (Φ, Ψ) also satisfies condition (3b) of Definition 3.1 of [8] i.e., it is *compatible* rather than merely almost compatible.

In [8] separate and different proofs are given of the compatibility of the pair (Φ, Ψ) in the cases of the real (K and J) and complex methods. But we can now see that these kinds of results, i.e. compatibility for the J method, the complex method and also for both $\pm$ methods are all consequences of the same arguments in the proof of Lemmas 3.11 and 6.2. Furthermore we can prove compatibility or almost compatibility for any other method generated by a pair of pseudolattices which admits differentiation. We do not deal directly with the K method in this approach, but the results of [19] indicate that it can be related to the J method. (The results of Section 8 may perhaps lead the way to an alternative approach to commutator results for the K method and for the newly revealed analogues of the K method in the context of pseudolattice pairs other than $\mathbf{l}^p$.)

We shall indicate further connections between our approach and the approach of [8] and its subsequent development in [9] in later sections of this paper.

6. HIGHER ORDER RESULTS

In this section, we present two higher order commutator theorems for our general method of interpolation. Our first theorem, involving derivation operators Ω_n (higher order analogues of Ω), is closely related to a result of [9] and to previously known results for the real and complex methods first obtained in [43, 53] (cf. also [7, 44]) Analogously to the first order case (cf. Section 5), the verification that a number of different interpolation methods satisfy the conditions required to use the arguments of [9] can be done simultaneously, by working in terms of general pseudolattice pairs.

Our second theorem is a variant of the first, dealing with higher order commutators which are defined in terms of translation operators $\mathscr{R}$ as well as the operators Ω_n. Here (again as in the first order case) it is not clear how to involve the approach developed in [8, 9].

In order to formulate and prove these results we start by recalling and elaborating upon the main step (Lemma 3.11) of the proof of Theorem 3.8.

Let $\vec{A}$ be a Banach pair. As we have seen above, there is a natural correspondence between elements in the space $\mathscr{J}(\mathbf{X}, \vec{A})$ and certain analytic functions defined on $\mathbb{A}$ with values on $A_0 + A_1$, and it will be convenient to express this more explicitly with the help of the following notation:

Definition 6.1. For each sequence $b = \{b_n\}_{n\in Z} \in \mathscr{J}(\mathbf{X}, \vec{A})$ we let f_b denote the analytic function given by $f_b(z) = \sum_{n\in Z} z^n b_n$.

The values of these analytic functions at a given $s \in \mathbb{A}$ are precisely the elements of the space $\vec{A}_{\mathbf{X},s}$.

Lemma 6.2. *Let* $\mathbf{X}$ *be a pair of pseudolattices which admits differentiation. Let* $\vec{A}$ *be a Banach pair, and let* $s \in \mathbb{A}$. *For each* $x \in A_0 + A_1$ *define*

$$N_s(x) = \inf\{\|b\|_{\mathscr{J}(\mathbf{X},\vec{A})} : x = f_b'(s) \ \textit{with}\ f_b(s) = 0\}.$$

Then,

(i) *there exists a constant* $c = c(s) > 0$ *such that for each* $x \in A_0 + A_1$ *satisfying* $N_s(x) < \infty$ *we have* $x \in \vec{A}_{\mathbf{X},s}$ *and*

$$\|x\|_{\vec{A}_{\mathbf{X},s}} \leqslant cN_s(x). \tag{6.1}$$

(ii) *Moreover, if the right-shift operator* S^{-1} *is bounded on* $\mathscr{J}(\mathbf{X}, \vec{A})$, *then there exists a constant* $c = c(s, \|S^{-1}\|) > 0$ *such that, for all* $x \in \vec{A}_{\mathbf{X},s}$, *we have*

$$\|x\|_{\vec{A}_{\mathbf{X},s}} \geqslant cN_s(x). \tag{6.2}$$

Proof. Let x be as in (i). Then, for each $\varepsilon > 0$ there exists $b \in \mathscr{J}(\mathbf{X}, \vec{A})$ such that $x = f_b'(s)$, with $f_b(s) = 0$, and moreover

$$N_s(x) + \varepsilon \geqslant \|b\|_{\mathscr{J}(\mathbf{X},\vec{A})}.$$

The proof of Lemma 3.11 shows that the function $g : \mathbb{A} \to A_0 + A_1$ defined by $g(z) = f_b(z)/(z - s)$ if $z \neq s$, and $g(s) = f_b'(s) = x$, is of the form

$$g(z) = f_{\{-D_{0,1/s}(b)\}_{n\in Z}}(z)$$

with (cf. (3.9))

$$\|\{-D_{0,1/s}(b)\}_{n\in Z}\|_{\mathscr{J}(\mathbf{X},\vec{A})} \leqslant c\|b\|_{\mathscr{J}(\mathbf{X},\vec{A})}.$$

Consequently, $x \in \vec{A}_{\mathbf{X},s}$ and

$$\|x\|_{\vec{A}_{\mathbf{X},s}} \leqslant \|\{-D_{0,1/s}(b)\}_{n\in Z}\|_{\mathscr{J}(\mathbf{X},\vec{A})} \leqslant c\|b\|_{\mathscr{J}(\mathbf{X},\vec{A})} \leqslant c(N_s(x) + \varepsilon).$$

We obtain (6.1) by letting $\varepsilon \to 0$. Now, to establish (ii), suppose that $x \in \vec{A}_{\mathbf{X},s}$. Then, given $\varepsilon > 0$, there exists $b \in \mathscr{J}(\mathbf{X}, \vec{A})$ such that $x = f_b(s)$ and $\|x\|_{\vec{A}_{\mathbf{X},s}} + \varepsilon \geqslant \|b\|_{\mathscr{J}(\mathbf{X},\vec{A})}$. Let $F(z) = (z - s)f_b(z) = zf_b(z) - sf_b(z)$. Then $F(s) = 0$ and $F'(s) = f_b(s) = x$. Moreover, since

$$F(z) = f_{S^{-1}(b)}(z) - sf_b(z) = f_{S^{-1}(b)-sb}(z),$$

it follows from our assumptions that

$$\|S^{-1}(b) - sb\|_{\mathscr{J}(\mathbf{X},\vec{A})} \leqslant (\|S^{-1}\| + |s|)\|b\|_{\mathscr{J}(\mathbf{X},\vec{A})}.$$

Consequently, setting $c = 1/(\|S^{-1}\| + |s|)$, we have

$$\begin{aligned}\|x\|_{\vec{A}_{\mathbf{X},s}} + \varepsilon &\geqslant c\|S^{-1}(b) - sb\|_{\mathscr{J}(\mathbf{X},\vec{A})} \\ &\geqslant cN_s(x).\end{aligned}$$

We conclude by letting $\varepsilon \to 0$. ∎

Remark 6.3. It follows from Lemma 6.2 that if $\mathbf{X}$ admits differentiation, and $\mathbf{X}$ is such that S^{-1} is bounded on $\mathscr{J}(\mathbf{X}, \vec{A})$ for each Banach pair $\vec{A}$, then for each $s \in \mathbb{A}$ we have

$$\vec{A}_{\mathbf{X},s} = \{x \in A_0 + A_1\colon\ x = f_b'(s) \text{ with } b \in \mathscr{J}(\mathbf{X}, \vec{A}) \text{ and } f_b(s) = 0\}$$

with

$$\|x\|_{\vec{A}_{\mathbf{X},s}} \approx N_s(x).$$

We shall now extend the previous lemma in an obvious fashion in order to take higher order cancellations (i.e. vanishing of higher order derivatives) into account.

LEMMA 6.4. *Suppose that the pseudolattice pair* $\mathbf{X}$ *admits differentiation. Let* $\vec{A}$ *be a Banach pair, let* $s \in \mathbb{A}$ *and* $n \in N$. *For* $x \in A_0 + A_1$ *define* $N_s^1(x) = N_s(x)$, *and for* $n > 1$ *let*

$$\begin{aligned}N_s^n(x) &= \inf\{\|b\|_{\mathscr{J}(\mathbf{X},\vec{A})} : x = f_b^{(n)}(s) \ \ \textit{with} \\ f_b(s) &= f_b'(s) = \cdots = f_b^{(n-1)}(s) = 0\}.\end{aligned}$$

Then,

(i) *there exists a constant* $c = c(s, n) > 0$ *such that for all* $x \in A_0 + A_1$ *with* $N_s^n(x) < \infty$, *we have* $x \in \vec{A}_{\mathbf{X},s}$ *and*

$$\|x\|_{\vec{A}_{\mathbf{X},s}} \leqslant cN_s^n(x). \tag{6.3}$$

(ii) *Moreover, if the shift operator* S^{-1} *is bounded on* $\mathscr{J}(\mathbf{X}, \vec{A})$, *then there exists a constant* $c = c(s, \|S^{-1}\|, n) > 0$ *such that for all* $x \in \vec{A}_{\mathbf{X},s}$ *we have*

$$\|x\|_{\vec{A}_{\mathbf{X},s}} \geqslant cN_s^n(x). \tag{6.4}$$

Proof. We shall first prove part (i) by successive applications of part (i) of Lemma 6.2. Let x be an arbitrary element of $A_0 + A_1$ with $N_s^n(x) < \infty$.

There exists an element b in $\mathscr{J}(\mathbf{X}, \vec{A})$ such that

$$f_b^{(n)}(s) = x \tag{6.5}$$

and

$$f_b^{(k)}(s) = 0 \qquad \text{for } k = 0, 1, \dots, n-1. \tag{6.6}$$

We shall define elements $h(j)$ of $\mathscr{J}(\mathbf{X}, \vec{A})$ for $j = 0, 1, \dots, n$. We first set $h(0) = b$. Then, using (6.6) for $k = 0$ we can apply Lemma 3.11 to obtain that the sequence $b(1) = -D_{0,1/s}(h(0))$ is in $\mathscr{J}(\mathbf{X}, \vec{A})$ with $\|h(1)\|_{\mathscr{J}(\mathbf{X},\vec{A})} \leqslant c(s)\|h(0)\|_{\mathscr{J}(\mathbf{X},\vec{A})}$ and the corresponding analytic function $f_{h(1)}$ satisfies $f_{h(1)}(s) = f'_{h(0)}(s)$ and $f_{h(1)}(z) = f_{h(0)}(z)/(z-s)$ for all $z \in \mathbb{A}\backslash\{s\}$. Using standard properties of analytic functions (in this case $A_0 + A_1$ valued ones) we see, furthermore, that $f_{h(1)}^{(k)}(s) = \frac{k!}{(k+1)!} f_{h(0)}^{(k+1)}(s)$ for all nonnegative integers k. In particular, we have

$$f_{h(1)}^{(k)}(s) = 0 \qquad \text{for } k = 0, 1, \dots, n-2.$$

We can now iterate this procedure: At the jth step, provided $f_{h(j-1)}(s) = 0$, we obtain $h(j)$ from $h(j-1)$ by setting $h(j) = -D_{0,1/s}(h(j-1))$. This gives $h(j) \in \mathscr{J}(\mathbf{X}, \vec{A})$ and

$$\|h(j)\|_{\mathscr{J}(\mathbf{X},\vec{A})} \leqslant c(s)\|h(j-1)\|_{\mathscr{J}(\mathbf{X},\vec{A})} \tag{6.7}$$

and an analytic function $f_{h(j)} : \mathbb{A} \to A_0 + A_1$ which satisfies $f_{h(j)}^{(k)}(s) = \frac{k!}{(k+1)!} f_{h(j-1)}^{(k+1)}(s)$ for all $k \geqslant 0$ and $f_{h(j)}(z) = f_{h(j-1)}(z)/(z-s)$ for all $z \in \mathbb{A}\backslash\{s\}$. This means that

$$f_{h(j)}^{(k)}(s) = 0 \qquad \text{for } k = 0, 1, n-j-1.$$

The final iteration occurs for $j = n$. Here we can use the fact that $f_{h(n-1)}(s) = 0$. It follows from (6.7) that $h(n) \in \mathscr{J}(\mathbf{X}, \vec{A})$ with

$$\|h(n)\|_{\mathscr{J}(\mathbf{X},\vec{A})} \leqslant c(s)^n \|b\|_{\mathscr{J}(\mathbf{X},\vec{A})}. \tag{6.8}$$

We also have

$$f_{h(n)}(s) = f'_{h(n-1)}(s) = \frac{1}{2} f''_{h(n-2)}(s) = \frac{1}{3!} f'''_{h(n-3)}(s) = \cdots = \frac{1}{n!} f_{h(0)}^{(n)}(s) = \frac{1}{n!} x.$$

This shows that $x \in \vec{A}_{\mathbf{X},s}$, and, after taking the infimum for all choices of b satisfying (6.5) and (6.6), we also have $\|x\|_{\vec{A}_{\mathbf{X},s}} \leqslant n! c(s)^n N_s^n(x)$. This completes the proof of part (i). At this point it will also be convenient to make an additional observation for later purposes, namely that for all $z \in \mathbb{A}\backslash\{s\}$ the function $f_{h(n)}$ satisfies

$$f_{h(n)}(z) = \frac{f_{h(n-1)}(z)}{(z-s)} = \frac{f_{h(n-2)}(z)}{(z-s)^2} = \cdots = \frac{f_{h(0)}(z)}{(z-s)^n}. \tag{6.9}$$

We now turn to showing (ii). Let x be an arbitrary element of $\vec{A}_{\mathbf{X},s}$. For arbitrary $\varepsilon > 0$ pick $b = \{b_k\}_{k\in\mathbb{Z}} \in \mathscr{J}(\mathbf{X}, \vec{A})$ such that $f_b(s) = x$ and $\|b\|_{\mathscr{J}(\mathbf{X},\vec{A})} \leqslant (1+\varepsilon)\|x\|_{\vec{A}_{\mathbf{X},s}}$. Now define another analytic function $F : \mathbb{A} \to A_0 + A_1$ by setting $F(z) = (z-s)^n f_b(z)$. Then $F = f_u$ for another sequence $u = \{u_k\}_{k\in\mathbb{Z}}$ which is a linear combination of powers of the shift operator S^{-1} applied to b. More specifically, $F(z) = \sum_{k=0}^{n} \binom{n}{k}(-s)^k \times z^{n-k} f_b(z)$ and, correspondingly, $u = \sum_{k=0}^{n} \binom{n}{k}(-s)^k S^{k-n} b$. This gives that $u \in \mathscr{J}(\mathbf{X}, \vec{A})$ with

$$\|u\|_{\mathscr{J}(\mathbf{X},\vec{A})} \leqslant \left(\sum_{k=0}^{n} \binom{n}{k} |s|^k \|S^{-1}\|^{n-k}_{\mathscr{J}(\mathbf{X},\vec{A}) \to \mathscr{J}(\mathbf{X},\vec{A})} \right) \|b\|_{\mathscr{J}(\mathbf{X},\vec{A})}.$$

Since $F^{(n)}(s) = n! f_b(s) = n! x$ and $F^{(j)}(s) = 0$ for $j = 0, 1, \ldots, n-1$, we obtain that

$$N_s^n(x) \leqslant \tfrac{1}{n!} \|u\|_{\mathscr{J}(\mathbf{X},\vec{A})} \leqslant c(n, s, \|S^{-1}\|)(1+\varepsilon)\|x\|_{\vec{A}_{\mathbf{X},s}}$$

for all choices of $\varepsilon > 0$ and the proof is complete. ∎

Remark 6.5. As indicated in the statement of Lemma 6.2 and, as is clear from its proof, the constants appearing in (6.1) and (6.2) depend on s and on the norm of S^{-1} on $\mathscr{J}(\mathbf{X}, \vec{A})$. But they also depend on our choice of the pseudolattice pair $\mathbf{X}$, as will other constants appearing in the other results of this section. Furthermore the norm $\|S^{-1}\|_{\mathscr{J}(\mathbf{X},\vec{A})}$ could, in principle, also depend on the choice of Banach pair $\vec{A}$. Throughout this section we will adopt the convention of not explicitly denoting this dependence on $\mathbf{X}$ and $\vec{A}$. We remark that the methods of Subsection A.1 can probably be used to enable these constants to be taken independent of the particular Banach pairs being used if the conditions which we impose are assumed to hold for all Banach pairs. This can certainly be done if, instead of requiring S^{-1} to be bounded on $\mathscr{J}(\mathbf{X}, \vec{A})$ for all Banach pairs $\vec{A}$, we impose the slightly stronger condition that $S^{-1} : \mathscr{X}_j(B) \to \mathscr{X}_j(B)$ is bounded for $j = 0, 1$ and each $B \in \mathbf{Ban}$. This can be shown using a simpler version of the proof of Corollary A.4.

Remark 6.6. Lemma 6.4 establishes that, in the terminology of Definition 2.1 of [9, p. 304], a certain system of interpolators is *compatible*. This is the system $(\Phi^0, \Phi^1, \Phi^2, \ldots, \Phi^n)$ which is defined, for any fixed pseudolattice couple $\mathbf{X}$ satisfying the hypotheses of Lemma 6.4 and any fixed $s \in \mathbb{A}$, by first choosing the functor H and the interpolator $\Phi^0 = \Phi$ to be as in Definition 5.1. Then the interpolator $\Phi^1 = \Psi$ is defined as in (5.2), and, generalizing this, Φ^k is defined for all $k = 0, 1, 2, \ldots,$ by setting

$$\Phi^k_{\vec{A}}(\{a_n\}_{n\in\mathbb{Z}}) = \frac{1}{k!}\frac{d^k}{dz^k}\left(\sum_{n\in\mathbb{Z}} z^n a_n\right)\Bigg|_{z=s}$$

for each Banach pair $\vec{A}$ and each $\{a_n\}_{n\in\mathbb{Z}} \in H(\vec{A}) = \mathscr{J}(\mathbf{X}, \vec{A})$.

We can also see that this same system satisfies the condition (2) of [9, p. 307]. After interchanging the roles of the indices $j-1$ and n, this corresponds exactly to the fact that the element $h(n)$ generated in part (i) of the proof of Lemma 6.4 satisfies

$$\frac{f_b^{(n+p)}(s)}{(n+p)!} = \frac{f_{h(n)}^{(p)}(s)}{p!} \qquad \text{for } p = 0, 1, 2, \ldots \ .$$

In order to state and prove our higher order commutator theorem we first give an extension of Definition 3.1 and introduce higher order derivation maps Ω.

Definition 6.7. Let $C_{\text{opt}} > 1$ be a fixed constant, let $\mathbf{X}$ be a Laurent compatible pair of pseudolattices $\mathbf{X}$, and let $s \in \mathbb{A}$. For a given Banach pair $\vec{B}$ and for each element $x \in \vec{B}_{\mathbf{X},s}$, we define the set $E(x)$ as before by

$$E(x) = \left\{ b = \{b_n\}_{n\in\mathbb{Z}} \in \mathscr{J}(\mathbf{X}, \vec{B}) : \sum_{n\in\mathbb{Z}} s^n b_n = x \right.$$
$$\left. \text{and } \|\{b_n\}_{n\in\mathbb{Z}}\|_{\mathscr{J}(\mathbf{X},\vec{B})} \leqslant C_{\text{opt}} \|x\|_{\vec{B}_{\mathbf{X},s}} \right\}$$

and select a fixed element $b \in E(x)$. We then define $\Omega_n x$ for each $n \in \mathbb{N}$ by

$$\Omega_n x = \frac{1}{n!} f_b^{(n)}(s).$$

In particular, for $n = 1$, we have

$$\Omega_1 x = \Omega x = \sum_{n\in\mathbb{Z}} n s^{n-1} b_n = f_b'(s).$$

Where necessary we shall use the notation $\Omega_{n,\vec{B}}$ to indicate the underlying pair $\vec{B}$ with respect to which these mappings are defined.

Remark 6.8. It should be stressed in the above definition that the *same* choice of $b \in E(x)$ is used for all $n \in \mathbb{N}$. Furthermore, for each $s' \neq s$ in $\mathbb{A}$, for later purposes, we can also define the translation operator $\mathscr{R}$ *consistently* with a given definition of the Ω_n's by setting $\mathscr{R}x = f_b(s')$ for each $x \in \vec{B}_{\mathbf{X},s}$ making the same choice as above of $b \in E(x)$.

The first theorem in this section can be considered as a higher order version of part (i) of Theorem 3.8. It is thus also essentially an extension of the higher order commutator theorems of [43, 53]. (To make a precise connection with these latter results would require versions of the results of Section 4 for higher order derivatives.) In view of Remark 6.6, it is in fact also a special case of Theorem 2.6 of [9, p. 308].

THEOREM 6.9. *Suppose that* $\mathbf{X}$ *admits differentiation. Let* $\vec{A}$ *and* $\vec{B}$ *be arbitrary Banach pairs and fix a point* $s \in \mathbb{A}$ *and a constant* $C_{\mathrm{opt}} > 1$. *Suppose that the shift operator* S^{-1} *is bounded on* $\mathscr{J}(\mathbf{X}, \vec{B})$.

(i) *For each* $n \in \mathbb{N}$, *let* $\Omega_{n,\vec{A}}$ *and* $\Omega_{n,\vec{B}}$ *denote derivation mappings corresponding to the above choices of* s, C_{opt} *and* $\mathbf{X}$, *for the pairs* $\vec{A}$ *and* $\vec{B}$, *respectively. Let* $T : \vec{A} \to \vec{B}$ *be a bounded linear operator, and, for each* n, *let* $[T, \Omega_n] = T\Omega_{n,\vec{A}} - \Omega_{n,\vec{B}}T$. *Let* $C_n^{\Omega}(T)$ *be defined inductively by*

$$C_1^{\Omega}(T) = [T, \Omega_1],$$

$$C_2^{\Omega}(T) = [T, \Omega_2] - \Omega_1 C_1^{\Omega}(T),$$

....

$$C_n^{\Omega}(T) = [T, \Omega_n] - \sum_{k=1}^{n-1} \Omega_{n-k} C_k^{\Omega}(T).$$

Then $C_n^{\Omega}(T)$ *maps* $\vec{A}_{\mathbf{X},s}$ *boundedly into* $\vec{B}_{\mathbf{X},s}$. *More precisely, there exists a constant* $c > 0$ *depending only on* s *and* n, *such that, whenever* $a \in \vec{A}_{\mathbf{X},s}$, *it follows that* $C_n^{\Omega}(T)a \in \vec{B}_{\mathbf{X},s}$, *and moreover*

$$\|C_n^{\Omega}(T)a\|_{\vec{B}_{\mathbf{X},s}} \leqslant c\|T\|_{\vec{A}\to\vec{B}}\|a\|_{\vec{A}_{\mathbf{X},s}}.$$

Remark 6.10. Naively it might at first seem more natural to seek such a result for the simpler commutator operator $[T, \Omega_n]$ rather than $C_n^{\Omega}(T)$. As we shall see, the "correcting terms" $\sum_{k=1}^{n-1} \Omega_{n-k} C_k^{\Omega}(T)$ which appear in the definition of $C_n^{\Omega}(T)$ are needed to guarantee that the derivatives of *all* orders less than n of an associated analytic function vanish at the point $z = s$. The definition of $C_n^{\Omega}(T)$ is in some sense "dictated" by a rather naturally defined sequence of functions $\{F_k\}$ which appears in the course of the proof, and in particular by formula (6.20) which these functions satisfy.

Proof. One way of proving this theorem is to invoke the arguments of [9, pp. 307–310], combined with Remark 6.6. However here we shall provide a more self-contained argument, which we also need because it provides most of the ingredients which will be used later for the proof of our second result.

Fix $a \in \vec{A}_{\mathbf{X},s}$ and let $u = \{u_k\}_{k\in\mathbb{Z}} \in E(a)$ and $v = \{v_k\}_{k\in\mathbb{Z}} \in E(Ta)$ be the elements chosen in the course of defining $\Omega_{n,\vec{A}}$, and $\Omega_{n,\vec{B}}$ for all $n \in \mathbb{N}$. Thus $u \in \mathscr{J}(\mathbf{X}, \vec{A})$ and $v \in \mathscr{J}(\mathbf{X}, \vec{B})$ and their norms in these spaces are bounded by $C_{\text{opt}}\|a\|_{\vec{A}_{\mathbf{X},s}}$ and $C_{\text{opt}}\|Ta\|_{\vec{B}_{\mathbf{X},s}} \leqslant C_{\text{opt}}C(\mathbf{X})\|T\|_{\vec{A}\to\vec{B}}\|a\|_{\vec{A}_{\mathbf{X},s}}$, respectively. Furthermore the associated functions f_u and f_v (in the notation of Definition 6.1) satisfy $f_u(s) = a$, $f_v(s) = Ta$ and $\Omega_{n,\vec{A}}a = \frac{1}{n!}f_u^{(n)}(s)$ and $\Omega_{n,\vec{B}}(Ta) = \frac{1}{n!}f_v^{(n)}(s)$ for each $n \in \mathbb{N}$. Let $w = Tu - v$, i.e. $w = \{w_k\}_{k\in\mathbb{Z}} = \{Tu_k - v_k\}_{k\in\mathbb{Z}}$ and let $F_1 = f_w = Tf_u - f_v$. We note for later use that

$$\tfrac{1}{k!}F_1^{(k)}(s) = T\Omega_{k,\vec{A}}a - \Omega_{k,\vec{B}}(Ta) = [T, \Omega_k]a \qquad \text{for each } k \in \mathbb{N}. \tag{6.10}$$

The preceding estimates show that

$$\|w\|_{\mathscr{J}(\mathbf{X},\vec{B})} \leqslant 2C_{\text{opt}}C(\mathbf{X})\|T\|_{\vec{A}\to\vec{B}}\|a\|_{\vec{A}_{\mathbf{X},s}}. \tag{6.11}$$

We also have

$$F_1(s) = Ta - Ta = 0$$

and, setting $k = 1$ in (6.10),

$$F_1'(s) = [T, \Omega_1]a = C_1^{\Omega}(T)a. \tag{6.12}$$

If we now apply Lemma 6.2 we simply recover part (i) of Theorem 3.8 which is also of course the case $n = 1$ for the present theorem. But the preceding steps are also the introduction to an argument which will provide a proof for all $n \geqslant 1$. The above function F_1 is the first in a sequence of analytic functions $F_k : \mathbb{A} \to B_0 + B_1$ which we will now construct. They will all have the following properties:

(i) $F_k = f_{b(k)}$ for some element $b(k) \in \mathscr{J}(\mathbf{X}, \vec{B})$ and

(ii) $F_k^{(j)}(s) = 0$ for $j = 0, 1, \ldots, k-1$, which in turn will imply, by Lemma 6.4, that

(iii) $F_k^{(k)}(s) \in \vec{B}_{\mathbf{X},s}$ and

$$\|F_k^{(k)}(s)\|_{\vec{B}_{\mathbf{X},s}} \leqslant c(k,s)\|b(k)\|_{\mathscr{J}(\mathbf{X},\vec{B})} \tag{6.13}$$

for some constant $c(k,s)$.

These conditions certainly hold for $k = 1$ with $b(1) = w$. We proceed and obtain F_k recursively for each $k \geqslant 2$ by setting

$$F_k(z) = \frac{1}{k}\left(F_{k-1}(z) - \frac{(z-s)^{k-1}}{(k-1)!}H_k(z)\right). \tag{6.14}$$

Here $H_k = f_{b^*(k)}$, where $b^*(k)$ is an element of $E(F_{k-1}^{(k-1)}(s))$, which ensures that $F_k^{(k-1)}(s) = 0$ and

$$\|b^*(k)\|_{\mathscr{J}(\mathbf{X},\vec{B})} \leqslant C_{\mathrm{opt}}\|F_{k-1}^{(k-1)}(s)\|_{\vec{B}_{\mathbf{X},s}}. \tag{6.15}$$

But, more specifically, $b^*(k)$ is selected to be the particular element of $E(F_{k-1}^{(k-1)}(s))$ for which

$$\Omega_{n,\vec{B}}(F_{k-1}^{(k-1)}(s)) = \frac{1}{n!}f_{b^*(k)}^{(n)}(s) = \frac{1}{n!}H_k^{(n)}(s) \qquad \text{for each } n \in \mathbb{N}. \tag{6.16}$$

It follows from (6.14) that the sequence $b(k)$ is obtained from the sequences $b(k-1)$ and $b^*(k)$ by applying a fixed linear combination of powers of the right shift S^{-1} and the identity operator I. More precisely we have,

$$b(k) = \frac{1}{k}\left(b(k-1) - \frac{(S^{-1} - sI)^{k-1}}{(k-1)!}b^*(k)\right),$$

and consequently, if $b^*(k)$ and $b(k-1)$ are both in $\mathscr{J}(\mathbf{X},\vec{B})$ then we can deduce that $b(k) \in \mathscr{J}(\mathbf{X},\vec{B})$ and

$$\|b(k)\|_{\mathscr{J}(\mathbf{X},\vec{B})} \leqslant C_0(k,s,\|S^{-1}\|)(\|b^*(k)\|_{\mathscr{J}(\mathbf{X},\vec{B})} + \|b(k-1)\|_{\mathscr{J}(\mathbf{X},\vec{B})}) \tag{6.17}$$

for some constant $C_0(k,s,\|S^{-1}\|)$. It follows easily from the defining formula (6.14) and these arguments that, if F_k has properties (i) and (ii) for all $k = 1,2,\ldots,j$, then the same is true for $k = j+1$ and so these properties hold for all $k \in \mathbb{N}$. Also, combining (6.13), (6.15) and (6.17) we see that, for all $k \geqslant 2$,

$$\|b(k)\|_{\mathscr{J}(\mathbf{X},\vec{B})} \leqslant C_1(k,s,C_{\mathrm{opt}},\|S^{-1}\|)\|b(k-1)\|_{\mathscr{J}(\mathbf{X},\vec{B})}$$

for some constant $C_1(k,s,C_{\mathrm{opt}},\|S^{-1}\|)$ depending on k, s, C_{opt} and $\|S^{-1}\|_{\mathscr{J}(\mathbf{X},\vec{B})\to\mathscr{J}(\mathbf{X},\vec{B})}$. Iterating this last estimate $k-1$ times gives that, for some other constant $C_2(k,s,C_{\mathrm{opt}},\|S^{-1}\|)$ depending on these same quantities,

$$\|b(k)\|_{\mathscr{J}(\mathbf{X},\vec{B})} \leqslant C_2(k,s,C_{\mathrm{opt}},\|S^{-1}\|)\|b(1)\|_{\mathscr{J}(\mathbf{X},\vec{B})}. \tag{6.18}$$

Since $b(1) = w$ we can combine this with (6.11) and (6.13) to give that

$$\|F_k^{(k)}(s)\|_{\vec{B}_{\mathbf{X},s}} \leqslant C_3(k, s, C_{\text{opt}}, C(\mathbf{X}), \|S^{-1}\|)\|T\|_{\vec{A}\to\vec{B}}\|a\|_{\vec{A}_{\mathbf{X},s}}$$

for all $k \geqslant 2$ and for yet another constant $C_3(k, s, C_{\text{opt}}, C(\mathbf{X}), \|S^{-1}\|)$ depending on the listed arguments. This estimate also holds of course for $k = 1$ (cf. (6.12) and part (i) of Theorem 3.8).

To complete the proof it remains to show that

$$F_k^{(k)}(s) = C_k^{\Omega}(T)a \qquad \text{for all } k \in \mathbb{N}. \tag{6.19}$$

The case $k = 1$ is already known (see (6.12)). To treat the cases where $k \geqslant 2$ we shall use the formula

$$F_k(z) = \frac{1}{k!}\left(F_1(z) - \sum_{m=1}^{k-1}(z-s)^m H_{m+1}(z)\right)$$

which is obvious for $k = 2$ and extends immediately to all $k \geqslant 2$ by induction. Differentiating k times at the point $z = s$ gives

$$\begin{aligned} F_k^{(k)}(s) &= \frac{1}{k!}\left(F_1^{(k)}(s) - \sum_{m=1}^{k-1}\binom{k}{k-m} m! H_{m+1}^{(k-m)}(s)\right) \\ &= \frac{1}{k!}F_1^{(k)}(s) - \sum_{m=1}^{k-1}\frac{1}{(k-m)!}H_{m+1}^{(k-m)}(s), \end{aligned}$$

and, after substituting using (6.10) and (6.16), this in turn gives that

$$F_k^{(k)}(s) = [T, \Omega_k]a - \sum_{m=1}^{k-1} \Omega_{k-m,\vec{B}}(F_m^{(m)}(s)). \tag{6.20}$$

If we know that $F_m^{(m)}(s) = C_m^{\Omega}(T)a$ for each $m = 1, 2, \ldots, k-1$ then we can deduce from (6.20) that $F_k^{(k)}(s) = C_k^{\Omega}(T)a$. Thus we have established (6.19) by induction and the proof is complete. ∎

Our second and last theorem in this section is a higher order translation result extending Theorem 3.8(ii), i.e. estimate (3.3):

$$\|T(\mathscr{R}_{\vec{A}}a) - \mathscr{R}_{\vec{B}}(Ta)\|_{\vec{B}_{\mathbf{X},s'}} \leqslant \tilde{C}|s-s'|\|T\|_{\vec{A}\to\vec{B}}\|a\|_{\vec{A}_{\mathbf{X},s}},$$

where $\mathscr{R}$ is a "translation" operator defined in terms of a second point s' chosen in $\mathbb{A}$. The idea is to replace the commutator $[T, \mathscr{R}]a$ appearing in (3.3) by a more elaborate related expression which tends to zero more quickly as $s' \to s$, i.e. its norm is bounded by $|s-s'|^n$ instead of merely

$|s - s'|$. Our proof of Theorem 6.9 suggests an appropriate definition of an analogue of the commutator $C_n^{\Omega}(T)$, which we shall denote by $C_n^{\mathscr{R}}(T)$. It involves operators Ω_k as well as $\mathscr{R}$ and these must all be chosen consistently (cf. Remark 6.8). The proof of Theorem 6.9 also essentially provides the proof of the following theorem:

THEOREM 6.11. *Suppose that* $\mathbf{X}$, $\vec{A}$, $\vec{B}$, s, C_{opt}, S^{-1} *and* $C_n^{\Omega}(T)$, *for each* $n \in \mathbb{N}$, *are defined as in the statement of Theorem* 6.9 *and satisfy the hypotheses of that theorem. For some* $s' \neq s$ *in* $\mathbb{A}$, *let* $\mathscr{R}_{\vec{A}}$ *and* $\mathscr{R}_{\vec{B}}$ *be the translation operators defined as in Definition* 3.1(iv), *consistently using the same representatives in* $\mathscr{J}(\mathbf{X}, \vec{A})$ *and* $\mathscr{J}(\mathbf{X}, \vec{B})$, *respectively*, *as are used for defining* $\Omega_{\vec{A},n}$ *and* $\Omega_{\vec{B},n}$. *Let* $[T, \mathscr{R}] = T\mathscr{R}_{\vec{A}} - \mathscr{R}_{\vec{B}}T$. *Let the commutators* $C_n^{\mathscr{R}}(T)$ *be defined by*

$$C_1^{\mathscr{R}}(T) = [T, \mathscr{R}],$$

$$C_2^{\mathscr{R}}(T) = \tfrac{1}{2}[T, \mathscr{R}] - (s' - s)\mathscr{R}_{\vec{B}}(C_1^{\Omega}(T)),$$

and, *in general*,

$$C_n^{\mathscr{R}}(T) = \frac{1}{n!}\left([T, \mathscr{R}] - \sum_{m=1}^{n-1} (s' - s)^m \mathscr{R}_{\vec{B}}(C_m^{\Omega}(T))\right).$$

Then $C_n^{\mathscr{R}}(T)a \in \vec{B}_{\mathbf{X},s'}$ *for all* $a \in \vec{A}_{\mathbf{X},s}$ *and*

$$\|C_n^{\mathscr{R}}(T)a\|_{\vec{B}_{\mathbf{X},s'}} \leqslant C'|s - s'|^n \|T\|_{\vec{A} \to \vec{B}} \|a\|_{\vec{A}_{\mathbf{X},s}} \tag{6.21}$$

for some constant C' *which depends on* $\mathbf{X}$, s, n, C_{opt} *and* $\|S^{-1}\|_{\mathscr{J}(\mathbf{X},\vec{B}) \to \mathscr{J}(\mathbf{X},\vec{B})}$ *but not on* T, a *or* s'.

Proof. We use the functions F_k introduced in the proof of Theorem 6.9. We note that, for each $k \in \mathbb{N}$, the value of the function $F_k(z)$ at $z = s'$ is precisely $C_k^{\mathscr{R}}(T)a$. We have $F_n = f_{b(n)}$ where $b(n) \in \mathscr{J}(\mathbf{X}, \vec{B})$. This ensures that $C_k^{\mathscr{R}}(T)a = F_n(s') \in \vec{B}_{\mathbf{X},s'}$. We next observe that, since $F_n^{(k)}(s) = 0$ for $k = 0, 1, \ldots, n-1$, we can apply the arguments of the proof of part (i) of Lemma 6.4 with b chosen to be $b(n)$. Thus we obtain an element $h(n) \in \mathscr{J}(\mathbf{X}, \vec{B})$ such that (cf. (6.9)) $F_n(z) = f_b(z) = (z - s)^n f_{h(n)}(z)$ for all $z \in \mathbb{A}$. It follows that

$$\|C_n^{\mathscr{R}}(T)a\|_{\vec{B}_{\mathbf{X},s'}} = \|(s' - s)^n f_{h(n)}(s')\|_{\vec{B}_{\mathbf{X},s'}} \leqslant |s' - s|^n \|h(n)\|_{\mathscr{J}(\mathbf{X},\vec{B})}.$$

Using (6.8) followed by (6.18) and (6.11), we see that this last expression is bounded above by

$$\begin{aligned}
&|s' - s|^n c(s)^n \|b(n)\|_{\mathscr{J}(\mathbf{X},\vec{B})} \\
&\quad \leqslant |s' - s|^n C_4(n, s, C_{\text{opt}}, \|S^{-1}\|)\|w\|_{\mathscr{J}(\mathbf{X},\vec{B})} \\
&\quad \leqslant |s' - s|^n C_5(n, s, C_{\text{opt}}, \|S^{-1}\|, \mathbf{X})\|T\|_{\vec{A}\to\vec{B}}\|a\|_{\vec{A}_{\mathbf{X},s}}
\end{aligned}$$

for suitable constants depending on the indicated parameters. This establishes (6.21). ∎

The results in this section lead naturally to the study of analogues of interpolation spaces of Lions–Schechter type in the context of our more general method. These spaces can be defined as follows:

Definition 6.12. Let $\vec{B}$ be a Banach pair, and suppose that the pseudolattice pair $\mathbf{X}$ admits differentiation. Then, for each nonnegative integer n, we define the space $\vec{B}^{(n)}_{\mathbf{X},s}$ as the set of all elements of the form $x = f_b^{(n)}(s)$, with $\{b_n\}_{n\in\mathbb{Z}} \in \mathscr{J}(\mathbf{X}, \vec{B})$, endowed with the natural quotient norm

$$\|x\|_{\vec{B}^{(n)}_{\mathbf{X},s}} = \inf\{\|b\|_{\mathscr{J}(\mathbf{X},\vec{B})} : x = f_b^{(n)}(s)\}.$$

If the shift operator S^{-1} is bounded on $\mathscr{J}(\mathbf{X}, \vec{B})$ then the spaces $\vec{B}^{(n)}_{\mathbf{X},s}$ are naturally nested, i.e.

$$\vec{B}^{(n-1)}_{\mathbf{X},s} \subset \vec{B}^{(n)}_{\mathbf{X},s} \qquad \text{for each } n \in \mathbb{N}. \tag{6.22}$$

(This is easy to show since $(z - s)f(z) \in \mathscr{J}(\mathbf{X}, \vec{B})$ whenever $f \in \mathscr{J}(\mathbf{X}, \vec{B})$.)

It is clear from the definitions that

$$\Omega_n : \vec{B}_{\mathbf{X},s} \to \vec{B}^{(n)}_{\mathbf{X},s}. \tag{6.23}$$

In fact the spaces $\vec{B}^{(n)}_{\mathbf{X},s}$ are closely related to the range spaces associated with derivation operators Ω. Under suitable mild conditions they can be identified with these range spaces, as we shall see, at least for $n = 1$, in the next section. Analogously to (6.23), for any bounded operator $T : \vec{A} \to \vec{B}$, each term in the formula which defines $C_n^{\Omega}(T)$ is clearly a bounded map from $\vec{A}_{\mathbf{X},s}$ into $\vec{B}^{(n)}_{\mathbf{X},s}$. This immediately implies also that

$$C_n^{\Omega}(T) : \vec{A}_{\mathbf{X},s} \to \vec{B}^{(n)}_{\mathbf{X},s}.$$

The cancellation properties of $C_n^{\Omega}(T)$ allow us to prove the much sharper commutator Theorem 6.9.

Let us also remark that, when specialized to the complex method, the $\vec{B}^{(n)}_{\mathbf{X},s}$ spaces defined above coincide with spaces (defined using the annulus $\mathbb{A}$) of Lions–Schechter type (cf. [7] and the references therein). The methods of Section 4 can be used to show that these spaces coincide with the usual Lions–Schechter spaces which are defined using the strip S instead of the annulus. For details see Subsection A.4 in Appendix A.

7. CHARACTERIZATION OF DOMAIN AND RANGE SPACES OF DERIVATION OPERATORS

Although the operators Ω are in general not bounded on the interpolation scales $\vec{A}_{\mathbf{X},s}$ we have shown in previous sections that they *commute* boundedly with bounded operators in the scale due to the cancellations that develop from the operation of taking commutators. A natural question in the theory is to describe the domain spaces associated with the operators Ω. The domain spaces turn out to be interpolation spaces themselves which are independent of any particular choice of Ω. They have been characterized for both the real and complex methods (cf. [19, 54]) and also in the more abstract context of [8]. In this section, we formulate some analogues of these previous results in the context of our construction. This amounts to giving new results in the case, for example, of the $\pm$ methods. We shall also consider the corresponding characterizations for range spaces of the operators Ω.

Although we shall occasionally use some results from Section 6, we shall deal here exclusively with *first-order* derivation operators, i.e. Ω will always be as defined in Definition 3.1(ii).

Let $\vec{A}$ be a Banach pair and let $\mathbf{X}$ be a pseudolattice pair which admits differentiation. For each $s \in \mathbb{A}$ and each mapping Ω obtained as in Definition 3.1(ii) we define

$$\mathrm{Dom}_{\mathbf{X},s}\,\Omega_{\vec{A}} = \{a \in \vec{A}_{\mathbf{X},s} : \Omega a \in \vec{A}_{\mathbf{X},s}\} \tag{7.1}$$

and let

$$\|a\|_{\mathrm{Dom}_{\mathbf{X},s}\,\Omega_{\vec{A}}} = \|a\|_{\vec{A}_{\mathbf{X},s}} + \|\Omega a\|_{\vec{A}_{\mathbf{X},s}}. \tag{7.2}$$

For our analysis in this section we need to make a specific choice of operators Ω which are homogeneous and so to ensure that $\|\lambda a\|_{\mathrm{Dom}_{\mathbf{X},s}\,\Omega_{\vec{A}}} = |\lambda|\|a\|_{\mathrm{Dom}_{\mathbf{X},s}\,\Omega_{\vec{A}}}$ for every scalar λ. That such a selection is possible in our context follows from the fact that (in the notation of Definition 3.1) we have $E(\lambda b) = \lambda E(b)$ for each nonzero $b \in \vec{A}_{\mathbf{X},s}$ and each nonzero $\lambda \in \mathbb{C}$.

If we define the interpolators Φ and Ψ over H in terms of $\mathbf{X}$ and s as in Section 5, then the domain space $\mathrm{Dom}(\Omega_{\vec{A}})$ associated with operators Ω introduced in [8, Definition 3.6, p. 205] coincides with the space $\mathrm{Dom}_{\mathbf{X},s}(\Omega_{\vec{A}})$ and their "norms" are equal. This enables us to deduce some properties of the spaces $\mathrm{Dom}_{\mathbf{X},s}(\Omega_{\vec{A}})$ and related spaces from results in [8].

Since the fact that $\mathbf{X}$ admits differentiation implies that the corresponding interpolators (Φ, Ψ) are almost compatible, Theorem 3.8 of [8, p. 205] shows that $\mathrm{Dom}_{\mathbf{X},s}\, \Omega_{\vec{A}}$ is a linear space and the expression in (7.2) defines a quasi-norm. Furthermore, this space coincides to within equivalence of (quasi-) norms with the *normed* space $\Phi_{\vec{A}}(\Psi_{\vec{A}}^{-1}(\vec{A}_{\mathbf{X},s}))$ which is independent of any particular choice of the operator Ω. This space consists of all $a \in A_0 + A_1$ such that, for some $b \in \mathscr{J}(\mathbf{X}, \vec{A})$,

$$a = f_b(s) \quad \text{and} \quad f_b'(s) \in \vec{A}_{\mathbf{X},s}. \tag{7.3}$$

The norm $\|a\|_{\Phi_{\vec{A}}(\Psi_{\vec{A}}^{-1}(\vec{A}_{\mathbf{X},s}))}$ can be taken to be the infimum of the quantities

$$\|b\|_{\mathscr{J}(\mathbf{X},\vec{A})} + \|f_b'(s)\|_{\vec{A}_{\mathbf{X},s}}$$

as b ranges over all elements having the above two properties (7.3). It is easy to check that the constants of equivalence between $\|\cdot\|_{\Phi_{\vec{A}}(\Psi_{\vec{A}}^{-1}(\vec{A}_{\mathbf{X},s}))}$ and $\|\cdot\|_{\mathrm{Dom}_{\mathbf{X},s}}(\Omega_{\vec{A}})$ depend only on C_{opt} and on the constant which appears in Lemma 3.11, i.e. on C_{opt}, $\mathbf{X}$ and s.

Our assumption that $\mathbf{X}$ is nontrivial ensures that, for the special couple $\vec{A} = (\mathbb{C}, \mathbb{C})$, we have

$$\Phi_{(\mathbb{C},\mathbb{C})}(\Psi_{(\mathbb{C},\mathbb{C})}^{-1}((\mathbb{C}, \mathbb{C})_{\mathbf{X},s})) = \mathbb{C}. \tag{7.4}$$

It is also clear (cf. also [8, Theorem 3.8]) that, for any Banach pairs $\vec{A}$ and $\vec{B}$, any linear operator $T : \vec{A} \to \vec{B}$ maps $\Phi_{\vec{A}}(\Psi_{\vec{A}}^{-1}(\vec{A}_{\mathbf{X},s}))$ to $\Phi_{\vec{B}}(\Psi_{\vec{B}}^{-1}(\vec{B}_{\mathbf{X},s}))$ with bound not exceeding $\|T\|_{\vec{A}\to\vec{B}} \max\{C(\mathscr{X}_0), C(\mathscr{X}_1)\}$. Exactly as in the proof of Theorem 2.14, this, together with (7.4), establishes the continuous embedding

$$A_0 \cap A_1 \subset \Phi_{\vec{A}}(\Psi_{\vec{A}}^{-1}(\vec{A}_{\mathbf{X},s})) \qquad \text{for each Banach pair } \vec{A} \tag{7.5}$$

and completes the proof that $\vec{A} \mapsto \Phi_{\vec{A}}(\Psi_{\vec{A}}^{-1}(\vec{A}_{\mathbf{X},s}))$ is an interpolation functor.

Under additional conditions on $\mathbf{X}$ we can give another description of the space $\mathrm{Dom}_{\mathbf{X},s}(\Omega_{\vec{A}})$.

DEFINITION 7.1. Let $\vec{A}$ be a Banach pair, and let $\mathbf{X}$ be a pair of pseudolattices which admits differentiation. Then for each $s \in \mathbb{A}$ we define

$$\vec{A}^{(-1)}_{\mathbf{X},s} = \{x : \exists b \in \mathscr{J}(\mathbf{X}, \vec{A}) \text{ s.t. } f_b(s) = x, f_b'(s) = 0\}$$

with

$$\|x\|_{\vec{A}^{(-1)}_{\mathbf{X},s}} = \inf\{\|b\|_{\mathscr{J}(\mathbf{X},\vec{A})} : f_b(s) = x, f_b'(s) = 0\}.$$

THEOREM 7.2. *Let $\vec{A}$ be a Banach pair, and let $\mathbf{X}$ be a pair of pseudolattices which admits differentiation and such that the shift operator S^{-1} is bounded on $\mathscr{J}(\mathbf{X}, \vec{A})$. Then, for each $s \in \mathbb{A}$ and for any corresponding choice of C_{opt} and Ω we have*

$$\text{Dom}_{\mathbf{X},s}\, \Omega_{\vec{A}} = \vec{A}^{(-1)}_{\mathbf{X},s}$$

in the sense that these two spaces coincide as sets, and moreover

$$\|x\|^{(-1)}_{\vec{A}_{\mathbf{X},s}} \approx \|x\|_{\text{Dom}_{\mathbf{X},s}\, \Omega_{\vec{A}}} \qquad \textit{for all } x \in \text{Dom}_{\mathbf{X},s}\, \Omega_{\vec{A}},$$

with constants of equivalence independent of x.

Proof. The boundedness of $S^{-1} : \mathscr{J}(\mathbf{X}, \vec{A}) \to \mathscr{J}(\mathbf{X}, \vec{A})$ implies, by part (ii) of Lemma 6.2 (cf. also Remark 6.3) that (Φ, Ψ) is compatible. So the result follows from part (b) of Theorem 3.8 of [8]. ∎

In the case of the complex method, i.e. when $\mathbf{X} = \{FC, FC\}$, the operator Ω was studied in [54] with the help of a sort of "linked product space" of Calderón and Lions–Schechter complex interpolation spaces. The following definition of a version of that space for general $\mathbf{X}$ has an apparent connection with the spaces $\vec{A}^{(-1)}_{\mathbf{X},s}$ and $\Phi_{\vec{A}}(\Psi^{-1}_{\vec{A}}(\vec{A}_{\mathbf{X},s}))$.

DEFINITION 7.3. Let $\vec{A}$ be a Banach pair and let $\mathbf{X}$ be a pair of pseudolattices which admits differentiation. For each $s \in \mathbb{A}$ we define

$$\vec{A}^{(0)\oplus(1)}_{\mathbf{X},s} = \{(x, y) : \exists b \in \mathscr{J}(\mathbf{X}, \vec{A}) \text{ s.t. } f_b(s) = x,\ f_b'(s) = y\}$$

with

$$\|(x,y)\|_{\vec{A}_{\mathbf{X},s}^{(0)\oplus(1)}} = \inf\{\|b\|_{\mathscr{J}(\mathbf{X},\vec{A})} : f_b(s) = x,\ f_b'(s) = y\}.$$

If $\mathbf{X}$ admits differentiation and $S^{-1} : \mathscr{J}(\mathbf{X},\vec{A}) \to \mathscr{J}(\mathbf{X},\vec{A})$ is bounded, so that the corresponding pair of interpolators (Φ,Ψ) is compatible, then Proposition 7.2 of [8] (which generalizes Lemmas (2.5) and (2.9) on [54, pp. 324–325], shows that $\vec{A}_{\mathbf{X},s}^{(0)\oplus(1)}$ coincides with the *twisted sum* $\vec{A}_{\mathbf{X},s} \oplus_\Omega \vec{A}_{\mathbf{X},s}$ which is defined to be the set of all $(x,y) \in \vec{A}_{\mathbf{X},s} \times (A_0 + A_1)$ such that $y - \Omega x \in \vec{A}_{\mathbf{X},s}$ for some homogeneously defined derivation operator $\Omega : \vec{A}_{\mathbf{X},s} \to A_0 + A_1$. The norm $\|(x,y)\|_{\vec{A}_{\mathbf{X},s}^{(0)\oplus(1)}}$ is also equivalent to the quantity

$$\|(x,y)\|_{\vec{A}_{\mathbf{X},s}\oplus_\Omega\vec{A}_{\mathbf{X},s}} = \|x\|_{\vec{A}_{\mathbf{X},s}} + \|y - \Omega x\|_{\vec{A}_{\mathbf{X},s}}. \tag{7.6}$$

For more about twisted sums in Banach space theory we refer to [1, 36, 38].

After having dealt with the domain spaces of mappings Ω, we close this section with a short discussion of the range spaces of these same mappings. As was already hinted at above, these range spaces are related to the Lions–Schechter type spaces $\vec{B}_{\mathbf{X},s}^{(1)}$ of Definition 6.12. Obviously, for each choice of Ω associated with $\vec{B}_{\mathbf{X},s}$ and each $x \in \vec{B}_{\mathbf{X},s}$, the element Ωx is in $\vec{B}_{\mathbf{X},s}^{(1)}$. However we cannot in general expect to have $\vec{B}_{\mathbf{X},s}^{(1)}$ coincide with the range of one particular choice of Ω, i.e. with the set $\{\Omega x : x \in \vec{B}_{\mathbf{X},s}\}$. It is not even clear a priori that such a set is a linear space, even if Ω is chosen homogeneously. (Given the large amount of freedom in the way Ω can be chosen, it is relatively straight forward to explicitly construct examples where Ω is chosen to ensure that this set is indeed not a linear space.)

Instead of taking the range of one particular Ω, let us consider the set of all Ωx with $x \in \vec{B}_{\mathbf{X},s}$ where now Ω also varies. If Ω ranges over an appropriately defined class of derivation mappings associated with $\vec{B}_{\mathbf{X},s}$, then we do obtain an identification with the whole of $\vec{B}_{\mathbf{X},s}^{(1)}$, and there is also a natural (quasi-) norm on this generalized range space which is equivalent to the norm of $\vec{B}_{\mathbf{X},s}^{(1)}$. We refer to [8, p. 207] for some other results about range spaces.

DEFINITION 7.4. Given Banach and Laurent compatible pseudolattice pairs $\vec{B}$ and $\mathbf{X}$, a point $s \in \mathbb{A}$ and a constant $C \geqslant 1$, we define $\mathrm{Ran}_{\mathbf{X},s}(\tilde{\Omega}_{\vec{B}}, C)$ to be the set of all elements x of the form $x = \Omega y$ where $y \in \vec{B}_{\mathbf{X},s}$ and Ω is some derivation map associated with $\vec{B}_{\mathbf{X},s}$ whose optimality constant

$C_{\mathrm{opt}} = C_{\mathrm{opt}}(\Omega)$ does not exceed C. For each $y \in \mathrm{Ran}_{\mathbf{X},s}(\tilde{\Omega}_{\vec{B}}, C)$, we define

$$\|y\|_{\mathrm{Ran}_{\mathbf{X},s}(\tilde{\Omega}_{\vec{B}},C)} = \inf\{\|x\|_{\vec{B}_{\mathbf{X},s}} : y = \Omega x,\ \ C_{\mathrm{opt}}(\Omega) \leqslant C\}. \tag{7.7}$$

In other words, $\mathrm{Ran}_{\mathbf{X},s}(\tilde{\Omega}_{\vec{B}}, C)$ is the set of all $y \in B_0 + B_1$ for which there exists an element $b \in \mathscr{J}(\mathbf{X}, \vec{B})$ which satisfies

$$y = f_b'(s) \quad \text{and} \quad \|b\|_{\mathscr{J}(\mathbf{X},\vec{B})} \leqslant C\|f_b(s)\|_{\vec{B}_{\mathbf{X},s}}. \tag{7.8}$$

Furthermore $\|y\|_{\mathrm{Ran}_{\mathbf{X},s}(\tilde{\Omega}_{\vec{B}},C)} = \inf\|f_b(s)\|_{\vec{B}_{\mathbf{X},s}}$, where the infimum is taken over all elements b satisfying (7.8).

An analogue of this space for the real method, i.e. corresponding essentially to the case where $\mathbf{X} = \mathbf{l}^p$, was introduced and studied in Section IV of [20]. The following result is thus an analogue of part (ii) of Theorem 4.1 of [20, pp. 188, 189]. It is proved in a rather different way.

THEOREM 7.5. *Let $\vec{B}$ be a Banach pair, and let $\mathbf{X}$ be a Laurent compatible pseudolattice pair. Let s be a point in $\mathbb{A}$. Suppose that there exists an element $u \in \mathscr{J}(\mathbf{X}, \vec{B})$ such that $f_u(s) \neq 0$ and $f_u'(s) = 0$. Then there exists a constant $C_* \geqslant 1$, depending only on s and u, such that, for each $C > C_*$,*

$$\mathrm{Ran}_{\mathbf{X},s}(\tilde{\Omega}_{\vec{B},C}) = \vec{B}^{(1)}_{\mathbf{X},s}$$

and the norm of $\vec{B}^{(1)}_{\mathbf{X},s}$ is equivalent to the expression in (7.7). *The constants of this equivalence depend only on s, u and C.*

Remark 7.6. The hypothesis about the existence of u is a mild one which clearly holds, for example, whenever $\mathbf{X}$ is such that all (or some) finitely supported $B_0 \cap B_1$ valued sequences are in $\mathscr{J}(\mathbf{X}, \vec{B})$. All examples of $\mathbf{X}$ which we have considered in this paper have this property. In fact it may also hold without our "usual" requirements that the pseudolattice pair $\mathbf{X}$ admits differentiation and that S^{-1} is bounded on $\mathscr{J}(\mathbf{X}, \vec{B})$. But if both these conditions do hold then the existence of u follows from (7.5) and Theorem 7.2.

Proof. It is obvious from the definition that, for every choice of $C > 1$, we have $\mathrm{Ran}_{\mathbf{X},s}(\tilde{\Omega}_{\vec{B}}, C) \subset \vec{B}^{(1)}_{\mathbf{X},s}$ and also $\|y\|_{\vec{B}^{(1)}_{\mathbf{X},s}} \leqslant C\|y\|_{\mathrm{Ran}_{\mathbf{X},s}(\tilde{\Omega}_{\vec{B}},C)}$ for each $y \in \mathrm{Ran}_{\mathbf{X},s}(\tilde{\Omega}_{\vec{B}}, C)$. We shall establish the reverse inclusion and the reverse inequality

$$\|y\|_{\mathrm{Ran}_{\mathbf{X},s}(\tilde{\Omega}_{\vec{B}},C)} \leqslant C'\|y\|_{\vec{B}^{(1)}_{\mathbf{X},s}} \tag{7.9}$$

for each $C > C_*$ where the constant C' is given by

$$C' = \left(\frac{2\|u\|_{\mathscr{J}(\mathbf{X},\vec{B})}}{C\|f_u(s)\|_{\vec{B}_{\mathbf{X},s}} - \|u\|_{\mathscr{J}(\mathbf{X},\vec{B})}} + 1 \right) \tag{7.10}$$

and we choose

$$C_* = \frac{\|u\|_{\mathscr{J}(\mathbf{X},\vec{B})}}{\|f_u(s)\|_{\vec{B}_{\mathbf{X},s}}}. \tag{7.11}$$

This also ensures, incidentally, that $C_* \geqslant 1$ and that the denominator in (7.10) is strictly positive.

Let y be an arbitrary element of $\vec{B}^{(1)}_{\mathbf{X},s}$ and choose an arbitrary small positive number ε which also satisfies $1 + \varepsilon < C'$. There exists $b \in \mathscr{J}(\mathbf{X}, \vec{B})$ such that $f'_b(s) = y$ and

$$\|b\|_{\mathscr{J}(\mathbf{X},\vec{B})} \leqslant (1+\varepsilon)\|y\|_{\vec{B}^{(1)}_{\mathbf{X},s}}.$$

If $\|b\|_{\mathscr{J}(\mathbf{X},\vec{B})} \leqslant C\|f_b(s)\|_{\vec{B}_{\mathbf{X},s}}$ then we obtain that $y \in \mathrm{Ran}_{\mathbf{X},s}(\tilde{\Omega}_{\vec{B}}, C)$ and also that

$$\|y\|_{\mathrm{Ran}_{\mathbf{X},s}(\tilde{\Omega}_{\vec{B}},C)} \leqslant \|f_b(s)\|_{\vec{B}_{\mathbf{X},s}} \leqslant \|b\|_{\mathscr{J}(\mathbf{X},\vec{B})} \leqslant (1+\varepsilon)\|y\|_{\vec{B}^{(1)}_{\mathbf{X},s}} \leqslant C'\|y\|_{\vec{B}^{(1)}_{\mathbf{X},s}},$$

as required. Otherwise, we have

$$\|b\|_{\mathscr{J}(\mathbf{X},\vec{B})} > C\|f_b(s)\|_{\vec{B}_{\mathbf{X},s}} \tag{7.12}$$

and this is the case where we need to use the special element u. We shall show (very easily!) that, for an appropriate choice of $\lambda > 0$, we have

$$\|\lambda u + b\|_{\mathscr{J}(\mathbf{X},\vec{B})} \leqslant C\|f_{\lambda u+b}(s)\|_{\vec{B}_{\mathbf{X},s}}. \tag{7.13}$$

and

$$\|f_{\lambda u+b}(s)\|_{\vec{B}_{\mathbf{X},s}} \leqslant C'\|b\|_{\mathscr{J}(\mathbf{X},\vec{B})}. \tag{7.14}$$

Since $f'_{\lambda u+b}(s) = y$, these last two estimates will imply that $y \in \mathrm{Ran}_{\mathbf{X},s}(\tilde{\Omega}_{\vec{B}}, C)$ with

$$\|y\|_{\mathrm{Ran}_{\mathbf{X},s}(\tilde{\Omega}_{\vec{B}},C)} \leqslant (1+\varepsilon)C'\|y\|_{\vec{B}^{(1)}_{\mathbf{X},s}}.$$

This will give (7.9) and so complete the proof.

In view of the estimates $\|\lambda u+b\|_{\mathscr{J}(\mathbf{X},\vec{B})}\leqslant\lambda\|u\|_{\mathscr{J}(\mathbf{X},\vec{B})}+\|b\|_{\mathscr{J}(\mathbf{X},\vec{B})}$ and (7.12) and

$$C\|f_{\lambda u+b}(s)\|_{\vec{B}_{\mathbf{X},s}}\geqslant C\lambda\|f_u(s)\|_{\vec{B}_{\mathbf{X},s}}-C\|f_b(s)\|_{\vec{B}_{\mathbf{X},s}}>C\lambda\|f_u(s)\|_{\vec{B}_{\mathbf{X},s}}-\|b\|_{\mathscr{J}(\mathbf{X},\vec{B})},$$

it is clear that (7.13) will hold for $\lambda>0$ satisfying

$$\lambda\|u\|_{\mathscr{J}(\mathbf{X},\vec{B})}+\|b\|_{\mathscr{J}(\mathbf{X},\vec{B})}=C\lambda\|f_u(s)\|_{\vec{B}_{\mathbf{X},s}}-\|b\|_{\mathscr{J}(\mathbf{X},\vec{B})}. \tag{7.15}$$

Since we are assuming that $C>C_*$, our definition (7.11) of C_* ensures that the number

$$\lambda=\frac{2\|b\|_{\mathscr{J}(\mathbf{X},\vec{B})}}{C\|f_u(s)\|_{\vec{B}_{\mathbf{X},s}}-\|u\|_{\mathscr{J}(\mathbf{X},\vec{B})}}$$

is positive and satisfies (7.15). Apart from giving (7.13) this also implies (recalling (7.10)) that

$$\begin{aligned}\|f_{\lambda u+b}(s)\|_{\vec{B}_{\mathbf{X},s}}&\leqslant\|\lambda u+b\|_{\mathscr{J}(\mathbf{X},\vec{B})}\\&\leqslant\left(\frac{2\|u\|_{\mathscr{J}(\mathbf{X},\vec{B})}}{C\|f_u(s)\|_{\vec{B}_{\mathbf{X},s}}-\|u\|_{\mathscr{J}(\mathbf{X},\vec{B})}}+1\right)\|b\|_{\mathscr{J}(\mathbf{X},\vec{B})}\\&=C'\|b\|_{\mathscr{J}(\mathbf{X},\vec{B})}.\end{aligned}$$

Thus we have established (7.14) and the proof indeed is now complete. ∎

8. EQUIVALENCE THEOREMS

The classical "equivalence theorem" in the theory of real interpolation states that two different constructions, usually referred to as the "J-method" and the "K-method" of Peetre, give rise to the same interpolation spaces. Earlier versions of these two methods appear as the two definitions of "espaces de moyennes" given in [41, pp. 9, 10]. Our definition here of the spaces $\vec{B}_{\mathbf{X},s}$ is of course modelled on the J-method and the first definition in [41, pp. 9, 10] (cf. also the equivalent "discretized" definition in [41, pp. 17, 18]). However it turns out that the second definition in [41], corresponding to the K-method, can also be generalized to our context here. Furthermore the condition introduced in Definition 3.4 for other purposes is also exactly what is required to prove a generalized version of the equivalence theorem, which will be the main result in this section. By analogy with the real method, our equivalence theorem can be expected to be a convenient tool for describing the duals of the spaces $\vec{B}_{\mathbf{X},s}$ and for obtaining reiteration theorems. It might ultimately be helpful, also for other purposes, to know that there are equivalent definitions in the style of the K-method for the complex method and the $\pm$ method.

DEFINITION 8.1. For each Banach pair $\vec{B}$ and pseudolattice pair $\mathbf{X}$ we define $\mathscr{K}(\mathbf{X}, \vec{B})$ to be the space of all couples of $B_0 + B_1$ valued sequences

$$\mathbf{b} = (\{b_{0,n}\}_{n\in\mathbb{Z}}, \{b_{1,n}\}_{n\in\mathbb{Z}})$$

such that the sequence $\{e^{jn} b_{j,n}\}_{n\in\mathbb{Z}}$ is in $\mathscr{X}_j(B_j)$ for $j = 0, 1$. This space is normed by

$$\|(\{b_{0,n}\}_{n\in\mathbb{Z}}, \{b_{1,n}\}_{n\in\mathbb{Z}})\|_{\mathscr{K}}(\mathbf{X}, \vec{B}) = \sum_{j=0}^{1} \|\{e^{jn} b_{j,n}\}_{n\in\mathbb{Z}}\|_{\mathscr{X}_j(B_j)}$$

and is of course a Banach space.

For each fixed $s \in \mathbb{A}$, let $\mathscr{K}_s(\mathbf{X}, \vec{B})$ be the subspace of $\mathscr{K}(\mathbf{X}, \vec{B})$ consisting of those couples of sequences $\mathbf{b}$ which satisfy the condition $s^n(b_{0,n} + b_{1,n}) = b_{0,0} + b_{1,0}$ for each $n \in \mathbb{Z}$. For each such element $\mathbf{b}$, let $\kappa(\mathbf{b}) = b_{0,0} + b_{1,0}$. Then we let $\vec{B}_{\mathbf{X},s:K}$ be the space of all elements $b \in B_0 + B_1$ which can be represented in the form $b = \kappa(\mathbf{b})$ for some $\mathbf{b} \in \mathscr{K}_s(\mathbf{X}, \vec{B})$. We define a seminorm on this space by setting

$$\|b\|_{\vec{B}_{\mathbf{X},s:K}} = \inf\{\|\mathbf{b}\|_{\mathscr{K}(\mathbf{X},\vec{B})} : \mathbf{b} \in \mathscr{K}_s(\mathbf{X}, \vec{B}), b = \kappa(\mathbf{b})\}.$$

THEOREM 8.2. *Let* $\mathbf{X}$ *be a Laurent compatible pair of pseudolattices and let* $\vec{B}$ *be an arbitrary Banach pair and let* s *be any point of* $\mathbb{A}$.

(i) *If* $\mathbf{X}$ *admits differentiation then* $\vec{B}_{\mathbf{X},s} \subset \vec{B}_{\mathbf{X},s:K}$ *and the embedding is continuous.*

(ii) *If the right-shift operator* S^{-1} *is bounded on* $\mathscr{X}_j(B_j)$ *for* $j = 0, 1$ *and if for each* $r \in (0, 1)$

$$\lim_{n\to-\infty} r^{|n|}\|b_n\|_{B_0} = 0 \qquad \text{for all } \{b_n\}_{n\in\mathbb{Z}} \in \mathscr{X}_0(B_0) \tag{8.1}$$

and

$$\lim_{n\to\infty} r^{n}\|b_n\|_{B_1} = 0 \qquad \text{for all } \{b_n\}_{n\in\mathbb{Z}} \in \mathscr{X}_1(B_1) \tag{8.2}$$

then $\vec{B}_{\mathbf{X},s:K} \subset \vec{B}_{\mathbf{X},s}$ *and the embedding is continuous.*

COROLLARY 8.3. *If* (2.4) *holds and if the shift operator S is an isometry on* $\mathscr{X}_j(B_j)$ *for* $j = 0, 1$ *then the spaces* $\vec{B}_{\mathbf{X},s}$ *and* $\vec{B}_{\mathbf{X},s:K}$ *coincide to within equivalence of norms.*

Proof. The corollary follows immediately from the theorem by Lemma 3.6.

The proof of the theorem is an easy adaptation of well-known proofs for the special case of the real method: First, for part (i), suppose that $b \in \vec{B}_{\mathbf{X},s}$ with $\|b\|_{\vec{B}_{\mathbf{X},s}} < 1$. Then there exists a sequence $\{b_n\}_{n\in\mathbb{Z}}$ in $\mathscr{J}(\mathbf{X}, \vec{B})$ with $\|\{b_n\}_{n\in\mathbb{Z}}\|_{\mathscr{J}(\mathbf{X},\vec{B})} < 1$, such that $b = \sum_{n\in\mathbb{Z}} s^n b_n$. Let us define $\mathbf{b} = (\{b_{0,n}\}_{n\in\mathbb{Z}}, \{b_{1,n}\}_{n\in\mathbb{Z}})$ by setting

$$b_{0,n} = s\sum_{k<0} s^k b_{n+k+1} \quad \text{and} \quad b_{1,n} = s\sum_{k\geqslant 0} s^k b_{n+k+1}.$$

Then $\{b_{0,n}\}_{n\in\mathbb{Z}} = sD_{0,1/s}(\{b_n\}_{n\in\mathbb{Z}})$. Furthermore $e^n b_{1,n} = (s/e)\sum_{k\geqslant 0} (s/e)^k \times e^{n+k+1} b_{n+k+1}$ so that $\{e^n b_{1,n}\}_{n\in\mathbb{Z}} = (s/e)D_{1,s/e}(\{e^n b_n\}_{n\in\mathbb{Z}})$. This establishes that $\mathbf{b} \in \mathscr{K}(\mathbf{X}, \vec{B})$ with norm not exceeding

$$C_1(\mathbf{X}, \vec{B}, s) := s\|D_{0,1/s}\|_{\mathscr{X}_0(B_0)\to\mathscr{X}_0(B_0)} + (s/e)\|D_{1,s/e}\|_{\mathscr{X}_1(B_1)\to\mathscr{X}_1(B_1)}.$$

But obviously we also have $s^n(b_{0,n} + b_{1,n}) = b$ for each $n \in \mathbb{Z}$. So $\mathbf{b} \in \mathscr{K}_s(\mathbf{X}, \vec{B})$ and $b = \kappa(\mathbf{b}) \in \vec{B}_{\mathbf{X},s:K}$ with (semi)norm not exceeding $C_1(\mathbf{X}, \vec{B}, s)$. So part (i) is proved.

Now, for part (ii), suppose that $b \in \vec{B}_{\mathbf{X},s:K}$ with $\|b\|_{\vec{B}_{\mathbf{X},s:K}} < 1$. Then there exists an element $\mathbf{b} = (\{b_{0,n}\}_{n\in\mathbb{Z}}, \{b_{1,n}\}_{n\in\mathbb{Z}})$ in $\mathscr{K}_s(\mathbf{X}, \vec{B})$ with $\|\mathbf{b}\|_{\mathscr{K}(\mathbf{X},\vec{B})} < 1$ such that $b = \kappa(\mathbf{b})$. We define the sequence $\{b_n\}_{n\in\mathbb{Z}}$ by setting $b_n = b_{0,n} - s^{-1}b_{0,n-1}$ for each $n \in \mathbb{Z}$. Then, since $s^n(b_{0,n} + b_{1,n}) = s^{n-1}(b_{0,n-1} + b_{1,n-1})$, we obtain that $b_n = s^{-1}b_{1,n-1} - b_{1,n}$ and consequently $b_n \in B_0 \cap B_1$. By the boundedness of the right-shift operator we deduce that $\{b_n\} \in \mathscr{J}(\mathbf{X}, \vec{B})$ with norm not exceeding $C_2(\mathbf{X}, \vec{B}, s) := 1 + s^{-1}\max_{j=0.1}\{e^j\|S^{-1}\|_{\mathscr{X}_j(B_j)\to\mathscr{X}_j(B_j)}\}$. By (8.1) we have that $\lim_{n\to-\infty} \|s^n b_{0,n}\|_{B_0} = 0$ and by (8.2) we also have that $\lim_{n\to\infty} \|s^n b_{1,n}\|_{B_1} = \lim_{n\to\infty} \|(s/e)^n e^n b_{1,n}\|_{B_1} = 0$. Then

$$\sum_{n\leqslant 0} s^n b_n = \sum_{n\leqslant 0} (s^n b_{0,n} - s^{n-1}b_{0,n-1}) = b_{0.0} - \lim_{n\to-\infty} s^n b_{0,n} = b_{0,0}$$

and

$$\sum_{n\geqslant 1} s^n b_n = \sum_{n\geqslant 1} (s^{n-1}b_{1,n-1} - s^n b_{1,n}) = b_{1.0} - \lim_{n\to\infty} s^n b_{1,n} = b_{1,0}.$$

So the series $\sum_{n\in\mathbb{Z}} s^n b_n$ converges to $b_{0,0} + b_{0,1} = b$ with respect to the norm of $B_0 + B_1$. This shows that $b \in \vec{B}_{\mathbf{X},s}$ with norm not exceeding $C_2(\mathbf{X}, \vec{B}, s)$. This completes the proof. ∎

APPENDIX A.

A.1. *Uniform Boundedness of the Operator Norms* $\|D_{j,\rho}\|_{\mathscr{X}_j(B)\to\mathscr{X}_j(B)}$ *and* $\|S^{\pm k}\|_{\mathscr{X}_j(B)\to\mathscr{X}_j(B)}$ *over the Class of all Banach Spaces* B

The key to studying these norms will be the following result:

Proposition A.1. *Let* $\mathscr{X}$ *be a pseudolattice and let* $V = \{v_{jk}\}_{j,k\in\mathbb{Z}}$ *be an infinite scalar matrix such that, for each* $B \in \mathbf{Ban}$,

(i) *for all* $\{b_n\}_{n\in\mathbb{Z}} \in \mathscr{X}(B)$, *the sum* $\sum_{k\in\mathbb{Z}} v_{nk}b_k$ *converges in* B *for each* $n \in \mathbb{Z}$ *and the sequence* $\{\sum_{k\in\mathbb{Z}} v_{nk}b_k\}_{n\in\mathbb{Z}}$ *is an element of* $\mathscr{X}(B)$, *and*

(ii) *the matrix* V *defines a bounded linear operator from* $\mathscr{X}(B)$ *into itself, which we will also denote by* V. *Then* $\sup_{B\in\mathbf{Ban}} \|V\|_{\mathscr{X}(B)\to\mathscr{X}(B)} < \infty$ *and there exists a Banach space* Y *such that*

$$\sup_{B\in\mathbf{Ban}} \|V\|_{\mathscr{X}(B)\to\mathscr{X}(B)} \leqslant C(\mathscr{X})^2 \|V\|_{\mathscr{X}(Y)\to\mathscr{X}(Y)},$$

where $C(\mathscr{X})$ *is the constant appearing in part* (iii) *of Definition* 2.1.

Proof. It will be convenient to present the main step of the proof as a separate lemma. ∎

Lemma A.2. *Let* $\{B_\gamma\}_{\gamma\in\Gamma}$ *be a family of complex Banach spaces indexed by an index set* Γ *and let* B_Γ *denote the Banach space of "bounded functions"* f *defined on* Γ *such that* $f(\gamma) \in B_\gamma$ *for each* $\gamma \in \Gamma$ *and such that the norm* $\|f\|_{B_\Gamma} = \sup_{\gamma\in\Gamma} \|f(\gamma)\|_{B_\gamma}$ *is finite. Then any pseudolattice* $\mathscr{X}$ *and operator* V *satisfying the hypotheses of Proposition* A.1 *also satisfy*

$$\sup_{\gamma\in\Gamma} \|V\|_{\mathscr{X}(B_\gamma)\to\mathscr{X}(B_\gamma)} \leqslant C(\mathscr{X})^2 \|V\|_{\mathscr{X}(B_\Gamma)\to\mathscr{X}(B_\Gamma)}. \quad \text{(A.1)}$$

To prove this lemma, we fix an arbitrary $\varepsilon > 0$ and an arbitrary $\beta \in \Gamma$ and let $\{b_n\}_{n\in\mathbb{Z}}$ be an element of $\mathscr{X}(B_\beta)$ such that $\|\{b_n\}_{n\in\mathbb{Z}}\|_{\mathscr{X}(B_\beta)} = 1$ and $\|V(\{b_n\}_{n\in\mathbb{Z}})\|_{\mathscr{X}(B_\beta)} \geqslant \|V\|_{\mathscr{X}(B_\beta)\to\mathscr{X}(B_\beta)} - \varepsilon$. For each $n \in \mathbb{Z}$ we denote $\tilde{b}_n = \sum_{k\in\mathbb{Z}} v_{nk}b_k$, i.e. $\{\tilde{b}_n\} = V(\{b_n\}_{n\in\mathbb{Z}})$. We shall need two norm one linear operators $P : B_\Gamma \to B_\beta$ and $Q : B_\beta \to B_\Gamma$. We define P by setting $Pf = f(\beta)$ for each $f \in B_\Gamma$. For the definition of Q, for each $b \in B_\beta$, we let Qb be the element of B_Γ such that $Qb(\beta) = b$ and, for each $\gamma \neq \beta$ in Γ, $Qb(\gamma)$ is the zero element of B_γ.

By part (iii) of Definition 2.1, the sequence $\{Qb_n\}_{n\in\mathbb{Z}}$ is an element of $\mathscr{X}(B_\Gamma)$ with $\|\{Qb_n\}_{n\in\mathbb{Z}}\|_{\mathscr{X}(B_\Gamma)} \leqslant C(\mathscr{X})$. It is also clear that $V(\{Qb_n\}_{n\in\mathbb{Z}})$ is the sequence $\{Q\tilde{b}_n\}_{n\in\mathbb{Z}}$. Furthermore, since PQ is the identity operator on B_β we obtain, applying (iii) of Definition 2.1 again, that

$$\begin{aligned}\|V(\{b_n\}_{n\in\mathbb{Z}})\|_{\mathscr{X}(B_\beta)} &= \|\{\tilde{b}_n\}_{n\in\mathbb{Z}}\|_{\mathscr{X}(B_\beta)} = \|\{PQ\tilde{b}_n\}_{n\in\mathbb{Z}}\|_{\mathscr{X}(B_\beta)}\\ &\leqslant C(\mathscr{X})\|\{Q\tilde{b}_n\}_{n\in\mathbb{Z}}\|_{\mathscr{X}(B_\Gamma)}\\ &\leqslant C(\mathscr{X})\|V\|_{\mathscr{X}(B_\Gamma)\to\mathscr{X}(B_\Gamma)}\|\{Qb_n\}_{n\in\mathbb{Z}}\|_{\mathscr{X}(B_\Gamma)}\\ &\leqslant C(\mathscr{X})^2\|V\|_{\mathscr{X}(B_\Gamma)\to\mathscr{X}(B_\Gamma)}.\end{aligned}$$

Since β and ε are arbitrary this gives (A.1) and proves the lemma.

We can now proceed with the proof of Proposition A.1. Let $\lambda = \sup_{B\in\mathbf{Ban}}\|V\|_{\mathscr{X}(B)\to\mathscr{X}(B)}$. If $\lambda = 0$ there is nothing to prove. Otherwise, whether or not λ is finite, let $\{\lambda_n\}_{n\in\mathbb{N}}$ be a sequence satisfying $0<\lambda_n<\lambda_{n+1}<\lambda$ and $\lim_{n\to\infty}\lambda_n = \lambda$. For each $n\in\mathbb{N}$, there exists a Banach space $B_n\in$ **Ban** such that $\|V\|_{\mathscr{X}(B_n)\to\mathscr{X}(B_n)}\geqslant\lambda_n$. Now let $\Gamma = \mathbb{N}$ and apply Lemma 9.2 to the family $\{B_n\}_{n\in\mathbb{N}}$. We choose Y to be the Banach space B_Γ. By hypothesis $V:\mathscr{X}(B_\Gamma)\to\mathscr{X}(B_\Gamma)$ is a bounded map, and so (9.1) gives the conclusions of the proposition. ∎

We can now apply Proposition 9.1 to obtain the results mentioned in Remarks 3.5 and 3.7.

Corollary A.3. *Let* $\mathbf{X} = \{\mathscr{X}_0,\mathscr{X}_1\}$ *be a fixed pair of pseudolattices which admits differentiation. Then*

(i) *for each* $\rho\in\mathbb{C}$ *with* $0<|\rho|<1$, *there exists a finite constant* $C_*(\mathbf{X},\rho)$ *depending only on* $\mathbf{X}$ *and* ρ *such that* $\sup_{B\in\mathbf{Ban}}\|D_{j,\rho}\|_{\mathscr{X}_j(B)\to\mathscr{X}_j(B)}\leqslant C_*(\mathbf{X},\rho)$ *for* $j = 0,1$.

(ii) *If, furthermore,* $\mathbf{X}$ *admits differentiation uniformly then, for each* α *and* β *with* $0<\alpha<\beta<1$, *the quantity*

$$C(\mathbf{X},\alpha,\beta) = \sup\{\|D_{j,\rho}\|_{\mathscr{X}_j(B)\to\mathscr{X}_j(B)} : \alpha\leqslant|\rho|\leqslant\beta,\ j = 0,1,\ B\in\mathbf{Ban}\}$$

is finite.

Proof. For $j = 0$ and each $n\in\mathbb{Z}$ we set $v_{nm} = \rho^{-m+n+1}$ for each $m<n+1$ and $v_{nm} = 0$ otherwise. Then $V = D_{0,\rho}$. If we choose $\mathscr{X} = \mathscr{X}_0$, then conditions (i)–(iii) of Definition 3.4 ensure that V satisfies conditions (i) and (ii) of Proposition A.1 and so $\sup_{B\in\mathbf{Ban}}\|D_{0,\rho}\|_{\mathscr{X}_0(B)\to\mathscr{X}_0(B)}<\infty$. An analogous argument for $j = 1$ completes the proof of part (i).

For part (ii), let Γ be the set of all triples $\gamma = (j, \rho, n)$ where $j \in \{0, 1\}$, $\alpha \leqslant |\rho| \leqslant \beta$ and $n \in \mathbb{N}$. For each such $\gamma = (j, \rho, n)$, we know by part (i) that $\sup\{\|D_{j,\rho}\|_{\mathscr{X}_j(B) \to \mathscr{X}_j(B)}\colon B \in \mathbf{Ban}\}$ is finite. So there exists a complex Banach space B_γ for which

$$\|D_{j,\rho}\|_{\mathscr{X}_j(B_\gamma) \to \mathscr{X}_j(B_\gamma)} \geqslant \sup\{\|D_{j,\rho}\|_{\mathscr{X}_j(B) \to \mathscr{X}_j(B)}\colon B \in \mathbf{Ban}\} - 1/n. \tag{A.2}$$

By hypothesis, the Banach space B_Γ constructed as in Lemma A.2 using this particular family $\{B_\gamma\}_{\gamma \in \Gamma}$ must satisfy

$$\sup\{\|D_{j,\rho}\|_{\mathscr{X}_j(B_\Gamma) \to \mathscr{X}_j(B_\Gamma)}\colon j = 0, 1,\ \alpha \leqslant |\rho| \leqslant \beta\} \coloneqq \lambda(\alpha, \beta, \mathbf{X}) < \infty.$$

So, for any fixed $j \in \{0, 1\}$ and $\rho \in \mathbb{C}$ satisfying $\alpha \leqslant |\rho| \leqslant \beta$, we can apply Lemma A.2 with $\mathscr{X} = \mathscr{X}_j$ and $V = D_{j,\rho}$ and obtain from (A.1) that

$$\sup_{\gamma \in \Gamma} \|D_{j,\rho}\|_{\mathscr{X}_j(B_\gamma) \to \mathscr{X}_j(B_\gamma)} \leqslant C(\mathscr{X}_j)^2 \lambda(\alpha, \beta, \mathbf{X}).$$

This, together with (A.2), gives that

$$\sup\{\|D_{j,\rho}\|_{\mathscr{X}_j(B) \to \mathscr{X}_j(B)}\colon B \in \mathbf{Ban}\} \leqslant \max_{j=0,1} C(\mathscr{X}_j)^2 \lambda(\alpha, \beta, \mathbf{X})$$

and completes the proof of (ii). ∎

Corollary A.4. *Suppose that* $\mathbf{X}$ *is a Laurent compatible pair satisfying the hypotheses of Lemma* 3.6. *Then these same hypotheses hold uniformly for all complex Banach spaces*, *i.e.*

$$\sum_{k>0} r^k \sup_{B \in \mathbf{Ban}} \|S^{-k}\|_{\mathscr{X}_0(B) \to \mathscr{X}_0(B)} < \infty \quad \textit{and}$$
$$\sum_{k>0} r^k \sup_{B \in \mathbf{Ban}} \|S^{k}\|_{\mathscr{X}_1(B) \to \mathscr{X}_1(B)} < \infty \tag{A.3}$$

Proof. Let $\mathbb{Z}_0 = \mathbb{Z} \backslash \mathbb{N}$ and $\mathbb{Z}_1 = \mathbb{N}$. Then (A.3) is equivalent to

$$\sum_{m \in \mathbb{Z}_j} r^{|m|} \sup_{B \in \mathbf{Ban}} \|S^m\|_{\mathscr{X}_j(B) \to \mathscr{X}_j(B)} < \infty \qquad \text{for } j = 0, 1. \tag{A.4}$$

Fix $j = 0$ or 1. For each $m \in \mathbb{Z}_j$, S^m has a matrix representation V which, by the hypotheses of Lemma 3.6, also satisfies the hypotheses of Proposition 9.1 for $\mathscr{X} = \mathscr{X}_j$. So Proposition 9.1 gives that

$$\sup_{B \in \mathbf{Ban}} \|S^m\|_{\mathscr{X}_j(B) \to \mathscr{X}_j(B)} \leqslant C(\mathscr{X}_j)^2 \|S^m\|_{\mathscr{X}_j(Y_m) \to \mathscr{X}_j(Y_m)} < \infty,$$

for some Banach space Y_m. Now let $\Gamma = \mathbb{Z}_j$ and let $Y = B_\Gamma$ where $B_m = Y_m$ for each $m \in \mathbb{Z}_j$. Then Lemma A.2 gives that

$$\|S^m\|_{\mathscr{X}_j(Y_m) \to \mathscr{X}_j(Y_m)} \leqslant \sup_{k \in \mathbb{Z}_j} \|S^m\|_{\mathscr{X}_j(Y_k) \to \mathscr{X}_j(Y_k)} \leqslant C(\mathscr{X}_j)^2 \|S^m\|_{\mathscr{X}_j(Y) \to \mathscr{X}_j(Y)}.$$

We deduce that

$$\begin{aligned} \sum_{m \in \mathbb{Z}_j} r^{|m|} \sup_{B \in \mathbf{Ban}} \|S^m\|_{\mathscr{X}_j(B) \to \mathscr{X}_j(B)} &\leqslant C(\mathscr{X}_j)^2 \sum_{m \in \mathbb{Z}_j} r^{|m|} \|S^m\|_{\mathscr{X}_j(Y_m) \to \mathscr{X}_j(Y_m)} \\ &\leqslant C(\mathscr{X}_j)^4 \sum_{m \in \mathbb{Z}_j} r^{|m|} \|S^m\|_{\mathscr{X}_j(Y) \to \mathscr{X}_j(Y)}. \end{aligned}$$

By hypothesis this last sum is finite, and so we obtain (A.4) and complete the proof. ∎

A.2. *Boundedness of the Constants $C_{\sigma,\sigma'}$ for Large $|\sigma - \sigma'|$*

Here we prove the estimate (4.26), i.e. that for each $\delta > 0$

$$\sup\{C_{\sigma,\sigma'} : \sigma, \sigma' \in \mathbb{S}, |e^\sigma - e^{\sigma'}| \geqslant \delta\} < \infty.$$

Let us first observe that

$$\begin{aligned} |e^\sigma - e^{\sigma'}| \geqslant \delta &\Rightarrow |e^{\sigma - \sigma'} - 1| \geqslant \frac{\delta}{e} \Rightarrow |e^{(\sigma-\sigma')/2} - e^{-(\sigma-\sigma')/2}| \geqslant \frac{\delta}{e\sqrt{e}} \\ &\Rightarrow \psi_1(\sigma - \sigma') \geqslant \frac{|e^{(\sigma-\sigma')^2}|\delta}{e\sqrt{e}|\sigma - \sigma'|} \geqslant \frac{e^{-(\mathrm{Im}(\sigma-\sigma'))^2}\delta}{e\sqrt{e}|\sigma - \sigma'|}. \end{aligned}$$

Then, since $\psi_1(0) = 1$ and $\psi_1(z) = 0$ only when $z = 2\pi k i$ for some nonzero integer k, we have that for each $\delta > 0$

$$\inf\{\psi_1(\sigma - \sigma') \colon \sigma, \sigma' \in \mathbb{S}, |\mathrm{Im}\,\sigma - \mathrm{Im}\,\sigma'| \leqslant 2\pi, |e^\sigma - e^{\sigma'}| \geqslant \delta\} > 0.$$

So, by (4.23), $C_{\sigma,\sigma'} = \gamma C(\sigma, \sigma')$ satisfies for each positive R and δ,

$$K_{\delta,R} = \sup\{C_{\sigma,\sigma'} \colon \sigma, \sigma' \in \mathbb{S}, |\mathrm{Im}\,\sigma - \mathrm{Im}\,\sigma'| \leqslant R, |e^\sigma - e^{\sigma'}| \geqslant \delta\} < \infty. \quad \text{(A.5)}$$

This reduces the proof of (4.26) to showing, for example, that

$$\sup\{C_{\sigma,\sigma'}\colon \sigma,\sigma'\in\mathbb{S}, |\mathrm{Im}\,\sigma-\mathrm{Im}\,\sigma'|>2\pi, |e^{\sigma}-e^{\sigma'}|\geqslant\delta\}<\infty \quad \text{for each } \delta>0. \tag{A.6}$$

One ingredient for the proof of (A.6) is the fact that, for all σ and σ' which are not "too close," the set $\mathscr{R}_{*,\infty}(b, C_{\mathrm{opt}}, \sigma, \sigma', \vec{B})$ is "trivial" in the sense that it is uniformly "comparable" with the unit ball $\mathscr{B}_{[\vec{B}]_{\mathrm{Re}\,\sigma'}}$ of $[\vec{B}]_{\mathrm{Re}\,\sigma'}$. More specifically, there exists a universal constant β such that, for all σ and σ' in $\mathbb{S}$ satisfying $|\mathrm{Im}\,\sigma-\mathrm{Im}\,\sigma'|>2\pi$, and for every $C_{\mathrm{opt}}\geqslant 1$ and every $\varepsilon>0$,

$$\begin{aligned} C_{\mathrm{opt}}\|b\|_{[\vec{B}]_{\mathrm{Re}\,\sigma}}\mathscr{B}_{[\vec{B}]_{\mathrm{Re}\,\sigma'}} &\subset \tilde{\mathscr{R}}_{*,\infty}(b, \beta C_{\mathrm{opt}}, \sigma, \sigma', \vec{B}) \\ &\subset (\beta+\varepsilon)C_{\mathrm{opt}}\|b\|_{[\vec{B}]_{\mathrm{Re}\,\sigma}}\mathscr{B}_{[\vec{B}]_{\mathrm{Re}\,\sigma'}}. \end{aligned} \tag{A.7}$$

(As in the statement of Theorem 4.2 we still require the constant C_{opt} to satisfy $C_{\mathrm{opt}}>1$.) For every choice of positive β and ε, the second inclusion in (A.7) is an immediate consequence of the definitions. Let us now prove the first inclusion. We shall use an entire function $\phi=\phi_{\xi,\eta,S}$ which is defined for any two distinct points ξ and η in $\mathbb{S}$ by

$$\phi_{\xi,\eta,S}(z)=\frac{z-\eta}{\xi-\eta}e^{(z-\xi)^2}=\frac{(z-\xi)+(\xi-\eta)}{\xi-\eta}e^{(z-\xi)^2}. \tag{A.8}$$

It satisfies $\phi(\xi)=1$ and $\phi(\eta)=0$ and $\sup_{z\in\mathbb{S}}|\phi(z)|$ is finite. In fact

$$\sup_{z\in S}|\phi(z)|\leqslant\frac{1}{\sqrt{2}\,e^2|\xi-\eta|}+e\leqslant\frac{1}{\sqrt{2}\,e^2|\mathrm{Im}\,\xi-\mathrm{Im}\,\eta|+e}.$$

(A different choice of $\phi_{\xi,\eta,\mathbb{S}}$ with similar properties and a smaller supremum could improve our estimate of β.) Given any $b'\in C_{\mathrm{opt}}\|b\|_{[\vec{B}]_{\mathrm{Re}\,\sigma}}\mathscr{B}_{[\vec{B}]_{\mathrm{Re}\,\sigma'}}$, we choose $f_1\in\mathscr{F}(\vec{B})$ with $f_1(\sigma')=b'$ and $\|f_1\|_{\mathscr{F}(\vec{B})}\leqslant C_{\mathrm{opt}}(1+\varepsilon)\|b\|_{[\vec{B}]_{\mathrm{Re}\,\sigma}}$. We also choose $f_2\in\mathscr{F}(\vec{B})$ with $f_2(\sigma)=b$ and $\|f_2\|_{\mathscr{F}(\vec{B})}\leqslant(1+\varepsilon)\|b\|_{[\vec{B}]_{\mathrm{Re}\,\sigma}}\leqslant C_{\mathrm{opt}}(1+\varepsilon)\|b\|_{[\vec{B}]_{\mathrm{Re}\,\sigma}}$. Set $f_3=\phi_{\sigma',\sigma},\mathbb{S}f_1+\phi_{\sigma,\sigma'},\mathbb{S}f_2$ (i.e. using two functions defined as in (A.8)). Then $f_3(\sigma)=f_2(\sigma)=b$ and $f_3(\sigma')=f_1(\sigma')=b'$

and

$$\|f_3\|_{\mathscr{F}(\vec{B})} \leqslant 2(1+\varepsilon)\left(\frac{1}{\sqrt{2}e^2|\mathrm{Im}\,\sigma - \mathrm{Im}\,\sigma'|} + e\right) C_{\mathrm{opt}}\|b\|_{[\vec{B}]_{\mathrm{Re}\,\sigma}}.$$

Since $|\mathrm{Im}\,\sigma - \mathrm{Im}\,\sigma'| > 2\pi$ and $\varepsilon > 0$ can be arbitrarily small, this shows that the first inclusion of (A.7) holds for

$$\beta = 2\left(\frac{1}{2\sqrt{2}e^2\pi} + e\right).$$

Now we use an obvious inclusion like the second inclusion in (A.7) followed by an application of the first inclusion in (A.7), but with C_{opt} replaced by $(1+\varepsilon)C_{\mathrm{opt}}$ and σ' replaced by any another point $\sigma'' \in \mathbb{S}$ which satisfies $\mathrm{Re}\,\sigma'' = \mathrm{Re}\,\sigma'$ and $|\mathrm{Im}\,\sigma - \mathrm{Im}\,\sigma''| > 2\pi$. This gives

$$\begin{aligned}\tilde{\mathscr{R}}_{*,\infty}(b, C_{\mathrm{opt}}, \sigma, \sigma', \vec{B}) &\subset (1+\varepsilon)C_{\mathrm{opt}}\|b\|_{[\vec{B}]_{\mathrm{Re}\,\sigma}}\mathscr{B}_{[\vec{B}]_{\mathrm{Re}\,\sigma'}} \\ &\subset \tilde{\mathscr{R}}_{*,\infty}(b, \beta(1+\varepsilon)C_{\mathrm{opt}}, \sigma, \sigma'', \vec{B}). \qquad \text{(A.9)}\end{aligned}$$

More specifically, we can choose $\sigma'' = \sigma' + 2\pi ki$ for some integer k. Then $e^{\sigma''} = e^{\sigma'}$. Furthermore, by choosing the integer k appropriately, we can have $2\pi < |\mathrm{Im}\,\sigma - \mathrm{Im}\,\sigma''| \leqslant 4\pi$. This enables us to apply (A.5) with $R = 4\pi$ and (4.8) to obtain that

$$\tilde{\mathscr{R}}_{*,\infty}(b, \beta(1+\varepsilon)C_{\mathrm{opt}}, \sigma, \sigma'', \vec{B}) \subset \tilde{\mathscr{R}}(b, K_{\delta,4\pi}\beta(1+\varepsilon)C_{\mathrm{opt}}, e^{\sigma}, e^{\sigma'}, \mathbf{FC}, \vec{B}). \qquad \text{(A.10)}$$

Combining (A.9) and (A.10) we obtain that the supremum in (A.6) is indeed finite (it can be taken to be any number larger than $K_{\delta,4\pi}\beta$) and so we have proved (4.26).

A.3. *Unboundedness of $C_{\sigma,\sigma'}$ as $e^{\sigma'}$ Tends to e^{σ}*

Inclusions (A.7) also provide us with a means of proving, as one would conjecture after Remark 4.3, that the constant $C_{\sigma,\sigma'}$ cannot remain bounded as $|e^{\sigma} - e^{\sigma'}|$ becomes arbitrarily small: Let us fix some $\sigma \in \mathbb{S}$ and let $\{\sigma'_n\}_{n\in\mathbb{N}}$ be some sequence in $\mathbb{S}$ which converges to $\sigma + 4\pi i$. We can suppose that $|\mathrm{Im}\,\sigma'_n - \mathrm{Im}\,\sigma| > 2\pi$ for all n, and so, by (A.7),

$$C_{\mathrm{opt}}\|b\|_{[\vec{B}]_{\mathrm{Re}\,\sigma}}\mathscr{B}_{[\vec{B}]_{\mathrm{Re}\,\sigma'}} \subset \tilde{\mathscr{R}}_{*,\infty}(b, \beta C_{\mathrm{opt}}, \sigma, \sigma'_n, \vec{B}).$$

This implies that there exist two distinct points $x \neq y$ which are contained in $\tilde{\mathscr{R}}_{*,\infty}(b, \beta C_{\text{opt}}, \sigma, \sigma', \vec{B})$ for all n. Now suppose that there exists some constant C such that $C_{\sigma,\sigma'_n} \leqslant C$ for all n. This implies that x and y are both contained in $\tilde{\mathscr{R}}(b, CC_{\text{opt}}, e^{\sigma}, e^{\sigma'_n}, \mathbf{FC}, \vec{B})$ for all n. We shall see that this leads to a contradiction by showing that $\tilde{\mathscr{R}}(b, CC_{\text{opt}}, e^{\sigma}, e^{\sigma'_n}, \mathbf{FC}, \vec{B})$ is contained in a ball in $B_0 + B_1$ whose centre is b and whose radius r_n tends to 0 as n tends to ∞: Consider the class $\hat{E}$ of functions $f : \bar{\mathbb{A}} \to B_0 + B_1$ of the form $f(z) = \sum_{k \in \mathbb{Z}} z^k b_k$ where $\{b_k\}_{k \in \mathbb{Z}} \in E(b, CC_{\text{opt}}, e^{\sigma}, \mathbf{FC}, \vec{B})$. Since $\|f(z)\|_{B_0+B_1} \leqslant CC_{\text{opt}}$ for all $z \in \partial\mathbb{A}$, it follows by Cauchy's formula that f' is bounded in $B_0 + B_1$ norm on compact subsets of $\mathbb{A}$. More specifically,

$$\sup\{\|f'(z)\|_{B_0+B_1} : f \in \hat{E}, |z - e^{\sigma}| \leqslant \delta\} = M_{\delta} < \infty$$

for each $\delta < \operatorname{dist}(e^{\sigma}, \partial\mathbb{A})$. Our sequence $\{\sigma'_n\}$ necessarily satisfies $|e^{\sigma'_n} - e^{\sigma}| \leqslant \delta$ for some such fixed δ and for all $n \in \mathbb{N}$. Since each $b' \in \tilde{\mathscr{R}}(b, CC_{\text{opt}}, e^{\sigma}, e^{\sigma'_n}, \mathbf{FC}, \vec{B})$ is of the form $b' = f(\sigma'_n)$ for some $f \in \hat{E}$, we see that $\|b' - b\|_{B_0+B_1} = \| \int_{e^{\sigma}}^{e^{\sigma'_n}} f'(z)\, dz\|_{B_0+B_1} \leqslant |e^{\sigma'_n} - e^{\sigma}| M_{\delta} = r_n$. This indeed gives the desired contradiction and shows that $\lim_{n \to \infty} C_{\sigma,\sigma'_n} = \infty$.

A.4. *The Coincidence of Lions–Schechter Spaces on the Strip with their Analogues on the Annulus*

In this appendix we will use some notation and definitions from the first part of Section 4.

For each nonnegative integer n, each $\theta \in (0,1)$, and each Banach pair $\vec{A}$, the Lions–Schechter interpolation space $[\vec{A}]_{\theta}^{(n)}$ is defined to be the set of all elements $a \in A_0 + A_1$ of the form $a = f^{(n)}(\theta)$ where $f \in \mathscr{F}(\vec{A})$. It is normed by the natural quotient norm.

Remark A.5. The original definitions given by Lions [39, 40], and Schechter [55] use a slightly modified version of Calderón's space $\mathscr{F}(\vec{A})$, which we may denote here by $\mathscr{F}_{LS}(\vec{A})$, whose elements $f : \bar{\mathbb{S}} \to A_0 + A_1$ are required to have the property that $t \mapsto f(j + it)$ is a continuous map of $\mathbb{R}$ into $A_0 + A_1$ for $j = 0, 1$. This is weaker than the condition required for Calderón's space, namely that $t \mapsto f(j + it)$ is continuous into A_j. As shown in [18] (at least for $n = 0$) replacing $\mathscr{F}(\vec{A})$ by $\mathscr{F}_{LS}(\vec{A})$ can sometimes give interpolation spaces which are strictly larger than $[\vec{A}]_{\theta}^{(n)}$. However the result to be presented in this subsection is valid for the interpolation spaces corresponding to either $\mathscr{F}(\vec{A})$ or $\mathscr{F}_{LS}(\vec{A})$.

PROPOSITION A.6. *For each nonnegative integer n, each $\theta \in (0,1)$ and each Banach pair $\vec{A}$, let $[\vec{A}]_{\theta}^{(n),\infty}$ denote the space obtained when $\mathscr{F}(\vec{A})$*

is replaced by $\mathscr{F}_\infty(\vec{A})$ *in the definition of* $[\vec{A}]_\theta^{(n)}$. *Then* $[\vec{A}]_\theta^{(n),\infty} = [\vec{A}]_\theta^{(n)}$ *and*

$$\|a\|_{[\vec{A}]_\theta^{(n),\infty}} \leqslant \|a\|_{[\vec{A}]_\theta^{(n)}} \leqslant C_n \|a\|_{[\vec{A}]_\theta^{(n),\infty}} \text{ for all } a \in [\vec{A}]_\theta^{(n)}, \tag{A.11}$$

where C_n *is a constant depending only on n. In particular,* $C_0 = C_1 = 1$.

Proof. As already mentioned at the beginning of Section 4, the case $n = 0$ is easy and well known. The case $n = 1$ is similar and essentially the same as Theorem 4.1. The case for general n is another slight modification of these two cases: Let us define the sequence $\{\alpha_m\}_{m \geqslant 0}$ by $\alpha_0 = 1$ and then recursively by

$$\alpha_m = -\frac{1}{m!} \sum_{k=0}^{m-1} \binom{m}{k} k! \alpha_k \frac{d^{m-k}}{dx^{m-k}} e^{x^2}|_{x=0}.$$

Then, for each nonnegative integer n, the function $\phi_n(z) = e^{z^2} \sum_{m=0}^n \alpha_m z^m$ satisfies $\phi_n(0) = 1$ and $\phi_n^{(m)}(0) = 0$ for each $m = 1, 2, \ldots, n$. Now, given any $a \in [\vec{A}]_\theta^{(n),\infty}$ and $\varepsilon > 0$, choose $f \in \mathscr{F}_\infty(\vec{A})$ such that $a = f^{(n)}(\theta)$ and $\|f\|_{\mathscr{F}(\vec{A})} \leqslant (1+\varepsilon)\|a\|_{[\vec{A}]_\theta^{(n),\infty}}$. For any $\delta > 0$ we define $g_\delta(z) = \phi_n(\delta(z-\theta))f(z)$. Then $g_\delta \in \mathscr{F}(\vec{A})$ and $g_\delta^{(n)}(\theta) = f^{(n)}(\theta) = a$. This shows that $[\vec{A}]_\theta^{(n),\infty} \subset [\vec{A}]_\theta^{(n)}$ and establishes the second inequality of (A.11) with $C_n = \inf_{\delta>0}(\sup\{|\phi_n(\delta z)| : \operatorname{Re} z \in [-1, 1]\})$. The remaining inequality and reverse inequality are obvious.

Corollary A.7 (cf. Lions [40, p. 3], Schechter [55, p. 122]). *The space* $[\vec{A}]_\theta^{(n)}$ *is continuously embedded in* $[\vec{A}]_\theta^{(n+1)}$ *and the embedding constant depends only on n and* θ.

Proof. Given any $a \in [\vec{A}]_\theta^{(n)}$ and $f \in \mathscr{F}(\vec{A})$ with $a = f^{(n)}(\theta)$ and $\|f\|_{\mathscr{F}(\vec{A})} \leqslant (1+\varepsilon)\|a\|_{[\vec{A}]_\theta^{(n)}}$ let $g(z) = (z-\theta)\phi_n(z-\theta)f(z)$ where ϕ_n is as in the preceding proof. Then $g \in \mathscr{F}(\vec{A})$ with $\|g\|_{\mathscr{F}(\vec{A})} \leqslant C(\theta, n)\|f\|_{\mathscr{F}(\vec{A})}$ and $g^{(n+!)}(\theta) = (n+1)\frac{d^n}{dz^n}(\phi_n(z-\theta)f(z))|_{z=\theta} = (n+1)f^{(n)}(\theta) = (n+1)a$. ∎

Proposition A.8. *For each nonnegative integer n, each* $\theta \in (0, 1)$ *and each Banach pair* $\vec{A}$, *let* $[\vec{A}]_\theta^{(n),2\pi}$ *denote the space obtained when* $\mathscr{F}(\vec{A})$ *is replaced by* $\mathscr{F}_{2\pi}(\vec{A})$ *in the definition of* $[\vec{A}]_\theta^{(n)}$. *Then* $[\vec{A}]_\theta^{(n)} = [\vec{A}]_\theta^{(n),2\pi}$ *and*

$$\frac{1}{C_n}\|a\|_{[\vec{A}]_\theta^{(n)}} \leqslant \|a\|_{[\vec{A}]_\theta^{(n),2\pi}} \leqslant C_n' \|a\|_{[\vec{A}]_\theta^{(n)}} \qquad \text{for all } a \in [\vec{A}]_\theta^{(n)} \tag{A.12}$$

where C_n is the constant appearing in (A.11) *and C'_n is another constant depending only on n.*

Proof. The case $n = 0$ goes back to [17] and the case $n = 1$ is essentially proved as part of Theorem 4.2. An elaboration of the same approach will also work for all $n \geqslant 1$:

Let $\phi : \mathbb{C} \to \mathbb{C}$ be the entire function defined by

$$\phi(0) = 1 \quad \text{and} \quad \phi(z) = e^{z^2}\left(\frac{e^z - 1}{z}\right)^{n+1} \qquad \text{for all } z \neq 0.$$

Then let $\psi(z) = \phi(z) \sum_{k=0}^{n} \beta_k z^k$, where the numbers β_k are defined by $\beta_0 = 1$ and then recursively by

$$\beta_m = -\frac{1}{m!} \sum_{k=0}^{m-1} \binom{m}{k} k! \beta_k \phi^{(m-k)}(0).$$

This ensures that $\psi(0) = 1$ and $\psi^{(m)}(0) = 0$ for $m = 1, 2, \ldots, n$. We can also see that, for each integer $m \neq 0$,

$$\psi(2\pi mi) = \psi'(2\pi mi) = \psi''(2\pi mi) = \cdots = \psi^{(n)}(2\pi mi) = 0,$$

since ϕ has this same property.

Now, given $a \in [\vec{A}]_\theta^{(n)}$ and $\varepsilon > 0$, we choose $f \in \mathscr{F}(\vec{A})$ such that $a = f^{(n)}(\theta)$ and $\|f\|_{\mathscr{F}(\vec{A})} \leqslant (1 + \varepsilon)\|a\|_{[\vec{A}]_\theta^{(n)}}$. Then define $F : \bar{\mathbb{S}} \to A_0 + A_1$ by

$$F(z) = \sum_{m \in \mathbb{Z}} \psi(z - \theta + 2\pi mi) f(z + 2\pi mi). \qquad \text{(A.13)}$$

By essentially the same arguments as in the proof of Theorem 4.2, F is an element of $\mathscr{F}_{2\pi}(\vec{A})$ with $\|F\|_{\mathscr{F}(\vec{A})} \leqslant C'_n \|f\|_{\mathscr{F}(\vec{A})}$ where $C'_n = \sup_{\operatorname{Re} z \in [-1,1]} \sum_{m \in \mathbb{Z}} |\psi(z + 2\pi mi)|$. The series in (A.13) can be differentiated term by term any number of times at each $z \in \mathbb{S}$. Thus the properties of ψ mentioned above imply that $F^{(n)}(\theta) = f^{(n)}(\theta) = a$. This proves that $[\vec{A}]_\theta^{(n)} \subset [\vec{A}]_\theta^{(n),2\pi}$ and the second inequality of (A.12). The first inequality and reverse inclusion follow trivially from Proposition A.6. ∎

Let us define $\mathscr{F}_{\mathbb{A}}(\vec{A})$ to be the space of functions $f : \bar{\mathbb{A}} \to A_0 + A_1$ such that $f(e^z) = F(z)$ for all $z \in \bar{\mathbb{S}}$ and some $F \in \mathscr{F}_{2\pi}(\vec{A})$ with norm $\|f\|_{\mathscr{F}_{\mathbb{A}}(\vec{A})} = \|F\|_{\mathscr{F}(\vec{A})} = \sup\{\|f(e^{j+it})\|_{A_j} : t \in [0, 2\pi], j = 0, 1\}$. For each $\theta \in (0, 1)$ and each nonnegative integer n we define $[\vec{A}]_\theta^{(n),\mathbb{A}}$ to be the space of elements $a \in A_0 + A_1$ of the form $a = f^{(n)}(e^\theta)$ for some $f \in \mathscr{F}_{\mathbb{A}}(\vec{A})$ with the natural quotient norm.

As has already been remarked (cf. (6.22)), for each n we have the continuous embeddings

$$[\vec{A}]_\theta^{(n),\mathbb{A}} \subset [\vec{A}]_\theta^{(n+1),\mathbb{A}} \tag{A.14}$$

and their proof is a simpler analogue of the proof of Corollary A.7.

THEOREM A.9. *For each $\theta \in (0,1)$ and each nonnegative integer n the space $[\vec{A}]_\theta^{(n),\mathbb{A}}$ coincides with $[\vec{A}]_\theta^{(n)}$ to within equivalence of norms and the constants of equivalence depend only on n and θ.*

Proof. Yet again the case $n = 0$ is known from [17] and the case $n = 1$ is essentially proved in Theorem 4.2. Let us now deal with general n: We can proceed by induction, assuming that the result is known for each $m = 0, 1, \ldots, n-1$. Suppose first that $a \in [\vec{A}]_\theta^{(n),\mathbb{A}}$. For any $\varepsilon > 0$, choose $f \in \mathscr{F}_{\mathbb{A}}(\vec{A})$ with $a = f^{(n)}(e^\theta)$. Define $F : \mathbb{S} \to A_0 + A_1$ by $F(z) = f(e^z)$. Then $F \in \mathscr{F}_{2\pi}(\vec{A})$ and $F^{(n)}(\theta) = \sum_{m=0}^n c_{m,\theta} f^{(m)}(e^\theta)$ for suitable constants $c_{m,\theta}$. So

$$a = \frac{1}{c_{n.\theta}} F^{(n)}(e^\theta) - \sum_{m=0}^{n-1} \frac{c_{m,\theta}}{c_{n,\theta}} f^{(m)}(\theta). \tag{A.15}$$

The first term on the right-hand side of (A.15) is an element of $[\vec{A}]_\theta^{(n),2\pi} = [\vec{A}]_\theta^{(n)}$ (cf. Proposition A.8) with norm bounded by a constant multiple of $\|F\|_{\mathscr{F}(\vec{A})} = \|f\|_{\mathscr{F}_{\mathbb{A}}(\vec{A})}$. The remaining terms, where m ranges from 0 to $n-1$, are elements of $[\vec{A}]_\theta^{(m),\mathbb{A}}$ and therefore, by our inductive assumption, of $[\vec{A}]_\theta^{(m)}$. Repeated applications of Corollary A.7 show that all of these elements are in $[\vec{A}]_\theta^{(n)}$ and that the norms of each of them can be bounded by constant multiples of $\|F\|_{\mathscr{F}(\vec{A})} = \|f\|_{\mathscr{F}_{\mathbb{A}}(\vec{A})}$. We conclude that $a \in [\vec{A}]_\theta^{(n)}$ and that $[\vec{A}]_\theta^{(n),\mathbb{A}}$ is continuously embedded in $[\vec{A}]_\theta^{(n)}$.

It remains to prove the reverse embedding: If $a \in [\vec{A}]_\theta^{(n)}$ then, by Proposition A.8, $a = F^{(n)}(\theta)$ for some $F \in \mathscr{F}_{2\pi}(\vec{A})$ with $\|F\|_{\mathscr{F}(\vec{A})} \leqslant const. \|a\|_{[\vec{A}]_\theta^{(n)}}$. Define $f : \bar{\mathbb{A}} \to A_0 + A_1$ by $f(z) = F(\operatorname{Log} z)$ where $\operatorname{Log} z$ denotes the principal branch of the complex logarithm. Then $f \in \mathscr{F}_{\mathbb{A}}(\vec{A})$ and $f^{(n)}(e^\theta) = \sum_{m=0}^n \tilde{c}_{m,\theta} F^{(m)}(\theta)$ for suitable constants $\tilde{c}_{m,\theta}$. The rest of the argument to show that $a \in [\vec{A}]_\theta^{(n),\mathbb{A}}$ is now almost exactly analogous to the first part of the proof, with (A.14) now playing the role of Corollary A.7. ∎

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