

CHAPTER 25

Quasi-Banach Spaces

Nigel Kalton*

Department of Mathematics, University of Missouri, Columbia, MO 65211, USA

E-mail: nigel@math.missouri.edu

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1. Introduction

The theory of the geometry of Banach spaces has evolved very rapidly over the past fifty years. By contrast the study of quasi-Banach spaces has lagged far behind, even though the first research papers in the subject appeared in the early 1940's ([18,6]). There are very sound reasons to want to develop understanding of these spaces, but the absence of one of the fundamental tools of functional analysis, the Hahn–Banach theorem, has proved a very significant stumbling block. However, there has been some progress in the non-convex theory and arguably it has contributed to our appreciation of Banach space theory.

A systematic study of quasi-Banach spaces only really started in the late 1950's and early 1960's with the work of Klee, Peck, Rolewicz, Waelbroeck and Zelazko. The efforts of these researchers tended to go in rather separate directions. The subject was given great impetus by the paper of Duren, Romberg and Shields in 1969 which demonstrated both the possibilities for using quasi-Banach spaces in classical function theory and also highlighted some key problems related to the Hahn–Banach theorem. This opened up many new directions of research. The 1970's and 1980's saw a significant increase in activity with a number of authors contributing to the development of a coherent theory. An important breakthrough was the work of Roberts in 1976 [73] and [75] who showed that the Krein–Milman Theorem fails in general quasi-Banach spaces by developing powerful new techniques. Quasi-Banach spaces (H_p -spaces when $p < 1$) were also used significantly in Alexandrov's solution of the inner function problem in 1982 [4]. During this period three books on the subject appeared by Turpin [86], Rolewicz [77] (actually an expanded version of a book first published in 1972) and the author, Peck and Roberts [56]. In the 1990's it seems to the author that while more and more analysts find that quasi-Banach spaces have uses in their research, paradoxically the interest in developing a general theory has subsided somewhat.

In this short article we will only give a glimpse of the theory, and we have tried to make the subject accessible for an audience which is primarily interested in and familiar with Banach space theory. There is no attempt to be encyclopaedic. Thus we will look carefully at problems related to the existence of closed subspaces which are very much in the spirit of the recent work of Gowers and Maurey [29] in Banach space theory. We will also consider the problem of characterizing the complemented subspaces of $L_p(0, 1)$ when $0 < p < 1$. In the last few sections we consider how the theory of lattices, analytic functions and tensor products alters in the non-convex setting.

Although this article is devoted to quasi-Banach spaces, much of the early theory was developed in the context of more general topological vector spaces or sometimes F -spaces (complete metric linear spaces). In some cases (such as the study of the Krein–Milman theorem for compact convex sets, see Section 5) restricting to quasi-Banach spaces loses nothing in terms of generality, and in most cases there is relatively little loss.

2. Preliminaries

In this section we will review a few elementary concepts and definitions. Further details can be found in one of the books [78,56].

Let us recall first that a quasi-norm $\|\cdot\|$ on vector space X over the field $\mathbb{K} = \mathbb{R}$ or \mathbb{C} is a map $X \rightarrow [0, \infty)$ with the properties:

- $\|x\| = 0$ if and only if $x = 0$.
- $\|\alpha x\| = |\alpha|\|x\|$ if $\alpha \in \mathbb{K}$, $x \in X$.
- There is a constant $C \geq 1$ so that if $x_1, x_2 \in X$ we have $\|x_1 + x_2\| \leq C(\|x_1\| + \|x_2\|)$.

The constant C is often referred to as the *modulus of concavity* of the quasi-norm. A very basic and important result is the *Aoki-Rolewicz theorem* ([6,77]). This result can be interpreted as saying that if $0 < p \leq 1$ is given by $C = 2^{1/p-1}$ then there is a constant B so that for any $x_1, \dots, x_n \in X$ we have

$$\left\| \sum_{k=1}^n x_k \right\| \leq B \left(\sum_{k=1}^n \|x_k\|^p \right)^{1/p} \tag{2.1}$$

It is then possible to replace $\|\cdot\|$ by an equivalent p -subadditive quasi-norm $\|\!\| \cdot \|\!\|$ so that

$$\|\!\|x_1 + x_2\|\!\| \leq (\|\!\|x_1\|\!\|^p + \|\!\|x_2\|\!\|^p)^{1/p}.$$

X is said to be p -normable if (2.1) holds. We will say that X is p -normed if the quasi-norm on X is p -subadditive. In general it is convenient to assume unless otherwise mention that a quasi-Banach space is p -normed for some $p > 0$.

The quasi-norm $\|\cdot\|$ induces a metric topology on X : in fact a metric can be defined by $d(x, y) = \|x - y\|^p$, when the quasi-norm is p -subadditive. X is called a *quasi-Banach space* if X is complete for this metric. Note that if we assume X is p -normed for some $p > 0$ then the quasi-norm is a continuous function for the metric topology. It is important to emphasize that the standard basic results of Banach space theory such as the Uniform Boundedness Principle, Open Mapping Theorem and Closed Graph Theorem which depend only on completeness apply to this type of space; however applications of convexity such as the Hahn-Banach theorem are not applicable.

If X and Y are quasi-Banach spaces then $\mathcal{L}(X, Y)$ denotes the space of bounded linear operators $T : X \rightarrow Y$ under the quasi-norm $\|T\| = \sup\{\|Tx\| : \|x\| \leq 1\}$. A special and important case is the dual space $X^* = \mathcal{L}(X, \mathbb{K})$ which is always a Banach space.

The best known examples of quasi-Banach spaces are the spaces ℓ_p and $L_p(0, 1)$, when $0 < p < 1$. These spaces are p -normable. It is readily seen that $\ell_p^* = \ell_\infty$ but $L_p(0, 1)^* = \{0\}$. Notice that ℓ_p has a *separating dual* while L_p has a *trivial dual*. This latter result is due to Day [18] in what is arguably the first paper on quasi-Banach spaces.

Another important example is the Hardy space H_p , i.e., the closed subspace of $L_p(\mathbb{T}, \frac{d\theta}{2\pi})$ spanned by the functions $\{e^{in\theta} : n \geq 0\}$. Although L_p has trivial dual, H_p has a *separating dual* ([22]).

If X has a separating dual then we can define an associated norm on X by the formula

$$\|x\|_c = \sup\{|x^*(x)| : \|x^*\| \leq 1\}.$$

It can be easily shown that $\|\cdot\|_c$ is the largest norm on X dominated by the original quasi-norm. The completion of X with this norm X_c is called the *Banach envelope* of X . In a natural sense X_c and X have the same dual space.

Many of the standard notions in Banach space theory can be carried through to quasi-Banach spaces. For example, the notions of (Rademacher) type and cotype (see [32]) can be defined in exactly the same way.

THEOREM 2.1. *Let X be a quasi-Banach space of type p where $0 < p \leq 2$. Then:*

- (1) ([40]) *If $p < 1$ then X is p -normable.*
- (2) ([37]) *If $p > 1$ then X is normable (i.e., a Banach space).*

We remark that there are non-locally convex spaces of type one which are not Banach spaces [40].

The Krivine–Maurey–Pisier theorems on finite representability of ℓ_p^n 's have analogues in this setting (we refer to [36,7] and [9]). In particular Dvoretzky's theorem always has an appropriate generalization (see [36] and [19]).

Let us also mention an important substitute for convexity in complex quasi-Banach spaces. We will say that a quasi-norm (which we assume r -subadditive for some $r < 1$) is *plurisubharmonic* if for any $x, y \in X$ then

$$\|x\| \leq \frac{1}{2\pi} \int_0^{2\pi} \|x + e^{i\theta}y\| d\theta.$$

For a discussion of this condition see [17]. It is important to note that the spaces L_p and their subspaces have plurisubharmonic norms. We will discuss this condition further in Section 8.

3. Linear subspaces and basic sequences

In this section we will discuss some very fundamental structure problems for quasi-Banach spaces concerning linear subspaces of quasi-Banach spaces. Many of the results and problems in this section are interesting in the category of F -spaces but we will restrict ourselves to quasi-Banach spaces for clarity. A fundamental and still unresolved problem is the following:

PROBLEM 3.1 (*The atomic space problem*). Does every quasi-Banach space have a proper closed infinite-dimensional subspace?

A quasi-Banach space X is called *atomic* if it has no proper closed infinite-dimensional subspaces. Very little is known about this problem. For a recent contribution in the context of F -spaces see [70]. Although Problem 3.1 remains elusive much progress has been made in understanding the structure of subspaces of quasi-Banach spaces. Before reviewing this progress we discuss the historical context for some of these ideas.

Since the failure of the Hahn–Banach theorem is a characteristic of non-locally convex spaces, it is natural that much of the early research in the area was devoted to trying to understand this phenomenon. A Banach space has a very rich dual space and this also

means that it has a very rich class of closed subspaces (each non-trivial continuous linear functional gives rise to a closed subspace of codimension one). Therefore an associated problem for quasi-Banach spaces is to find (infinite-dimensional) proper closed subspaces.

It is clear from Day's result that it is possible to find a closed subspace E of L_p when $0 < p < 1$ and a continuous linear functional $e^* \in E^*$ which cannot be extended; indeed E can be taken to be one-dimensional. This construction will work in any space X which fails to have a separating dual. In the space ℓ_p for $0 < p < 1$ more work is required but Peck [68] gave a similar example of the failure of the Hahn–Banach theorem. Later, Duren, Romberg and Shields [23] found a representation of the dual of H_p for $0 < p < 1$ and used it to show that the Hahn–Banach theorem also fails in these spaces. Their work led them to formulate a conjecture. They defined a quasi-Banach space X (or more generally an F-space) to have the *Hahn–Banach Extension property (HBEP)* if whenever e^* is a continuous linear functional on a closed subspace E of X then e^* has an extension $x^* \in X^*$. They also defined the notion of a *proper closed weakly dense (PCWD)* subspace as a proper closed subspace E so that the quotient X/E has trivial dual. They then asked whether a quasi-Banach space with (HBEP) is necessarily locally convex and whether a any non-locally convex quasi-Banach space has a PCWD-subspace. It is easy to see that if X has HBEP then it must have a separating dual and every quotient must have HBEP; hence if X has a PCWD-subspace it cannot have HBEP. These two questions had a considerable impact on the theory because they focused attention on *the problem of subspaces*. In effect HBEP is equivalent to the statement that the weak and norm topologies have the same closed subspaces.

After important contributions in [79] and [71] the first of these problems was resolved in [33]:

THEOREM 3.2. *A quasi-Banach space X has HBEP if and only if X is locally convex (i.e., a Banach space).*

The method of proof of Theorem 3.2 relies on the construction of basic sequences. Of course, there is no guarantee that quasi-Banach spaces will contain basic sequences (unlike Banach spaces). In fact an atomic space (if it exists) would be an immediate counterexample; but we will later show how to construct a quasi-Banach space without a basic sequence. However it is natural to start by imitating as far as the possible the classical Bessaga–Pełczyński basic sequence selection techniques. It soon becomes clear that the role of the weak (or weak*) topology can be replaced by any weaker Hausdorff vector topology τ on X so that X has an equivalent τ -lower-semi-continuous quasi-norm. We will call such a topology *polar*.

PROPOSITION 3.3 (Basic sequence selection criterion). *Let X be a quasi-Banach space and suppose (x_n) is a sequence so that $\lim x_n = 0$ for some polar vector topology τ but $\inf \|x_n\| > 0$. Then (x_n) has a subsequence which is basic.*

We recall that a sequence (x_n) in a quasi-Banach space X is called a *Markushevich basis* if $[x_n] = X$ and there is a bi-orthogonal sequence (x_n^*) so that (x_n^*) separate the points of X .

We will say that (x_n) is a *Markushevich basic sequence* if it is a Markushevich basis for its closed linear span. An immediate corollary of this proposition is:

PROPOSITION 3.4 (Markushevich basic sequence selection criterion). *Let X be a quasi-Banach space and suppose (x_n) is a sequence so that $\lim x_n = 0$ for some weaker Hausdorff vector topology τ but $\inf \|x_n\| > 0$. Then (x_n) has a subsequence (y_n) which is a Markushevich basic sequence and whose bi-orthogonal sequence (y_n^*) in $[y_n]^*$ satisfies $\sup \|y_n^*\| < \infty$.*

An alternative approach to this result was given by Drewnowski [21]. We are now a position to indicate a proof of Theorem 3.2:

PROOF OF THEOREM 3.2. Since X has (HBEP) it is clear that X^* separates points and therefore the Banach envelope norm $\|, \|_c$ induces a weaker Hausdorff vector topology on X . We argue that it cannot be a strictly weaker topology than the quasi-norm topology. Indeed, if it is strictly weaker, then using Proposition 3.4 one can find a sequence (x_n) such that $\|x_n\|_c < 4^{-n}$ but $\|x_n\| = 1$ for all n and (x_n) is a Markushevich basis for its closed linear span E with bi-orthogonal functionals (x_n^*) satisfying $\sup \|x_n^*\| < \infty$. Then we can define $e^* \in E^*$ by $e^* = \sum_{n=1}^{\infty} 2^{-n} x_n^*$. Suppose e^* can be extended to a bounded linear functional $f^* \in X^*$. Then $f^*(4^n x_n) = 2^n$ but $\|4^n x_n\|_c \leq 1$ for all n . This contradiction shows that X coincides with its Banach envelope. □

Once this is established it is not too difficult to prove a companion result for PCWD subspaces [37]:

THEOREM 3.5. *Let X be a quasi-Banach space with a separating dual. If X has no PCWD subspace then X is locally convex.*

Notice however that the hypothesis of a separating dual is required here. We will see later that this hypothesis is necessary: there exist non-locally convex quasi-Banach spaces which do not have any quotient with trivial dual.

Let us now return to the discussion of basic sequences. Theorems 3.3 and 3.4 yield some characterizations of spaces with basic sequences:

THEOREM 3.6. *Let X be a separable infinite-dimensional quasi-Banach space. Then the following conditions on X are equivalent:*

- (i) X contains a basic sequence.
- (ii) There is descending sequence (L_n) of infinite-dimensional closed subspaces of X with $\bigcap_{n=1}^{\infty} L_n = \{0\}$.
- (iii) There is a family \mathcal{L} of infinite-dimensional closed subspaces such that $\bigcap \{L: L \in \mathcal{F}\}$ is infinite-dimensional for any finite subset \mathcal{F} of \mathcal{L} but $\bigcap \{L: L \in \mathcal{L}\} = \{0\}$.
- (iv) There is a strictly weaker Hausdorff vector topology on X .

These implications are relatively easy. The equivalence of (ii) and (iii) simply follows from the Lindelof property for separable metric spaces. That (i) implies (ii) is trivial. For

(ii) implies (iv) simply consider the vector topology on X induced by the semi-quasi-norms $x \rightarrow d(x, L_n)$ for $n = 1, 2, \dots$. Thus the only implication with any difficulty here is that (iv) implies (i). Let τ be a Hausdorff vector topology on X , which is strictly weaker than the original quasi-norm topology qn . Let τ^* be a maximal Hausdorff vector topology on X strictly weaker than qn (such a topology must exist). Let τ^{**} be the quasi-norm topology on X defined by taking the τ -closure of the original unit ball as a new unit ball. Then the maximality of τ means that either $\tau^{**} = \tau^*$ or $\tau^{**} = qn$. But the former case means that the identity $i : (X, \tau^*) \rightarrow (X, qn)$ is almost continuous and a form of the Closed Graph Theorem comes into play: one deduces that $\tau^* = qn$ a contradiction. It follows that $\tau^{**} = qn$ and so τ^* is a polar topology. One can use the Lindelof property to construct a weaker metrizable Hausdorff vector topology ρ which is still polar. Then an application of Theorem 3.3 completes the proof.

The last condition leads to the definition of a minimal space as any quasi-Banach space which does not have any weaker Hausdorff vector topology. A separable quasi-Banach space is minimal if and only if it contains no basic sequence (separability is redundant here, but that requires a little more explanation). Obviously an atomic space must be minimal but we shall see that the converse is false.

Let us now illustrate the problem by considering an arbitrary separable Banach space X . Let \mathcal{L} be a maximal collection of infinite-dimensional closed subspaces of X with the property that any finite intersection is infinite-dimensional. Let $E = \bigcap \{L : L \in \mathcal{L}\}$. There are three possibilities:

- $E = \{0\}$. Then by Theorem 3.6 X is non-minimal.
- E is infinite-dimensional. Then E is atomic.
- $\dim E < \infty$. In this case X/E contains a basic sequence, but X could still be minimal.

The third possibility suggests a way of constructing a minimal space with no atomic subspace. It is even possible to hope for an example where $\dim E = 1$ and X/E is a Banach space. Obviously one needs that X is not a Banach space: this brings into focus a distinct problem which also received a considerable amount of attention in the 1970's: the three space problem for Banach spaces, which is discussed in the next section. It will turn out that there is a counterexample of this nature and it is closely related to the recent work of Gowers and Maurey [29,28].

THEOREM 3.7 ([52]). *There is a quasi-Banach space which does not contain a basic sequence.*

We will postpone discussion of this theorem to Section 4. In view of Theorem 3.7 it is possible to ask whether such examples can be created in classical spaces such as L_p when $p < 1$. In fact there are two positive results which show that every subspace of L_p has a basic sequence. The first result is due to Bastero [8] who show that the theory of Krivine–Maurey stability can be extended to quasi-Banach spaces. This shows that:

THEOREM 3.8. *If X is closed subspace of L_p when $0 < p < 1$ then X contains a subspace isomorphic to ℓ_r for some $p \leq r \leq 2$.*

The second result of Tam [84] gives a general and important criterion for the existence of basic sequences. Notice that this result also includes the case of subspaces of L_p .

THEOREM 3.9. *Let X be a complex quasi-Banach space with a plurisubharmonic quasi-norm. Then X contains a basic sequence.*

To conclude this section, we note that in [30] an example is created of a sequence (f_n) contained in L_p when $0 < p < 1$ so $\inf_{j \neq k} \|f_j - f_k\|_p > 0$ and every subsequence $(f_n)_{n \in M}$ is fundamental in L_p .

4. The three-space problem and minimal extensions

We now turn our attention to a central problems of the area in the 1970's, the *three-space problem for local convexity* which asked if there is a non-locally convex quasi-Banach space X with a closed subspace E such that both E and X/E are locally convex. This problem belongs to a family of three-space problems for which we refer to [12].

THEOREM 4.1. *There is a non-locally convex quasi-Banach space X with a subspace E of dimension one so that X/E is isomorphic to ℓ_1 .*

Theorem 4.1 is due independently (and essentially simultaneously) to the author, Ribe and Roberts [38,72] and [75]. However the examples created in each case were very different.

Suppose X is a quasi-Banach space. We will say that Y is a *minimal extension* of X if there is a subspace E of Y with $\dim E = 1$ and $Y/E \approx X$. We will say that Y is the *trivial extension* (or that Y *splits*) if L is complemented, i.e., $Y \approx L \oplus X$ in the natural way. We say ([55]) that X is a \mathcal{K} -space if every minimal extension of X is trivial.

We now describe a general construction of a minimal extension (first used in [38] and [72]). Let us suppose X is a quasi-Banach space (over the field \mathbb{K}). Let X_0 be any fixed dense linear subspace of X (of course $X_0 = X$ is a possible choice). We say that a map $F : X_0 \rightarrow \mathbb{K}$ is *quasilinear* if:

- (1) $F(\alpha x) = \alpha F(x)$ for $x \in X_0$ and $\alpha \in \mathbb{K}$.
- (2) There is a constant K so that

$$|F(x_1 + x_2) - F(x_1) - F(x_2)| \leq K(\|x_1\| + \|x_2\|), \quad x_1, x_2 \in X.$$

We then define a quasi-norm on $\mathbb{K} \oplus X_0$ by

$$\|(\alpha, x)\|_F = |\alpha - F(x)| + \|x\|, \quad x \in X_0, \alpha \in \mathbb{K}.$$

The completion of $\mathbb{K} \oplus X_0$ for this quasi-norm is a minimal extension of X which we denote $\mathbb{K} \oplus_F X$. Conversely every minimal extension of X is isomorphic (as an extension) to $\mathbb{K} \oplus_F X$ for a suitable quasilinear map F (see [37]). It is clear that if F and G are any two quasilinear maps on X_0 then F and G define equivalent quasi-norms on $\mathbb{K} \oplus X_0$ if and only if there is a constant C so that $|F(x) - G(x)| \leq C\|x\|$ for every $x \in X_0$. In this case

we will say that F and G are *equivalent*. The minimal extension $\mathbb{K} \oplus_F X$ splits if and only if there is a linear map $G : X_0 \rightarrow \mathbb{K}$ equivalent to F , i.e., G satisfies estimate of the form

$$|F(x) - G(x)| \leq C \|x\|, \quad x \in X_0. \tag{4.2}$$

In this way \mathcal{K} -spaces are characterized in terms of an approximation property. We refer to [11] for related results. We also note the connection with the concept of Hyers–Ulam functional stability (see [31]); the question is essentially is whether a functional which satisfies a perturbation of the functional equation for linear maps is itself a perturbation of a linear map. Let us note that if X is a Banach space then (4.2) is equivalent to an estimate of the form

$$\left| F(x_1 + \cdots + x_n) - \sum_{i=1}^n F(x_i) \right| \leq C' \sum_{i=1}^n \|x_i\|, \quad x_1, \dots, x_n \in X_0. \tag{4.3}$$

PROOF OF THEOREM 4.1. To prove Theorem 4.1 we follow the construction of Ribe [72] of a space now known as the *Ribe space*. According to the preceding discussion, it is enough to exhibit a function $F : c_{00} \rightarrow \mathbb{K}$ defined on the dense subspace c_{00} of all finitely supported sequences in ℓ_1 , which is quasilinear and fails to satisfy (4.3). Ribe’s example is the functional

$$F(x) = \sum_{k=1}^{\infty} x_k \log |x_k| - \left(\sum_{k=1}^{\infty} x_k \right) \log \left| \sum_{k=1}^{\infty} x_k \right|$$

(where $0 \log 0 := 0$). A slight modification yielding an equivalent quasi-norm is the functional

$$\Lambda(x) = \sum_{k=1}^{\infty} x_k \log \frac{|x_k|}{\|x\|}. \tag{4.4}$$

To see that (4.3) does not hold it is enough to compute $F(e_1 + \cdots + e_n) - \sum_{k=1}^n F(e_k) = -n \log n$. □

The Ribe space immediately produces the necessary example for Theorem 4.1. Also as observed by Roberts [75] it gives an example of a non-locally convex space with no quotient with trivial dual; this gives a counterexample to complement Theorem 3.5.

At this point let us mention an important open problem:

PROBLEM 4.2. Classify those Banach spaces which are \mathcal{K} -spaces (i.e., so that every minimal extension is trivial). Is it true that a Banach space X is a \mathcal{K} -space if and only if X^* has non-trivial cotype?

There is some body of evidence to support the conjecture in Problem 4.2. The known results are:

THEOREM 4.3 ([38]). *Suppose X is a Banach space with non-trivial type. Then every minimal extension of X is trivial.*

THEOREM 4.4 ([58]). *Suppose X is a Banach space which is the quotient of an \mathcal{L}_∞ -space. Then every minimal extension of X is trivial.*

It is perhaps worth noting that the latter theorem can be restated in terms of a stability theorem for set functions.

THEOREM 4.5 ([58]). *There is a universal constant so that whenever \mathcal{A} is an algebra of subsets of some set Ω and $F : \mathcal{A} \rightarrow \mathbb{R}$ is a set function satisfying:*

$$|F(A \cup B) - F(A) - F(B)| \leq 1 \quad \text{if } A \cap B = \emptyset$$

then there is an additive set function μ with

$$|F(A) - \mu(A)| \leq K$$

for every $A \in \mathcal{A}$.

There are some non-locally convex \mathcal{K} -spaces:

THEOREM 4.6 ([38]). *If $0 < p < 1$ then every minimal extension of ℓ_p or L_p splits.*

Let us return to the case of ℓ_1 . As we have seen it is possible to characterize minimal extensions of ℓ_1 via quasilinear maps on c_{00} . It turns out that it is possible up to equivalence to characterize quasilinear maps in a very convenient form. To understand this let us first note that it is only necessary to specify a quasilinear map F on the positive cone c_{00}^+ of c_{00} since any map obeying the conditions for quasilinearity on the positive cone can be extended by the formula

$$F(x) = F(x^+) - F(x^-),$$

where $x^+ = \max(x, 0)$ and $x^- = \max(-x, 0)$. This extension is then unique up to equivalence.

Let X be a Banach sequence space, i.e., a space of sequences equipped with a norm $\|\cdot\|_X$ such that

- The basis vectors $e_n \in X$.
- If $\xi \in X$ and $|\eta_k| \leq |\xi_k|$ for every k then $\eta \in X$ and $\|\eta\|_X \leq \|\xi\|_X$.
- For every $n \in \mathbb{N}$ the linear functional $\eta \rightarrow \eta_n$ is continuous.
- If ξ is a sequence such that $n \in \mathbb{N}$ $S_n \xi = (\xi_1, \dots, \xi_n, 0, \dots) \in X$ and $\sup \|S_n \xi\|_X < \infty$ then $\xi \in X$ and $\|\xi\|_X = \sup_{n \in \mathbb{N}} \|S_n \xi\|_X$.

The last condition here is usually called the Fatou property. We can now define an associated quasilinear map on c_{00}^+ by the formula

$$\Phi_X(x) = \sup_{\|\xi\|_X \leq 1} \sum_{k=1}^{\infty} x_k \log |\xi_k|. \tag{4.5}$$

This functional was introduced under the name *indicator function of X* in [51] and later under the name *entropy function of X* in [66] where it plays an important role in the solution of the distortion problem. The fact that it is quasilinear is first observed in [51]. By way of illustration consider the case $X = \ell_1$, when as easy calculation gives that $\Phi_X = \Lambda$ where Λ is defined by Eq. (4.4). Note that $\Phi_{\ell_p} = \frac{1}{p}\Lambda$ and $\Phi_{\ell_\infty} = 0$.

The entropy functions yield an important source of minimal extensions of ℓ_1 . They do not completely classify minimal extensions because each is convex on the positive cone. A complete classification is however obtained in [51]:

THEOREM 4.7. *Let $F : c_{00} \rightarrow \mathbb{K}$ be a quasilinear map (for the ℓ_1 -norm). Then there exists a positive α and a Banach sequence space X so that F is equivalent to $\alpha(\Phi_X - \Phi_{X^*})$ where X^* is the Köthe-dual of X .*

PROOF OF THEOREM 3.7. We return to Theorem 3.7. As suggested in the discussion it is reasonable to hope for an example of a minimal extension of ℓ_1 with no basic sequence. If Y is this minimal extension and L is the kernel of the quotient map onto ℓ_1 this requires that every infinite-dimensional closed subspace of Y contains L . Clearly this is very much related to the construction of Gowers and Maurey [29] of a Banach space where any two infinite-dimensional subspaces almost intersect. Now if we write Y in the form $\mathbb{K} \oplus_F \ell_1$ where F is defined on c_{00} , then we can translate our requirement to a condition on F . This is that F cannot be equivalent to a linear functional on any infinite-dimensional subspace of c_{00} . This in turn is equivalent to the requirement that for every infinite-dimensional subspace E of c_{00} we have:

$$\sup\{|F(x)| : x \in E, \|x\| \leq 1\} = \infty.$$

In this language, this is a type of *distortion* problem similar to distortion problem for Banach spaces solved in [66]. □

Clearly Theorem 4.7 suggests we should try use a functional of the type $F = \Phi_X$ where X is a suitably exotic Banach sequence space. The correct choice is a space used by Gowers [28] which is a modification of the original Gowers–Maurey construction in [29]. The proof that such an example works requires technical calculations similar in spirit to work in [29]; we refer to [52].

One final remark on this example is in order: one can find a dense subspace which has HBEP. Thus in Theorem 3.2 it is necessary to assume that X is complete; it should be noted that in [71] a number of results are proved for metrizable topological vector spaces with HBEP, without the assumption of completeness.

5. The Krein–Milman theorem

A classic problem in non-locally convex spaces asks whether every compact convex subset of a quasi-Banach space has an extreme point. This can be traced back as least as far as [61], and probably much further. Although, at first sight, this problem is unrelated to the three-space problem of the preceding section, in retrospect their negative solutions require very much the same constructions.

THEOREM 5.1. *There is a compact convex subset of L_p when $0 < p < 1$ which has no extreme points.*

Theorem 5.1 is due to Roberts [73] and [74]. Roberts's original construction is contained in [74] and this only gives a compact convex subset of some quasi-Banach space without extreme points, while [73] contains a much simplified approach and the theorem as stated. Since [73] is not readily available, a good reference for this argument is [56]. Roberts then used the key ideas in his proof of Theorem 5.1 to prove Theorem 4.1. In this article we will take the opposite direction, which in hindsight seems the right way to look at things.

We now turn to the proof of Theorem 5.1. The approach used by Roberts in both [73] and [75] is through the notion of a *needlepoint*. If X is a quasi-Banach space we say that $x \in X$ is a needlepoint if given $\varepsilon > 0$ there exists a finite set F so that $x \in \text{co } F$, $\|v\| < \varepsilon$ if $v \in F$ and for any $y \in \text{co } F$ there exists $0 \leq \alpha \leq 1$ with $\|y - \alpha x\| < \varepsilon$. X is called a *needlepoint space* if every point of X is a needlepoint. The main ingredients of the proof are:

PROPOSITION 5.2. *If X is a needlepoint space then X contains a compact convex set K with no extreme points.*

PROPOSITION 5.3. *The space L_p for $0 < p < 1$ is a needlepoint space.*

In [75] it is simply shown that there is a needlepoint space: the fact that L_p has this property is proved in [73].

For the proof of Proposition 5.2 we refer to [56]. The construction is in fact a quite logical inductive argument using at each stage that every point of the space is a needlepoint.

We will describe an approach to Proposition 5.3 which uses the notion of minimal extensions. In fact, it is very easy to see that if Y is a non-trivial minimal extension of a Banach space X so that $X \approx Y/L$ where $\dim L = 1$ then every $e \in L$ is a needlepoint. Thus we have an easy direct construction of a non-zero needlepoint by using the Ribe space. Of course, the Ribe space is not a needlepoint space; in fact any needlepoint space must have trivial dual. However we next note:

THEOREM 5.4 ([42]). *The Ribe space is isomorphic to a subspace of L_p when $0 < p < 1$.*

It is in fact instructive to see that the Ribe space is a rather natural subspace of L_p . Let (ξ_n) be a sequence of independent random variables each with the Cauchy distribution, i.e.,

$$\lambda(\xi_n \in B) = \frac{1}{\pi} \int_B \frac{dx}{1+x^2}$$

so that

$$\int e^{is\xi_n(t)} dt = e^{-|s|}.$$

Then consider the space E generated by the constant function 1 and the sequence $\{|\xi_n|\}_{n=1}^\infty$. It may be shown that if $\alpha_0, \alpha_1, \dots \in \mathbb{R}$ is a finitely supported sequence then

$$\left\| \alpha_0 + \sum_{k=1}^\infty \alpha_k |\xi_k| \right\|_p \sim |\alpha_0 - \Lambda((\alpha_k)_{k=1}^\infty)| + \sum_{k=1}^\infty |\alpha_k|,$$

where Λ is defined in (4.4). For details we refer to [42]. This implies that E is isomorphic to the Ribe space and the constant function 1 is a needlepoint of L_p .

To complete the argument we need only note that L_p is a *transitive space*, i.e., given any $f, g \in L_p$ with $f \neq 0$ there is a bounded linear operator $T : L_p \rightarrow L_p$ with $Tf = g$. As the image of a needlepoint is necessarily a needlepoint this means that L_p is a needlepoint space. This then completes the proof of Proposition 5.3 and hence of Theorem 5.1.

There has been some investigation of geometric properties of a compact convex set K which guarantee the existence of extreme points. One way to formulate this idea is to consider a Banach space X and a compact linear operator $T : X \rightarrow Y$ where Y is a quasi-Banach space. Let $K = \overline{T(B_X)}$. Then K is a symmetric compact convex set in Y . We can then consider geometric conditions on X so that K is affinely homeomorphic to a compact convex subset of a locally convex space; this is equivalent to the existence of a separating family of continuous affine functions on K . Two results of this nature are known:

THEOREM 5.5. *Suppose X is a Banach space and $T : X \rightarrow Y$ is bounded linear operator. Then the collection of continuous affine functions on $K = \overline{T(B_X)}$ separates the points of K if either of the following conditions hold:*

- (1) ([26]) X contains no subspace isomorphic to ℓ_1 .
- (2) ([58]) T is compact and X is an \mathcal{L}_∞ -space.

Finally an example is created in [50] of a compact convex set which is *not* affinely homeomorphic to a subset of $L_0[0, 1]$. It should be pointed out that Proposition 3.2 of [50] has an error: the set K is not convex (see [5]); however the original set K_0 or its symmetrization can be used in its place.

6. Operators and the structure of L_p -spaces when $0 < p < 1$

In this section we will treat some results on operators and their representations and discuss the isomorphic structure of the spaces $L_p[0, 1]$.

If X is a quasi-Banach space with trivial dual then the algebra $\mathcal{L}(X)$ may, in fact, be rather small, as one does not have the rich class of finite-rank operators. Let us say that a space X is *rigid* if $\mathcal{L}(X) = \mathbb{C}I$. In [57] the following result is proved (see also [90] for a quasi-normed but incomplete example of a rigid space):

THEOREM 6.1. *If $0 < p < 1$ then L_p has a subspace X so that every quotient of X is rigid.*

In a recent preprint, Roberts shows there are many rigid spaces by showing:

THEOREM 6.2 ([76]). *Every separable p -normable quasi-Banach space with trivial dual is the quotient of a separable rigid p -normable quasi-Banach space.*

One of the classical results on non-locally convex spaces is a theorem of Williamson [92] which says essentially that the theorem of the Fredholm alternative remains valid. If X has trivial dual so that X^* reduces to $\{0\}$ then this implies that any compact operator $K : X \rightarrow X$ has only zero in its spectrum. Later Pallaschke [67] observed that if X is also transitive then K must itself be the zero operator, since if $K \neq 0$ one can find an endomorphism $T : X \rightarrow X$ so that TK has one in its spectrum. In particular this result applies to L_p when $p < 1$. This result was generalized by Turpin [86] to a wider class of spaces and in [56] Pallaschke's result is extended to strictly singular operators. These results suggested a question (due to Pełczyński): is it possible to find a compact endomorphism of a space with trivial dual? In effect, this is equivalent to a more general question: does there exist a quasi-Banach space X with a trivial dual and a non-zero compact operator $K : X \rightarrow Y$ where Y is any quasi-Banach space? If such an example can be constructed we can suppose $K(X)$ dense in Y so that $Y^* = \{0\}$ and consider the map $(x, y) \rightarrow (0, Kx)$ on $X \oplus Y$. Let us say then that a space X *admits compact operators* if there is a non-zero compact operator $T : X \rightarrow Y$ where Y is some quasi-Banach space.

Pełczyński's question was resolved in [59]:

THEOREM 6.3. *There is a quasi-Banach space with trivial dual which admits compact operators.*

The proof used some classical function theory and the earlier results of Duren, Romberg and Shields [23]. We would like however to indicate a slightly different proof based on [60] and the results of Section 3. The key observation is that if X is a quasi-Banach space whose unit ball B_X is compact for some (Hausdorff) vector topology τ , then one can mimic the proof of the Banach–Dieudonné theorem to show that the topology $\hat{\tau}$ which is defined to be the finest topology on X agreeing with τ on bounded sets is a vector topology. Let us take $X = \ell_2(\ell_p^n)$, where $0 < p < 1$. Then X has certain features of a reflexive Banach space; in particular, its unit ball is weakly compact. If we use the weak topology w for τ then $\hat{\tau}$

is the “bounded weak” topology bw . We show next that every closed subspace E of X is bw -closed. In fact if not we can find a sequence $x_n \in B_X$ converging in the bw -topology to a point $x \notin E$. Using Theorem 3.3 or even a simple gliding hump argument we can suppose $(x_n - x)$ is a basic sequence equivalent to a block basis of the original basis. But then passing to a further subsequence we can suppose it is equivalent to the canonical basis of ℓ_2 . But there is a linear functional φ on its closed linear span with $\varphi(x_n - x) = 1$ for all n , which produces a contradiction. Now an application of Theorem 3.5 gives a subspace E so that X/E has trivial dual; in this space the bw -topology factors to a quotient topology for which the unit ball is compact. We remark that this topology by its construction is locally p -convex and so easily provides examples of compact operators into p -normable quasi-Banach spaces. For full details see pp. 140–146 of [56].

Let us note that Sisson [81] showed that there is an example of a rigid space admitting compact operators; this can now also be deduced from Theorem 6.2.

The examples indicated above leave open the question of whether specific spaces admit compact operators. Of course the obvious example is the space L_p when $0 < p < 1$. This space does not admit compact operators; in fact more is true:

THEOREM 6.4 ([35]). *Suppose $0 < p < 1$. Let $T : L_p \rightarrow Y$ be a non-zero operator into some quasi-Banach space Y (or even a topological vector space). Then there is an infinite-dimensional subspace H of L_p which is isomorphic to ℓ_2 and so that $T|_H$ is an isomorphism.*

This result also holds in a wide class of non-locally convex Orlicz spaces. Let us sketch a very simple proof that there cannot be a compact operator $K : L_p \rightarrow Y$ where we assume Y has an r -subadditive quasi-norm. Let $f \in L_p$ and suppose r_n are the Rademacher functions. If $K(fr_n)$ has any cluster point it must be zero since for any subsequence $\frac{1}{n}f(r_{k_1} + \dots + r_{k_n})$ converges to zero. Hence $\lim_{n \rightarrow \infty} \|K(fr_n)\| = 0$. Let $A_n = \{r_n = 1\}$ and $B_n = \{r_n = -1\}$. Then

$$Kf = 2K(f\chi_{A_n}) - K(fr_n) = K(fr_n) - 2K(f\chi_{B_n}).$$

Thus

$$\begin{aligned} \|Kf\| &\leq 2^{1-1/r} \liminf_{n \rightarrow \infty} (\|Kf\chi_{A_n}\|^r + \|Kf\chi_{B_n}\|^r)^{1/r} \\ &\leq 2^{1-1/p} \liminf_{n \rightarrow \infty} (\|Kf\chi_{A_n}\|^p + \|Kf\chi_{B_n}\|^p)^{1/p} \\ &\leq 2^{1-1/p} \|K\| \|f\|_p. \end{aligned}$$

This yields $\|K\| \leq 2^{1-1/p} \|K\|$, i.e., $K = 0$.

It is very possible that Theorem 6.4 is not the best result here. Under special hypotheses one can do much better:

THEOREM 6.5. *Suppose $0 < r < p < 1$. Then:*

- (1) ([39]) Let $T : L_p \rightarrow L_p$ be a non-zero operator. Then there is a subspace E of L_p so that $E \approx L_p$ and $T|_E$ is an isomorphism.
- (2) ([41]) Let $T : L_p \rightarrow L_r$ be a non-zero operator. Then if $p < q \leq 2$ there is a subspace E of L_p with $E \approx L_q$ and $T|_E$ is an isomorphism.

These results follow from representation theorems which we discuss shortly. Let us remark at this point that part (2) extends to operators taking values in *natural* which we will discuss in Section 7.

PROBLEM 6.6. Suppose $0 < p < 1$ and let $T : L_p \rightarrow Y$ be any non-zero operator. Suppose $p < q \leq 2$. Does it follow that there is a subspace $E \approx L_q$ (or even ℓ_q) so that $T|_E$ is an isomorphism?

We will now turn to some basic questions on the structure of the spaces $L_p[0, 1]$ when $0 < p < 1$. Some of these questions can be regarded as analogues of similar questions for the Banach spaces L_p when $p \geq 1$. However, the theory has a different flavor. A simple example is the fact that the quotient of L_p by a one-dimensional subspace is never isomorphic to L_p when $p < 1$ [55]. In fact this is an immediate consequence of the observation in Theorem 4.6 that every minimal extension of L_p splits.

A key result is the concrete representation of operators on $L_p[0, 1]$ given in [39]. This result which was inspired by an earlier similar result for the case $p = 0$ due to Kwapien [62] has a somewhat similar form also in the case $p = 1$ [39]. Results of similar type have been studied for operators $T : L_p \rightarrow L_0$ in [41] and also for operators on general rearrangement-invariant spaces [44] and [47].

THEOREM 6.7. Suppose $T : L_p[0, 1] \rightarrow L_p[0, 1]$ is a bounded operator where $0 < p < 1$. Then there is a sequence of Borel functions $a_n : [0, 1] \rightarrow \mathbb{K}$ and Borel maps $\sigma_n : [0, 1] \rightarrow [0, 1]$ so that:

- (1) $|a_n(s)| \geq |a_{n+1}(s)|$ for $n = 1, 2, \dots$ and $s \in [0, 1]$.
- (2) $\sigma_n(s) \neq \sigma_m(s)$ when $m \neq n$ and $s \in [0, 1]$.
- (3) $\sum_{n=1}^{\infty} |a_n(s)|^p < \infty$ for almost every $s \in [0, 1]$.
- (4) $\sum_{n=1}^{\infty} \int_{\sigma_n^{-1}B} |a_n(s)|^p ds \leq \|T\|^p \lambda(B)$ for every Borel set $B \subset [0, 1]$.
- (5) If $f \in L_p$ then

$$Tf(s) = \sum_{n=1}^{\infty} a_n(s) f(\sigma_n s) \quad s\text{-a.e.}$$

Conversely if (a_n) and (σ_n) are given satisfying (1)–(4) then (5) defines a bounded operator on L_p .

This theorem rather easily yields Theorem 6.5(1). It also gives a simple proof that L_p is a primary space, i.e., if $L_p = X \oplus Y$ then either X or Y must be isomorphic to L_p . (In fact essentially the same proof can be given in the case $p = 1$; see [39] and [24].)

However an intriguing question, closely related to a similar question in the case $p = 1$, is:

PROBLEM 6.8. Is $L_p[0, 1]$ a prime space when $0 < p < 1$, i.e., is any infinite-dimensional complemented subspace isomorphic to L_p ?

It is well-known that the spaces ℓ_p are prime when $0 < p < 1$. This is due to Stiles [82], by a proof similar to that of Pełczyński for the case $p \geq 1$. In [39] it is shown that the space $L_0[0, 1]$ is prime by using Kwapien's representation of operators for this case. The case $0 < p < 1$ is however more difficult and remains open. The natural way to attack this problem is to take an arbitrary projection P on L_p and use Theorem 6.7 to represent it. It turns out that one must show that there is a Borel set B of positive measure and $n \in \mathbb{N}$ so that $|a_n| > 0$ on B and σ_n is one-to-one on B . In the case $p = 0$ this final step can be completed rather easily but in the case $0 < p < 1$ it is not quite so clear. In [42] a detailed study of this and related problems was undertaken and a curious but unsatisfactory result was obtained:

THEOREM 6.9. *Suppose $0 < p < 1$. Then L_p has at most two non-trivial complemented subspaces up to isomorphism.*

If the second mysterious complemented subspace were to exist it would have some remarkable properties. For example, there would be an averaging projection of the space of vector-valued functions $L_p(Z) = L_p([0, 1]; Z)$ onto its subspace of constant functions.

PROBLEM 6.10. Suppose $0 < p < 1$ and X is any quasi-Banach space. Is it possible that there is an averaging projection on $L_p(X)$?

PROBLEM 6.11. Suppose $1 \leq p < \infty$ and suppose X is a quasi-Banach space so that there is an averaging projection on $L_p(X)$. Is X locally convex?

We remark that in [39] it is shown that there is no averaging projection on $L_p(L_p)$. Some partial results on Problem 6.11 are given in [45].

7. Lattices and natural spaces

We next present some basic facts in the theory of quasi-Banach lattices. It turns out that some interesting complications arise in the theory, associated with problems of convexity. Let us note first that the discussion of pp. 40–41 of [64] of homogeneous functions applies verbatim to quasi-Banach lattices. In this way we can define the notions of p -convexity and q -concavity as in [64] or [32] for quasi-Banach lattices. It is clear, for example, that the fundamental examples L_p when $0 < p < 1$ are then p -convex lattices. However this situation is very special, and there are examples of quasi-Banach lattices which fail to be p -convex for any $p > 0$. These issues are discussed in [46] and [16].

The issues involved relate to the Maharam submeasure problem, for which we refer to the article [25]. Suppose Ω is a set and Σ is a σ -algebra of subsets of Ω . A set function $\phi : \Sigma \rightarrow [0, \infty)$ is called a *submeasure* if we have

- $\phi(\emptyset) = 0$,
- $\phi(A) \leq \phi(B)$ if $A \subset B$,
- $\phi(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} \phi(A_n)$ for every $(A_n)_{n=1}^{\infty} \subset \Sigma$.

ϕ is called a *continuous* or *Maharam submeasure* if $A_n \downarrow \emptyset$ implies $\phi(A_n) \downarrow 0$. The unsolved Maharam submeasure problem asks whether every Maharam submeasure is equivalent to a measure, in the sense that if ϕ is a Maharam submeasure, then there is a measure μ so that $\mu(A) = 0$ if and only if $\phi(A) = 0$. See [58] for a partial result.

A submeasure ϕ is called *pathological* if whenever μ is a measure with $\mu \leq \phi$ then $\mu = 0$. This is equivalent to the fact that given any $\varepsilon > 0$ there exist $B_1, \dots, B_n \in \Sigma$ so that

$$\frac{1}{n} \sum_{k=1}^n \chi_{B_k} \geq (1 - \varepsilon) \chi_{\Omega} \tag{7.6}$$

and

$$\max_{1 \leq k \leq n} \phi(B_k) \leq \varepsilon. \tag{7.7}$$

Pathological submeasures have been constructed by several authors, but the simplest example seems to be that due to Talagrand [83]. The Maharam problem quoted above is equivalent to asking whether a pathological submeasure can be continuous.

Let us suppose then that (Ω, Σ, ϕ) is a submeasure space. Suppose $0 < r < \infty$. Let X be the space of all Σ -measurable functions f so that:

$$\|f\|_X = \left(\int_0^{\infty} \phi(|f| > t^r) dt \right)^{1/p} < \infty.$$

It is not hard to show that X is a quasi-Banach lattice. However if ϕ is pathological then X cannot be p -convex for any $p > 0$; this follows routinely from (7.6) and (7.7).

It is clear that if X has a p -subadditive quasi-norm then we have $(p, 1)$ -convexity, i.e.,

$$\left\| \sum_{k=1}^n |x_k| \right\| \leq \left(\sum_{k=1}^n \|x_k\|^p \right)^{1/p}, \quad x_1, \dots, x_n \in X.$$

It is shown in [16] that we then have

PROPOSITION 7.1. *Suppose $0 < p < 1$ and that X is a p -normable quasi-Banach space (with a p -subadditive quasi-norm). Then if $0 < q < r \leq 1$ with $1/q - 1/r = 1/p - 1$ we*

have:

$$\left\| \left(\sum_{k=1}^n |x_k|^r \right)^{1/r} \right\| \leq \left(\sum_{k=1}^n \|x_k\|^q \right)^{1/q}, \quad x_1, \dots, x_n \in X.$$

In [46] the notion of *lattice-convexity* or *L-convexity* is introduced. A quasi-Banach lattice X is *lattice-convex* or *L-convex* if there exists $C > 0$ so that if $u, x_1, \dots, x_n \in X$ with $\max_k |x_k| \leq |u|$ but $\frac{1}{n} \sum_{k=1}^n |x_k| \geq |u|$ then $\|u\| \leq C \max_k \|x_k\|$.

THEOREM 7.2 ([46]). *Let X be a p -normable quasi-Banach lattice. The following conditions are equivalent:*

- (1) X is lattice-convex.
- (2) X is r -convex for some $r > 0$.
- (3) X is r -convex for every $0 < r < p$.

There are special situations which guarantees lattice-convexity:

THEOREM 7.3 ([46,54]). *Let X be a quasi-Banach lattice with non-trivial cotype. Then X is lattice-convex.*

We remark that in [46] this theorem is deduced from a result on the Maharam submeasure problem proved in [58]. In [54] a simpler direct proof is given.

THEOREM 7.4 ([46]). *Let X be a quasi-Banach lattice which is isomorphic to a subspace of a lattice-convex quasi-Banach lattice. Then X is lattice-convex.*

Based on this theorem, we introduce the class of *natural spaces*. A quasi-Banach space X is called *natural* if it is linearly isomorphic to a subspace of a lattice-convex quasi-Banach lattice. In practice this implies that X is isomorphic to an ℓ_∞ -product of spaces of the type $L_p(\mu)$ where $0 < p < \infty$ is fixed. It may be shown that a p -normable space is natural if and only if it is finitely representable in the space weak L_p or $L_{p,\infty}$. The motivation for this definition is that most spaces that arise in analysis are natural. Natural spaces are relatively good spaces to work in. We mention, for example, that natural spaces must always contain a basic sequence. In fact it is not difficult to see that if X is separable and natural then there is a one-one operator $T : X \rightarrow L_p(0, 1)$. If T is not an isomorphism then X is not minimal and we can apply Theorem 3.6 while if T is an isomorphism then Theorem 3.8 applies. Of course this means that the example in Theorem 3.7 is not natural. There are some simpler examples of non-natural spaces. For example, the spaces $L_p(\mathbb{T})/H_p$ for $0 < p < 1$ (essentially proved in [41]) and the Schatten ideals \mathcal{C}_p for $0 < p < 1$ fail to be natural [49].

The importance of lattice-convexity is that this assumption allows us to use many of the powerful techniques available in the study of Banach lattices. Let us mention the example

of *square-function arguments*. It is well-known that in a Banach lattice with non-trivial cotype one has an estimate of the form (see [32] and [64]):

$$\mathbb{E} \left(\left\| \sum_{k=1}^n \varepsilon_k x_k \right\| \right) \approx \left\| \left(\sum_{k=1}^n |x_k|^2 \right)^{1/2} \right\|.$$

It follows from Theorem 7.3 above and similar arguments that this estimate works in any quasi-Banach lattice with cotype. We also mention the Krivine generalization of Grothendieck’s theorem (see [64], p. 93), which remains valid (with a different constant) even for operators between lattice-convex quasi-Banach lattices.

One of the most interesting lines of application of these ideas is in the study of uniqueness of unconditional bases in certain natural quasi-Banach spaces (see [88]). By combining well-established techniques from Banach space theory with the additional information that if X has an unconditional basis (u_n) then it simultaneously an unconditional basis in the Banach envelope, it is possible to prove some quite powerful uniqueness results for a wide class of spaces with unconditional bases. This line of research was initiated in [34] where it is shown that the spaces ℓ_p and many Orlicz sequence spaces for $0 < p < 1$ have unique unconditional basis. A general uniqueness criterion was developed in [53] which was improved in a recent paper of [93]:

THEOREM 7.5. *Let X be a natural quasi-Banach space with a normalized unconditional basis (u_n) . Suppose that:*

- (1) X is isomorphic to $X \oplus X$.
- (2) *There exists $q < 1$ so that if $\sum_{n=1}^\infty a_n u_n$ converges then $\sum_{n=1}^\infty |a_n|^q < \infty$. Then for any normalized unconditional basis (v_n) of X there is a permutation π of \mathbb{N} so that (u_n) and $(v_{\pi(n)})$ are equivalent.*

A weaker predecessor of this result was used in [53] to show that $H_p(\mathbb{T}^m)$ and $H_p(T^n)$ are isomorphic for $0 < p < 1$ if and only if $m = n$. For further results on uniqueness see [63,1] and [2].

8. Analytic functions and applications

Let X be a complex quasi-Banach spaces. Let Ω be an open subset of the complex plane \mathbb{C} . Then a function $F : \Omega \rightarrow X$ is called *analytic* if for every $z_0 \in \Omega$ there exists $\delta > 0$ and $x_n \in X$ for $n \geq 0$ so that if $|z - z_0| < \delta$ then

$$F(z) = \sum_{n=0}^\infty x_n z^n.$$

This definition of an X -valued analytic function was first employed by Turpin [86]. It is rather easy to see that other possible definitions based on complex differentiability do not work satisfactorily in quasi-Banach spaces. For example, if \mathbb{D} is the open unit

disk with standard area measure then the map $F : \mathbb{C} \rightarrow L_p(\mathbb{D})$ where $p < 1$ defined by $F(z)(w) = (z - w)^{-1}$ is actually complex differentiable but does not have a local power series expansion, and indeed is not infinitely differentiable.

The basic theory of analytic functions was developed by Turpin [86], who noticed that if K is a compact subset of Ω then the convex hull of $F(K)$ remains bounded in X . This enables one to show that there is a factorization of f through a Banach space. More precisely, if Ω_0 is an open relatively compact subset of Ω then there is a Banach space Y , a one-one bounded injection $j : Y \rightarrow X$ and an analytic function $G : \Omega_0 \rightarrow Y$ so that $F = j \circ G$. Since the theory of Banach space-valued analytic functions is very well understood one can use this device to prove many of the basic desired properties of analytic functions. For example, if f is analytic on a disk $\{z : |z - z_0| < r\}$ then f has a (necessarily unique) power-series expansion valid throughout the disk.

However, the theory of analytic functions is by no means as clean as for Banach spaces. The first obstacle is the Maximum Modulus Principle. A simple example due to Alexandrov [3] shows what can happen. Consider the function $F : \mathbb{D} \rightarrow L_p(\mathbb{T})$ defined by $F(z)(e^{i\theta}) = e^{-i\theta} (1 - e^{-i\theta} z)^{-1}$. This map is plainly analytic and extends continuously to a function on the closed unit disk $\bar{\mathbb{D}}$. Its power series expansion is given by $F(z) = \sum_{n=0}^{\infty} e^{-i(n+1)\theta} z^n$. Consider the subspace $H_p(\mathbb{T})$ and let $Q : L_p \rightarrow L_p/H_p$ be the quotient map. Then $Q \circ F$ is analytic into L_p/H_p and $\|Q(F(0))\| = 1$. However on the boundary if $|z| = 1$ then $QF(z) = 0$, i.e., $F(z) \in H_p$. To see this rewrite $F(z)$ as $-\bar{z}(1 - \bar{z}e^{i\theta})^{-1}$.

Of course if X has an equivalent plurisubharmonic quasi-norm then for any analytic function $F : \Omega \rightarrow X$ the map $z \mapsto \|F(z)\|$ is subharmonic and so we have a Maximum Modulus Principle. In fact this property essentially characterizes spaces for which a form of Maximum Modulus Principle holds. Let us say that X is *A-convex* if there is a constant C so that for every X -valued polynomial $F(z) = \sum_{k=0}^n x_k z^k$ we have

$$\|F(0)\| \leq C \max_{|z|=1} \|F(z)\|.$$

Of course this will imply that the same conclusion holds for any continuous function F on the closed unit disk $\bar{\mathbb{D}}$ which is analytic in the interior, so that the space L_p/H_p is an example of a non-A-convex space. It is shown in [49] that X is A-convex if and only if X has an equivalent plurisubharmonic quasi-norm.

Since the spaces L_p and ℓ_p are A-convex it follows trivially that natural spaces are A-convex. Of course by Theorem 3.9 A-convex spaces always contain basic sequences and so this yields another proof that natural spaces also must contain basic sequences. However, it should be noted that the Schatten classes C_p for $0 < p < 1$ are A-convex but fail to be natural (see [49]).

The treatment of analytic functions valued in a non-A-convex space requires different techniques, but it turns out that the theory is still quite rich. The key ingredient is an atomic decomposition theorem due to Coifman and Rochberg [15]. In this paper, the authors proved some very general atomic decompositions for certain Bergman spaces. As a by-product they extended the results of [23] and [80] to calculate the p -envelope of H_r when $0 < r < 1$ (this is the p -normed analogue of the Banach envelope). Let us denote this

space $B_{r,p}$. It turns out that $B_{r,p}$ consists of the space of all analytic functions on \mathbb{D} such that

$$\|f\|_{r,p} = \left(\int_{\mathbb{D}} |f(w)|^p (1 - |w|^2)^{p/r-2} dA(w) \right)^{1/p} < \infty,$$

where dA is standard area measure $dx dy$. The key to their proof of this is the following atomic decomposition:

THEOREM 8.1. *There is a constant $C = C(p, r)$ so that if $\psi \in B_{r,p}$ then there exist $z_k \in \mathbb{D}$ and $\alpha_k \in \mathbb{C}$ for $k \in \mathbb{N}$ so that $(\sum_{k=1}^{\infty} |\alpha_k|^p)^{1/p} \leq C \|\psi\|_{r,p}$ and*

$$\psi(w) = \sum_{k=1}^{\infty} \alpha_k (1 - |z_k|^2)^{\nu+1-\sigma} (1 - wz_k)^{-(\nu+2)},$$

where $\sigma = 1/r - 1$ and $\nu = [\sigma]$.

Let us illustrate how this can be used to establish some basic estimates (cf. [48]). Suppose X is a p -normable space and that $F : \mathbb{D} \rightarrow X$ is a polynomial. Suppose $F(z) = x_0 + x_1z + \dots + x_nz^n$. Fix $0 < r < p$ and define a linear operator $T : B_{r,p} \rightarrow X$ by

$$T(f) = \sum_{k=0}^n x_k \frac{(\nu + 1)!}{(\nu + k + 1)!} f^{(k)}(0).$$

Then

$$T((1 - wz_k)^{-\nu+2}) = F(z_k).$$

It follows by applying Theorem 8.1 rather crudely we can get an estimate that

$$\|T\| \leq C \max_{z \in \mathbb{D}} \|F(z)\|.$$

However

$$T(w^k) = \frac{k!(\nu + 1)!}{(\nu + k + 1)!} x_k$$

so that

$$\|x_k\| \leq C \binom{\nu + k + 1}{k} \|T\| \|w^k\|_{B_{r,p}}.$$

If we choose r so that $\sigma \in \mathbb{N}$ so that $\sigma = \nu$ then this gives a Cauchy-type estimate which is valid for any function F analytic in the open unit disk,

$$\|F^{(k)}(0)\| \leq C k! k^{1/p-1} \max_{z \in \mathbb{D}} \|F(z)\|. \tag{8.8}$$

Once one has these Cauchy estimates certain other basic principles of complex analysis follow. For example, it is clear that Liouville's theorem holds (i.e., a bounded analytic function is constant). This was first noted in [48] although an earlier weaker version for functions analytic on the Riemann sphere was observed by Turpin [86]. Let us also note there is an annular Maximum Modulus Principle:

THEOREM 8.2 ([49]). *For any $0 < r < 1$ and any $0 < p \leq 1$ there is a constant $C = C(r, p)$ so that if X is a p -normed quasi-Banach space and $F: \mathbb{D} \rightarrow X$ is an analytic function then*

$$\|F(0)\| \leq C \sup_{r \leq |z| < 1} \|F(z)\|.$$

PROOF OF THEOREM 8.2. We will indicate a proof quite different from that of [49]. Suppose the theorem is false for some $0 < r < 1$ and $0 < p < 1$. Then we may find a sequence of analytic functions $F_k: \mathbb{D} \rightarrow X_k$ where each X_k is p -normed and such that

$$\sup_{|z| < 1} \|F_k(z)\| = \|F_k(w_k)\| = 1$$

for some w_k with $|w_k| < r$ but

$$\sup_{r \leq |z| < 1} \|F_k(z)\| < 2^{-k}.$$

Define

$$G_k(z) = F\left(\frac{w_k - z}{1 - \bar{w}_k z}\right).$$

A routine power-series calculation shows these functions are analytic on \mathbb{D} and now we have:

$$\sup_{|z| < 1} \|G_k(z)\| = \|G_k(0)\| = 1$$

and

$$\|G_k(z)\| < 2^{-k}, \quad \text{if } |z| > \frac{2r}{1+r^2}.$$

Now define $G: \mathbb{D} \rightarrow \ell_\infty(X_k)$ by $G(z) = (G(z_k))_{k=1}^\infty$. The fact that G is also analytic follows from the Cauchy estimates 8.8 which imply that the obvious power series expansion converges to G . Now let $c_0(X_k)$ be the subspace of the ℓ_∞ -product of sequences $(x_k)_{k=1}^\infty$ with $\lim \|x_k\| = 0$ and let Q be the quotient map onto $\ell_\infty(X_k)/c_0(X_k)$. Then $Q \circ G$ is also analytic on \mathbb{D} and vanishes on the annulus $2r(1+r^2)^{-1} < |z| < 1$. However the standard fact that the zeros of an analytic function are isolated unless the function vanishes

identically remains valid in a quasi-Banach space; this is an easy consequence of the factorization principle of Turpin used above. Hence $Q \circ G$ vanishes identically on \mathbb{D} which gives a contradiction.

In fact much more detailed information about analytic functions is obtained in [48] and [49]. It is shown that there is an intimate relationship with the theory of integration in quasi-normed spaces which was developed by Turpin and Waelbroeck (see, e.g., [86] or [89]). We will not go into these topics here, but we would like to isolate one specific result on the degree of failure of the Maximum Modulus Principle.

Let us recall the example of Alexandrov in the space L_p/H_p namely $G(z) = Q \circ F(z)$ where $Q : L_p(\mathbb{T}) \rightarrow L_p/H_p$ is the quotient map and $F(z)(e^{i\theta}) = e^{-i\theta} (1 - ze^{-i\theta})^{-1}$. As remarked above this function G extends continuously to the closed disk and vanishes identically on the boundary. One can ask for the rate of decay of G near the boundary. It can be shown that

$$\|G(z)\| \leq C(1 - |z|)^{1/p-1}, \quad z \in \mathbb{D},$$

where C is some constant. □

The point of interest here is G is the “worst” possible such function. More precisely (see [48]):

THEOREM 8.3. *Suppose $0 < p < 1$ and that X is a p -normable quasi-Banach space. Suppose $F : D \rightarrow X$ is an analytic function. Then:*

- (1) *If $\lim_{|z| \rightarrow 1} (1 - |z|)^{1-1/p} \|F(z)\| = 0$ then F vanishes identically.*
- (2) *If F does not vanish identically and $\|F(z)\| \leq C(1 - |z|)^{1/p-1}$ for some constant C and all $z \in \mathbb{D}$ then there is a non-zero linear operator $T : L_p/H_p \rightarrow X$.*

9. Tensor products and algebras

In this section we apply some of the ideas in Sections 7 and 8. Let us first note that if X, Y, Z are Banach spaces then the collection of all continuous bilinear maps $B : X \times Y \rightarrow Z$ is quite rich, because of the presence of continuous linear functionals. Clearly some restrictions must be imposed when we pass to quasi-normed spaces. In fact there are no non-zero linear operators from L_p to a space which is q -normable if $q > p$ so to construct a non-zero bilinear map $B : L_p \times L_p \rightarrow X$ requires that X must already be p -normable at best. In this case there is an easy example namely $B : L_p(0, 1) \times L_p(0, 1) \rightarrow L_p(0, 1)^2$ given by $B(f, g) = f \otimes g$ where $f \otimes g(x, y) = f(x)g(y)$.

Let us make the problem more precise. Suppose X is p -normed and Y is q -normed where $0 < p, q \leq 1$. For $0 < r < 1$ and $u = \sum_{k=1}^n x_k \otimes y_k \in X \otimes Y$ define the r -tensor (semi-)quasi-norm by

$$\|u\|_r = \sup \left\{ \left\| \sum_{k=1}^n B(x_k, y_k) \right\| \right\},$$

where B runs over all bilinear maps $B : X \times Y \rightarrow Z$ where Z is r -normed and $\|B\| \leq 1$. From the above remarks, it is clear that $\|\cdot\|_r$ may reduce to 0 for certain choices of X, Y and r , e.g., if $X = Y = L_p$ and $p < r < 1$. The question is to determine the optimal condition on r which ensures $\|\cdot\|_r$ is a *tensor quasi-norm*, i.e., $\|u\|_r = 0$ implies $u = 0$ and $\|x \otimes y\| = \|x\| \|y\|$.

This problem was first considered by Turpin [87] who proved the following theorem:

THEOREM 9.1. *Suppose $0 < p \leq 1$ and $0 < q \leq 1$. Then $\|\cdot\|_r$ is a tensor quasi-norm on $X \otimes Y$ provided $1/r \geq 1/p + 1/q - 1$.*

In [48] this result was shown to be best possible:

THEOREM 9.2. *Suppose $0 < p \leq 1$ and $0 < q \leq 1$. Suppose $0 < r < 1$ and that there is a non-zero bilinear map $B : L_p/H_p \times L_q/H_q \rightarrow Z$ where Z is r -normable. Then $1/r \geq 1/p + 1/q - 1$.*

PROOF OF THEOREM 9.1. Let us indicate the proof following [16]; we consider the case of real spaces, but the argument extends to complex spaces. We need only consider the case $1/r = 1/p + 1/q - 1$. Suppose X is p -normed. Let $X^\#$ denote the algebraic dual of X (i.e., all linear functionals with no continuity restrictions). Let \widehat{X} denote the lattice of all real functions φ on $X^\#$ so that

$$\|\varphi\|_{\widehat{X}} = \inf \left\{ \left(\sum_{k=1}^n \|x_k\|^p \right)^{1/p} : |\varphi(x^\#)| \leq \sum_{k=1}^n |x^\#(x_k)| \ \forall x^\# \in X^\# \right\} < \infty.$$

One then checks that \widehat{X} is a p -normed quasi-Banach lattice (strictly speaking one must first quotient by the ideal of functions φ with $\|\varphi\|_{\widehat{X}} = 0$). Furthermore we can embed X isometrically into \widehat{X} via the map $x \rightarrow \hat{x}$ where $\hat{x}(x^\#) = x^\#x$. We make a similar construction of a q -normed quasi-Banach lattice \widehat{Y} of functions on $Y^\#$.

Next we create an r -normed quasi-Banach lattice Z of functions on $X^\# \times Y^\#$ by defining

$$\|h\|_{\widehat{Z}} = \inf \left\{ \left(\sum_{k=1}^n \|x_k\|^r \|y_k\|^r \right)^{1/r} : |h(x^\#, y^\#)| \leq \sum_{k=1}^n |x^\#(x_k)| |y^\#(y_k)| \ \forall x^\#, y^\# \right\}.$$

There is an obvious bilinear map $B : X \times Y \rightarrow Z$ given by $B(x, y) = \hat{x} \otimes \hat{y}$ where $\hat{x} \otimes \hat{y}(x^\#, y^\#) = \hat{x}(x^\#)\hat{y}(y^\#)$. Clearly $\|B\| = 1$. The main point to establish is that $\|B(x, y)\| = \|x\| \|y\|$. This will show that $\|x \otimes y\|_r = \|x\| \|y\|$ and this can be used to show that $\|\cdot\|_r$ is a quasi-norm on $X \otimes Y$ (we will omit this part of the argument).

Suppose then that

$$|\hat{x} \otimes \hat{y}| \leq \sum_{k=1}^n |\hat{x}_k \otimes \hat{y}_k|.$$

This implies that if $y^\# \in Y^\#$ then in \widehat{X} we have

$$|y^\#(y)\hat{x}| \leq \sum_{k=1}^n |y^\#(y_k)| |\hat{x}_k|$$

so that

$$|y^\#(y)| \|x\| \leq \left(\sum_{k=1}^n |y^\#(y_k)|^p \|x_k\|^p \right)^{1/p}$$

which translates as a statement in $Y^\#$, i.e.,

$$\|x\| |\hat{y}| \leq \left(\sum_{k=1}^n \|x_k\|^p |y_k|^p \right)^{1/p}.$$

Now Y is q -normed and so we can use Proposition 7.1 to deduce that

$$\|x\| \|y\| \leq \left(\sum_{k=1}^n \|x_k\|^r \|y_k\|^r \right)^{1/r}$$

which proves that $\|B(x, y)\| = \|x\| \|y\|$. □

PROOF OF THEOREM 9.2. We recall the construction of a function $G_p: \mathbb{D} \rightarrow L_p/H_p$ which is analytic and satisfies $\|G_p(z)\| \leq C(1 - |z|)^{1/p-1}$. In fact it is easy to show that the function $G = G_p$ constructed in Section 8 also has the property that the closed linear span of its range is the entire space L_p/H_p . Assume $B: L_p/H_p \times L_q/H_q \rightarrow Z$ is a bounded bilinear form when Z is s -normed for $s > 1/p + 1/q - 1$. Then if $|\zeta| = 1$ then analytic function $H(z) = B(G_p(z), G_q(\zeta z))$ on \mathbb{D} satisfies $\lim_{|\zeta| \rightarrow 1} \|H(z)\| (1 - |z|)^{1-1/s} = 0$ and so by Theorem 8.3 vanishes identically. Hence $B(G_p(w), G_q(z)) = 0$ if $|w| = |z|$ and $w, z \in \mathbb{D}$. If $|w| < 1$ then $z \rightarrow B(G_p(w), G_q(z))$ is analytic in \mathbb{D} and vanishes on the circle $|z| = |w|$ so again vanishes identically. Combining we obtain that B vanishes identically. □

Now let us turn our attention to the problem of (complex) quasi-Banach algebras. The fact that much of the basic theory of Banach algebras can be carried over to quasi-Banach algebras was discovered in the 1960's by Zelazko [94] and [95]. A key fact here (see [56], p. 124) is that if A is a commutative quasi-Banach algebra then the spectral radius

$$\rho(x) = \lim_{n \rightarrow \infty} \|x^n\|^{1/n} \tag{9.9}$$

is actually a seminorm. From this it is relatively easy to build the standard theory of the spectrum and in particular to justify the name spectral radius. In particular note that in any commutative quasi-Banach algebra A with identity we must have that A^* is non-trivial (and indeed there are continuous multiplicative linear functionals).

Let us now turn to non-commutative algebras. We always assume that A has an identity. It is clear, in view of the preceding discussion of tensor products, that since multiplication defines a bilinear map $A \times A \rightarrow A$ there must be some restrictions on A . For example, we cannot have A isomorphic to L_p/H_p when $p < 1$. We can explain some of the restrictions by noticing that even in the non-commutative case the spectral radius is a “nice” function (although obviously not in general a seminorm).

THEOREM 9.3 ([49]). *Let A be a quasi-Banach algebra with identity. Then the spectral radius $x \rightarrow \rho(x)$ is a plurisubharmonic function.*

PROOF. This is quite easy. Suppose $F : \mathbb{C} \rightarrow X$ is a polynomial. We use the Annular Maximum Modulus Principle of Theorem 8.2. If $r < 1$ there is a constant $C = C(r, A)$ so that

$$\|F(0)^n\| \leq C \sup_{r \leq |z| \leq 1} \|F(z)^n\|$$

and so

$$\|F(0)^n\|^{1/n} \leq C^{1/n} \sup_{r \leq |z| \leq 1} \|F(z)^n\|^{1/n}.$$

From this it follows that

$$\rho(F(0)) \leq \max_{r \leq |z| \leq 1} \|F(z)\|.$$

Now by standard techniques using outer functions this implies the stronger inequality

$$\log \rho(F(0)) \leq \int_0^{2\pi} \log \|F(e^{i\theta})\| \frac{d\theta}{2\pi}.$$

Applying this to $F(z)^n$ and taking limits yields that $\log \rho$ is plurisubharmonic and hence also ρ is plurisubharmonic. □

The existence of a non-trivial plurisubharmonic function on A is a significant restriction (cf. [49]). Once one has this information one can go further in extending the theory of Banach algebras to this more general setting. For example, Dilworth and Ransford showed that the set of invertible elements is pseudo-convex and extended the Johnson theorem on the uniqueness of the complete norm topology to quasi-Banach algebras [20].

We conclude by raising a question which seems rather interesting, and relates to the preceding discussion:

PROBLEM 9.4. Let A be any quasi-Banach algebra with identity. Can A have trivial dual?

Of course we may assume that $A = \mathcal{L}(X)$ for some quasi-Banach space X . In fact in all known examples $\mathcal{L}(X)$ has non-trivial dual (even if $X = L_p/H_p$ when $p < 1$). One reason

we would like to understand this question is that if Problem 9.4 has a negative solution then Problem 6.10 must also have a negative solution and this would imply that $L_p(0, 1)$ is a prime space. However our evidence for a negative solution is rather slim at present.

10. Final remarks

To conclude let us mention a few topics that perhaps fall into domain of this chapter but we have been forced to omit for lack of space.

Local theory. One area of increased interest recently has been the local theory of quasi-Banach spaces. It is rather surprising that many of the major achievements of the local theory of Banach spaces can be extended in a reasonable form to quasi-Banach spaces (so that *convexity* is not really relevant!). See, for example, [27,10] and [65].

Topological classification. The problem of topological classification for separable quasi-Banach spaces is open. Cauty [13] has shown that there are examples of separable F -spaces which are not homeomorphic to a Hilbert space. It is apparently unknown if this can be done for a quasi-Banach space. Very recently Cauty [14] has shown that, surprisingly, the Schauder Fixed Point theorem holds for any compact convex set in an F -space; this settles a problem going back over seventy years.

The uniform classification of quasi-Banach spaces is largely unexplored, but see [91] for a recent result.

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Edited by

W.B. JOHNSON

Texas A&M University, College Station, Texas, USA

J. LINDENSTRAUSS

The Hebrew University of Jerusalem, Jerusalem, Israel



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