The convexity type of quasi-Banach spaces

by

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1. Introduction

Let X be a quasi-Banach space. We define (cf. [3])

$$b_n = \sup_{\|x_i\| \le 1} \inf_{\varepsilon_i = \pm 1} \| \varepsilon_1 x_1 + \ldots + \varepsilon_n x_n \| .$$

Then as we define the <u>convexity type</u> p of X by

$$\frac{1}{p} = \lim_{n \to \infty} \frac{\log b_n}{\log n}$$

(the limit exists: see Section 4). The main result of this paper is that $0 < p \le 2$ and that p is the smallest number such that ℓ_p is weakly finitely representable in X (see Section 4 for the definition of weak finite representability). A corollary is that ℓ_1 is weakly finitely representable in any non-locally convex quasi-Banach space (and indeed is finitely representable if the norm is r-subadditive for some $r > 0$). We also solve a problem of Turpin by showing that if the galb of a quasi-Banach space contains ℓ_p strictly then it contains ℓ_q for some $q > p$.

For Banach spaces the convexity type p satisfies $1 \le p \le 2$. It was introduced by Maurey and Pisier [5], but with a slightly different definition (Definition 2.2); it turns out to be the supremum of all q such that X is of type q-Rademacher. The main result quoted above can in this case be combining results of Rosenthal [7] with Theorem 2.1 of Maurey-Pisier [5]. However it does not appear possible to simply generalize this approach to quasi-Banach spaces; see the discussion in Section 4.

In Section 3, we extend recent results of Krivine [4] and Rosenthal [7] on the finite representability of ℓ_p to quasi-Banach spaces. Naturally, most of their arguments go through with only minor modifications; we hope therefore that the reader will understand when we elect to simply sketch the idea of the proof, or refer to the papers of Krivine and Rosenthal.

Our main results are proved in Section 4.

2. Notation

As usual, a quasi-norm on a real vector space X is a mapping $x \mapsto \|x\|$ of X into R such that

(i) $\|x\| \geq 0$ with equality if and only if $x = 0$, $x \in X$,

(ii) $\|tx\| = |t| \, \|x\|$, $t \in R$, $x \in X$,

(iii) there is a constant k (the modulus of concavity of $\|\cdot\|$) such that

$$\|x+y\| \leq k(\|x\| + \|y\|) \qquad x,y \in X .$$

A quasi-norm induces a locally bounded topology on X and conversely any locally bounded topology on X may be described by a quasi-norm. A complete quasi-normed space is called a quasi-Banach space.

Two quasi-norms $\|\cdot\|$ and $\|\cdot\|^*$ are said to be equivalent if there exist $0 < m \leq M < \infty$ such that

$$m\|x\| \leq \|x\|^* \leq M\|x\| \qquad x \in X.$$

A quasi-norm $\|\cdot\|$ is p-subadditive where $0 < p \leq 1$ if

(iv) $$\|x+y\|^p \leq \|x\|^p + \|y\|^p \qquad x,y \in X .$$

A quasi-Banach space with a p-subadditive quasi-norm is called a p-Banach space. A fundamental theorem of Aoki and Rolewicz ([6] p.61) asserts that every quasi-norm is equivalent to a quasi-norm which is p-subadditive for some p, $0 < p \leq 1$. Of course a quasi-Banach space in X is isomorphic to a p-Banach space if and only if it is (locally) p-convex, i.e. possesses a bounded neighbourhood of 0, V say, such that if $x,y \in V$ and $|a|^p + |b|^p \leq 1$ then $ax + by \in V$.

At this point we observe the following technical lemma:

Lemma 2.1. *Let X be a locally bounded space and let U be a bounded symmetric neighbourhood of 0; then given $0 < \varepsilon < 1$ there exists $\delta > 0$ and $r > 0$ such that for some closed r-convex neighbourhood of 0, V, we have* $U \subset V \subset (1+\delta)V \subset U + \varepsilon U$.

Proof. Define $\delta = \varepsilon/8k^3$, where k is the modulus of concavity of U, i.e. $U + U \subset kU$. Then if we can choose W r-convex so that $U \subset W \subset U + \delta U$, we have

$$(1+\delta)W \subset (1+\delta)(U+\delta U) = U + \delta(U+U) + \delta^2 U$$
$$\subset U + k^2\delta U,$$

and hence

$$(1+\delta)\overline{W} \subset U + 4k^2\delta U + 4k^2\delta U$$
$$\subset U + 8k^3\delta U = U + \varepsilon U.$$

Hence choosing $V = \overline{W}$ we are home. Thus it remains to show that for some r-convex set W, $U \subset W \subset U + \delta U$.

Since X is locally bounded, by the Aoki-Rolewicz theorem, there is a $p > 0$ such that the closed p-convex hull $co_p U$ of U is bounded. Suppose for all $r > 0$, $co_r U \not\subset U + \delta U$. Choose for each r, $z_r \in co_r U \setminus (U+\delta U)$, where

$$z_r = \sum_{i=1}^{\infty} a_{ri}\, u_{ri}$$

and each sum is finitely non-zero, $a_{r1} \geq a_{r2} \geq \dots \geq 0$, and $u_{ri} \in U$.

Then

$$\sum_{i=2}^{\infty} a_{ri}\, u_{ri} \notin \delta U.$$

However

$$\sum_{i=2}^{\infty} a_{ri}\, u_{ri} \in \left(\sum_{i=2}^{\infty} a_{ri}^p\right)^{\frac{1}{p}} co_p U$$

and

$$\left(\sum_{i=2}^{\infty} a_{ri}^p\right) \leq \left(a_{r2}^{p-r} \sum_{i=2}^{\infty} a_{ri}^r\right)^{\frac{1}{p}}$$
$$\leq a_{r2}^{1-\frac{r}{p}}.$$

However

$$2a_{r2}^r \leq a_{r1}^r + a_{r2}^r \leq 1$$

and hence

$$\left(\sum_{i=2}^{\infty} a_{ri}^{p}\right) \leq \left(\frac{1}{2}\right)^{\frac{1}{r}-\frac{1}{p}} \to 0,$$

as $r \to 0$. For suitable r we have

$$\left(\frac{1}{2}\right)^{\frac{1}{r}-\frac{1}{p}} co_p(U) \subset \delta U$$

and thus we have a contradiction.

A linear map $T: X \to Y$, where X and Y are quasi-normed, is continuous if and only if

$$\|T\| = \sup(\|Tx\| : \|x\| \leq 1) < \infty.$$

If N is a closed subspace of a quasi-normed space X, then X/N may be quasi-normed by the formula

$$\|x+N\| = \inf_{y\in N} \|x+y\|.$$

3. Finite representability

In this section we extend standard Banach space ideas developed in [1], [2], [4] and [7] to quasi-normed spaces. For technical reasons it is convenient to assume the quasi-norm r-subadditive for some $r > 0$. The precise condition that is required is that given $\varepsilon > 0$, there exists $\delta > 0$ such that if $\|x\| \leq 1$ and $\|y\| \leq \delta$ then $\|x+y\| \leq 1+\varepsilon$. This is implied by r-subadditivity; we shall restrict ourselves to r-Banach spaces.

Suppose X is a quasi-Banach space. Then a quasi-Banach space Y is said to be finitely representable in X if given $\varepsilon > 0$ and any finite-dimensional subspace E of Y there is a subspace F of X with $\dim F = \dim E$ and an isomorphism $T: E \to F$ such that $\|T\| . \|T^{-1}\| \leq 1+\varepsilon$.

Suppose I is an infinite set and that $\mathcal{U}$ is a free ultrafilter on I. Then the space $\ell_I^\infty(X)$ is defined to be the space of all bounded maps $x: I \to X$ quasi-normed by

$$\|x\| = \sup_{i\in I} \|x_i\|.$$

Let $c_{0,\mathcal{U}}(X)$ be the space of all $x \in \ell_I^\infty(X)$ such that

$$\lim_{\mathcal{U}} \|x_i\| = 0.$$

Then $c_{0,\mathcal{U}}(X)$ is a closed subspace of $\ell_I^\infty(X)$. We define the ultrapower $X_{\mathcal{U}}^I$ to be the quotient of $\ell_I^\infty(X)$ by $c_{0,\mathcal{U}}(X)$. For $x \in \ell_I^\infty(X)$ define

$$\|x\|_{\mathcal{U}} = \|qx\|$$

where $q : \ell_I^\infty(X) \to X_{\mathcal{U}}^I$ is the quotient map. If the quasi-norm on X is r-subadditive then it is easy to see that

$$\|x\|_{\mathcal{U}} = \lim_{\mathcal{U}} \|x_i\|.$$

Under these circumstances we easily obtain the following generalization of a result familiar in Banach space theory ([1], [4]).

Theorem 3.1. Suppose X is an r-Banach space and $0 < r \leq 1$. Then a quasi-Banach space Y is finitely representable in X if and only if Y is isometric to a closed subspace of some ultrapower of X.

The next result is due essentially to Krivine [4], although our proof differs slightly.

Theorem 3.2. Let X be an r-Banach space and let $\mathcal{U}$ be a free ultra-filter on $\mathbb{N}$. Suppose (x_n) is a sequence in X. Then there is a sequence (e_n) in some ultra-power of X such that

$$\Big\| \sum_{i=1}^{n} t_i e_i \Big\| = \lim_{m_n \mathcal{U}} \lim_{m_{n-1} \mathcal{U}} \cdots \lim_{m_1 \mathcal{U}} \Big\| \sum_{i=1}^{n} t_i x_{m_i} \Big\|.$$

Proof. Let $I = \mathbb{N}^{\mathbb{N}}$ and let A_k be the sub-algebra of $\ell_I^\infty(\mathbb{R})$ of all real functions f which depend only on the first k co-ordinates, i.e.

$$f((\ell_n)) = f((m_n))$$

whenever $\ell_n = m_n$ for $1 \leq n \leq k$. Let $A = \bigcup_k (A_k : k \in \mathbb{N})$. Then there is a norm-one multiplicative linear functional ψ on A defined by

$$\psi(f) = \lim_{\ell_k \mathcal{U}} \cdots \lim_{\ell_1 \mathcal{U}} f((\ell_n)) \qquad f \in A_k.$$

It now follows that there is an ultra-filter $\mathcal{V}$ on I such that

$$\lim_{\mathcal{V}} f(i) = \psi(f) \qquad f \in A.$$

Consider $X_{\mathcal{V}}^I$ and define $e_k \in \ell_I^\infty(X)$ by

$$e_k((\ell_n)) = x_{\ell_k}.$$

Then

$$\left\| \sum_{i=1}^{n} t_i e_i \right\|_{\mathcal{V}} = \lim_{m_n \, \mathcal{U}} \cdots \lim_{m_1 \, \mathcal{U}} \left\| \sum_{i=1}^{n} t_i x_{m_i} \right\|.$$

Identifying e_k with its equivalence class in $X^I_{\mathcal{V}}$ the proof is complete.

A sequence (e_n) in a quasi-Banach space with property

$$\left\| \sum_{i=1}^{n} t_i e_i \right\| = \left\| \sum_{i=1}^{n} t_i e_{m_i} \right\|$$

whenever $m_1 < m_2 < \ldots < m_n$, is called a <u>spreading sequence</u> (cf. [7]). Clearly the sequence (e_n) of Theorem 3.2 is spreading; we shall refer to it as a <u>spreading model</u> for (x_n).

<u>Lemma 3.3</u>. *If (e_n) is a spreading sequence, and $\|e_1 - e_2\| = \delta > 0$, then*

$$\left\| \sum_{i=1}^{n} t_i e_i \right\| \geq \frac{\delta}{2k} \max_{1 \leq i \leq n} |t_i|$$

where k is the modulus of concavity of $\|\cdot\|$.

<u>Proof</u>. See Theorem I.1 of [4].

If (e_n) is a spreading sequence in a Banach space then Brunel and Sucheston [1] show that $(e_{2n} - e_{2n+1})$ is an unconditional basic sequence. For quasi-Banach spaces however it is not clear that this sequence is even basic. Fortunately, however, this fact may be replaced by the arguments of Theorem I.1 of [4] and Theorem 1.4 of [7]. We call a sequence (h_n), in a quasi-normed space X, <u>sign-invariant</u> provided

$$\left\| \sum_{i=1}^{n} \varepsilon_i t_i h_i \right\| = \left\| \sum_{i=1}^{n} t_i h_i \right\|$$

whenever $t_1, \ldots, t_n \in R$ and $\varepsilon_i = \pm 1$, $i = 1, 2, \ldots, n$.

If (h_n) is sign-invariant, there is a constant c, depending on the modulus of concavity of the quasi-norm such that

$$\left\| \sum_{i=1}^{n} s_i t_i h_i \right\| \leq c \max_{1 \leq i \leq m} |s_i| \left\| \sum_{i=1}^{n} t_i h_i \right\|$$

whenever $s_1, \ldots, s_n, t_1, \ldots, t_n \in \mathbb{R}$.

Lemma 3.4 Suppose (e_n) is a spreading sequence in an r-Banach space X and that $\|e_1 - e_2\| > 0$. Let $f_n = e_{2n} - e_{2n+1}$ and suppose $m(n) \uparrow \infty$ is such that

$$\|\sum_{i=1}^{m(n)} f_i\| = M_n \to \infty .$$

Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$ and define $g_k \in X_{\mathcal{U}}^{\mathbb{N}}$ by:

$$g_k(n) = \frac{1}{M_n} \sum_{i=(k-1)m_n+1}^{k\,m(n)} f_i .$$

Then any spreading model for (g_k) is sign-invariant..

The proof is exactly as given Theorem I.1 of [4] and Theorem 1.4 of [7]. We shall also need another result from the proofs of these theorems.

Lemma 3.5 Suppose (e_n) is a spreading sequence in an r-Banach space X and that $\|e_1 - e_2\| > 0$. If $f_n = e_{2n} - e_{2n+1}$ and

$$\sup_n \|\sum_{i=1}^{n} f_i\| < \infty$$

the closed linear span of (f_n) is isomorphic to c_o.

Proof. If $\|e_1 - e_2\| > \delta$ then

$$\|\sum_{i=1}^{n} t_i f_i\| \geq \frac{\delta}{2k} \max_{1\leq i\leq n} |t_i|$$

where k is the modulus of concavity of $\|\cdot\|$. Conversely, if

$$\|\sum_{i=1}^{n} f_i\| \leq M \qquad n \in \mathbb{N}$$

then

$$\|\sum_{i=1}^{n} \varepsilon_i f_i\| \leq 2kM$$

whenever $\varepsilon_i = \pm 1$, and hence

$$\|\sum_{i=1}^{n} t_i f_i\| \leq C \max_{1\leq i\leq n} |t_i| \qquad t_1,\ldots,t_n \in \mathbb{R},$$

where C is a constant depending only on M and k.

We now come to an important and fundamental result due to Krivine [4] for $p \geq 1$; in fact the generalization to the case of quasi-Banach spaces goes through almost verbatim.

Theorem 3.6 *Suppose* X *is an* r-*Banach space which is isomorphic to* ℓ_p *where* $r \leq p < \infty$. *Then* ℓ_p *is finitely representable in* X.

Proof. (Sketch). Let (x_n) be a basis of X equivalent to the usual basis of ℓ_p. We can construct a spreading model (e_n) for (x_n) as in Theorem 3.2, and then (e_n) is equivalent to the usual basis of ℓ_p. In particular if $f_n = e_{2n} - e_{2n+1}$, then

$$\|\sum_{i=1}^{n} f_i\| \geq c n^{1/p}$$

where $c > 0$ and hence we can appeal to Lemma 3.4 to produce a sign-invariant spreading sequence (h_n) which is also equivalent to the unit vector basis of ℓ_p. It is then possible to duplicate the proofs of Theorems 2.2 and 2.3 of [7], (also Theorem III.1 of [4]). The fact that (h_n) is not a 'lattice' sequence in the sense of [7] turns out to be unimportant - sign-invariance is sufficient. Some form of continuity of the quasi-norm is necessary, but it is sufficient that it be r-subadditive (this property obviously carries over to ultrapowers).

We should, of course, stress at this point that we have omitted a very substantial argument here.

Corollary 3.7 *If* X *is an* r-*Banach space and* ℓ_p $(0 < p < \infty)$ *is isomorphic to a subspace of an ultrapower of* X, *then* ℓ_p *is finitely representable in* X.

4. Representability of ℓ_p $(0 < p \leq 2)$

Let X be an infinite-dimensional quasi-Banach space. We define $a_n = a_n(X)$ by

$$a_n = \sup\left\{\|\sum_{i=1}^{n} x_i\| : \|x_i\| \leq 1,\ 1 \leq i \leq n\right\}$$

and $b_n = b_n(X)$ by

$$b_n = \sup_{\|x_i\|\leq 1} \inf_{\varepsilon_i = \pm 1} \|\sum_{i=1}^{n} \varepsilon_i x_i\| .$$

Then (cf. [3]) the sequences $\{a_n\}$ and $\{b_n\}$ are monotone increasing and submultiplicative. It follows that

$$\lim_{n\to\infty} \frac{\log b_n}{\log n} = \alpha$$

and

$$\lim_{n\to\infty} \frac{\log a_n}{\log n} = \beta$$

exist, where $0 \leq \alpha \leq \beta < \infty$. We shall define $p = p(X)$, the <u>convexity type</u> of X by

$$p = \alpha^{-1}.$$

so that, at least for the moment, $0 < p \leq \infty$.

We observe that $p(X)$ is an isomorphic inveriant of X and that if Y is finitely represented in X then $p(Y) \geq p(X)$. The following results are essentially given in [3] Lemma 2.4 and Theorem 2.5.

Proposition 4.1 $$\lim_{n\to\infty} \frac{\log a_n}{\log n} = \max\left(1, \frac{1}{p}\right).$$

Proposition 4.2 $p(X) > 1$ *if and only if* X *is isomorphic to a B-convex Banach space.*

It follows from Proposition 4.2 and the Dvoretzky theorem that $0 < p(X) \leq 2$ in general. However we shall not assume Dvoretzky's theorem in advance, but prefer to deduce from our argument. We therefore replace it with the following lemma.

Lemma 4.3 *If* $\dim X = \infty$ *then* $p(X) < \infty$.

Proof. By Lemmas 3.4 and 3.5 there is a quasi-Banach space Y with an unconditional basis which is finitely represented in X. Then $p(Y) \geq p(X)$. Y is isomorphic to a space $\tilde{Y}$ with a lattice-normed unconditional basis, i.e. a basis (e_n) such that

$$\left\| \sum_{i=1}^{n} s_i t_i e_i \right\| \leq \max_{1\leq i\leq n} |s_i| \left\| \sum_{i=1}^{n} t_i e_i \right\|.$$

Thus if $p(X) = \infty$, we have $p(\widetilde{Y}) = \infty$. Now we apply the argument of Theorem II.1 of [4]. Suppose there exists an $r < \infty$ such that

$$\| \sum_{i=1}^{n} t_i e_i \| \geq C\left(\sum_{i=1}^{n} |t_i|^r\right)^{1/r} \qquad t_1,\dots,t_n \in \mathbb{R}$$

where $C > 0$. Then

$$b_n \geq C n^{1/r}$$

and hence $p(\widetilde{Y}) \leq r$. Thus we deduce from the argument of Theorem II.1 of [4] that c_0 is finitely representable in $\widetilde{Y}$, and hence that ℓ_1 is also finitely representable in $\widetilde{Y}$. This implies $p(\widetilde{Y}) \leq 1$, and we have a contradiction.

Let us note at this point that if X is a Banach space, then the quantity $p(X)$ is identical to the quantity $p_X = p^X$ considered by Maurey and Pisier ([5], Théorème 2.1); the definition here is a minor variation on the definition of p_X in [5]. Maurey and Pisier show that if $p = p_X$ then the injection $\ell_1 \to \ell_p$ is finitely factorizable in X, i.e. given $n \in \mathbb{N}$ and $\delta > 0$ there exist $e_1,\dots,e_n \in X$ such that

$$(1-\delta)\left(\sum_{i=1}^{n} |\alpha_i|^p\right)^{1/p} \leq \| \sum_{i=1}^{n} \alpha_i e_i \| \leq \sum_{i=1}^{n} |\alpha_i| ,$$

for $\alpha_1,\dots,\alpha_n \in \mathbb{R}$. They then show that this implies that ℓ_p is finitely representable in X, using (essentially) Theorem 3.3 of Rosenthal [7]. This will be our main result for r-Banach spaces.

However we have not attempted here simply to follow the Maurey-Pisier argument, as for non-locally convex spaces, Rosenthal's result loses some of its force: one must assume that ℓ_∞ is not finitely representable in X. This restriction for Banach spaces is insignificant, but is important for quasi-Banach spaces. We also note that, in any case, Maurey and Pisier use the fact (false for non-locally convex spaces) that if (e_n) is a spreading sequence then $(e_{2n} - e_{2n+1})$ is an unconditional basic sequence.

Thus we shall follow a different line of proof. We prove first two preparatory lemmas, which essentially replace Théorème 2.1 of [5].

Lemma 4.4 Suppose $s > p$. Then there is a sequence (z_n) in some ultrapower of X such that for some M_0 we have

$$\|\sum_{n\in A} \eta_n z_n\| \geq n^{1/s}$$

whenever $A \subset \mathbb{N}$, $|A| = 2^m$, $m \geq M_0$, and $\eta_n = \pm 1$, $n \in A$.

Proof. Let us define q by

$$\frac{2}{q} = \frac{1}{p} + \frac{1}{s}.$$

Then $r \leq p < q < s$. Let

$$\gamma = -\frac{r}{p}\frac{(q-p)}{(q-r)}$$

and choose M_0 so that

$$\left(\frac{1}{2}\right)^{\gamma M_0} \leq \frac{1}{2}\left[1 - \left(\frac{1}{2}\right)^{\gamma}\right]\left(\frac{r}{q}\right)^{\frac{qr}{q-r}}.$$

Now suppose $M \geq M_0$. Then there exists $N \geq 2^{M+1}$ such that for $1 \leq k \leq N$

$$\frac{b_N}{b_k} \geq \left(\frac{N}{k}\right)^{1/q}$$

since otherwise we would have $b_n = O(n^{1/q})$.

Hence there exists a set $(x_1,\dots,x_N)$ of vectors in some ultrapower of X such that

$$\|x_i\| \leq 1 \qquad 1 \leq i \leq N$$

$$\|\sum_{i=1}^{N} \eta_i x_i\| \geq b_N \qquad \eta_i = \pm 1, \quad 1 \leq i \leq N.$$

Suppose $M_0 \leq m \leq M$ and let $(A_1,\dots,A_d)$ be a maximal collection of disjoint subsets of $(1,2,\dots,N)$ with $|A_i| = 2^m$ $(1 \leq i \leq d)$, such that for a suitable choice of signs $\gamma_j = \pm 1$

$$\|\sum_{j\in A_i} \gamma_j x_j\| < 2^{m/s}.$$

Then let $B_m = \{1,2,\dots,N\} \setminus \bigcup_{i=1}^{d} A_i$. If $C \subset B_m$ and $|C| = 2^m$

$$\|\sum_{j\in C} \gamma_j x_j\| \geq 2^{m/s}$$

whenever $\gamma_j = \pm 1$. We have $|B_m| = N - d2^m$.

For a suitable choice of signs $\eta_1 \dots \eta_d$

$$\|\sum_{i=1}^{d} \eta_i \sum_{j\in A_i} \gamma_j x_j\| \leq 2^{m/s} b_d$$

and for suitable $(\gamma_i : i \in B_m)$

$$\|\sum_{i\in B_m} \gamma_i x_i\| \leq b_{N-d.2^m}.$$

Hence

$$b_N^r \leq b_{N-d.2^m}^r + 2^{r\frac{m}{s}} b_d^r$$

i.e.

$$1 \leq \left(\frac{b_{N-d.2^m}}{b_N}\right)^r + 2^{r\frac{m}{s}}\left(\frac{b_d}{b_N}\right)^r$$

$$\leq \left(1 - \frac{d.2^m}{N}\right)^{\frac{r}{q}} + \left(\frac{d.2^m}{N}\right)^{\frac{r}{q}} 2^{mr\left(\frac{1}{s}-\frac{1}{q}\right)}$$

$$\leq 1 - \frac{r}{q}.\left(\frac{d.2^m}{N}\right) + \left(\frac{d.2^m}{N}\right)^{\frac{r}{q}} 2^{mr\left(\frac{1}{s}-\frac{1}{q}\right)}$$

by the Mean Value Theorem, since $r < q$.

Thus

$$\left(\frac{d.2^m}{N}\right)^{1-\frac{r}{q}} \leq \left(\frac{q}{r}\right) 2^{mr\left(\frac{1}{s}-\frac{1}{q}\right)}$$

and

$$\frac{d.2^m}{N} \leq \left(\frac{q}{r}\right)^{\frac{qr}{q-r}} 2^{-m\gamma}.$$

Let $B = \bigcap_{m=M_0}^{M} B_m$. Then

$$|B| \geq N\left(1 - \left(\frac{q}{r}\right)^{\frac{qr}{q-r}} \sum_{m=M_0}^{\infty} 2^{-m\gamma}\right)$$

$$\geq N\left(1 - \left(\frac{q}{r}\right)^{\frac{qr}{q-r}} 2^{-M_0\gamma}\left(1 - \left(\frac{1}{2}\right)^{\gamma}\right)^{-1}\right)$$

$$\geq \frac{1}{2}N \geq 2^M .$$

If $C \subset B$ and $|C| = 2^m$ for $m \geq M_0$ then

$$\|\sum_{i \in C} \eta_i x_i\| \geq |C|^{1/s} \qquad \eta_i = \pm 1,\ 1 \leq i \leq 2^m .$$

By passing to a further ultrapower we obtain the result.

__Lemma 4.5__ *If* $s > p$, *there is a sign-invariant spreading sequence* (u_n) *in some ultrapower of* X *such that*

$$\|\sum_{i=1}^{n} u_i\| \geq c n^{1/s} \qquad n \in \mathbb{N},$$

where $c > 0$.

__Proof__. Again choose q such that

$$\frac{2}{q} = \frac{1}{p} + \frac{1}{s} .$$

Choose a spreading sequence (e_n) in an ultrapower of X by Theorem 3.2 and Lemma 4.4 so that if $m \geq M_1$

$$\|\sum_{i=1}^{2^m} \eta_i e_i\| \geq 2^{m/q}$$

for $\eta_i = \pm 1$, $1 \leq i \leq 2^m$. It is clear that $\|e_i - e_j\| \geq \delta > 0$ if $i \neq j$, since otherwise $e_i = e$ for all i, and by suitable choice of signs

$$\sum_{i=1}^{2^m} \eta_i e_i = 0 \qquad m \geq 1 .$$

Now let $f_i = e_{2i} - e_{2i+1}$ $(i \geq 1)$.

Let

$$\theta_m = \|\sum_{i=1}^{2^m} f_i\| .$$

Then $\theta_m 2^{-m/s} \to \infty$ and so there is an increasing sequence $m(n)$ such that

$$\theta_{m(n)} 2^{-\frac{m(n)}{s}} \leq \theta_\ell 2^{-\ell/s} \cdot \text{ for } \ell \geq m(n) .$$

Let

$$g_{nk} = \frac{1}{\theta_{m(n)}} \sum_{i=(k-1)2^m+1}^{k.2^m} f_i .$$

Then $\|g_{nk}\| = 1 , \ 1 \leq k < \infty$ and

$$\| \sum_{k=1}^{2^\ell} g_{n\,k} \| = \frac{\theta_{m(n)+\ell}}{\theta_{m(n)}} \geq . 2^{\ell/s} \qquad 1 \leq \ell < \infty .$$

Thus by Lemma 3.4 we may find a sign-invariant sequence (u_n) in an ultrapower of X such that

$$\| \sum_{i=1}^{2^\ell} u_i \| \geq 2^{\ell/s} \qquad 1 \leq \ell < \infty .$$

If $2^\ell < n < 2^{\ell+1}$

$$\begin{aligned} \| \sum_{i=1}^{n} u_i \| &\geq a \| \sum_{i=1}^{2^\ell} u_i \| \\ &\geq a . 2^{\ell/s} \\ &\geq c n^{1/s} \end{aligned}$$

where $a > 0$ and $c > 0$ are constant depending only on the modulus of concavity of X.

Theorem 4.6 Suppose X is an infinite-dimensional r-Banach space. Then the convexity type p of X satisfies

(i) $0 < p \leq 2$,

(ii) if $0 < q \leq 2$, ℓ_q is finitely representable in X if and only if $q \geq p$.

Proof. Let us first remark that if X is a Banach space, this may be deduced easily from Lemma 4.5 and Theorem 3.3 of [7], since it may be assumed without loss of generality that c_0 is not finitely representable in X, since $p(X) \geq 1$. However if X is not locally convex, then $p(X)$ may be less than one, and this assumption becomes restrictive. This explains the following rather more involved argument (see the remarks above).

Suppose $\infty > s > \tau > t > p$ and choose from Lemma 4.5 a spreading sign-invariant sequence (u_n) in an ultrapower Y of X such that

$$\rho_n = \|\sum_{i=1}^{n} u_i\| \geq c n^{1/t}$$

where $c > 0$. Since $\rho_n n^{-\frac{1}{\tau}} \to \infty$, there is an increasing sequence $m(n)$ such that

$$\rho_{m(n)} m(n)^{-\frac{1}{\tau}} \geq \rho_\ell \ell^{-\frac{1}{\tau}} \qquad 1 \leq \ell \leq m(n).$$

Now let E_n be the subspace of $L_s(0,1)$ spanned by the characteristic functions $\chi_{k,n}$ of $\left(\frac{k-1}{m(n)}, \frac{k}{m(n)}\right)$ and define $T_n : E_n \to Y$ by

$$T_n\left(\sum_{k=1}^{m(n)} \beta_k \chi_{k,n}\right) = \frac{1}{\rho_{m(n)}} \sum_{k=1}^{m(n)} \beta_k u_k .$$

Since (u_k) is sign-invariant there is a constant $M < \infty$ such that

$$\|\sum_{i=1}^{n} a_i c_i u_i\| \leq M \max_{1\leq i\leq n} |a_i| \, \|\sum_{i=1}^{n} c_i u_i\| .$$

Hence if $\varphi, \psi \in E_n$ then

$$\|T_n(\varphi\psi)\| \leq M\|\varphi\|_\infty \|T_n \psi\| .$$

In particular if $\varphi \in E_n$ and $\|\varphi\|_s = 1$, then

$$\|T_n \varphi\| \leq 2M\|T_n \varphi^*\|$$

where

$$\varphi^* = \sum_{k=-\infty}^{\infty} 2^k \chi_{A_k}$$

and $A_k = \{\xi : 2^{k-1} < |\varphi(\xi)| \leq 2^k\}$. Now

$$1 \geq \int_{A_k} |\varphi(\xi)|^s d\xi$$

$$\geq 2^{(k-1)s} \lambda(A_k)$$

and hence $\lambda(A_k) \leq \max(1, 2^{-(k-1)s}) \qquad -\infty < k < \infty$.

Hence

$$\|T_n\varphi^*\|^r \le \sum_{k=-\infty}^{\infty} 2^{kr} \|T_n\chi_{A_k}\|^r$$

$$\le \sum_{k=-\infty}^{\infty} 2^{kr} \rho_{m(n)}^{-r} \rho_{m(n)}^{r} \lambda(A_k)$$

$$\le \sum_{k=-\infty}^{\infty} 2^{kr} \lambda(A_k)^{r/\tau}$$

$$\le \sum_{k=-\infty}^{-1} 2^{kr} + \sum_{k=0}^{\infty} 2^{kr-(k-1)\frac{sr}{\tau}}$$

$$= C^r < \infty$$

where C is independent of n. Thus $\|T_n\| \le 2MC$ for all n.

Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$ and let $\Pi_{\mathcal{U}}(E_n)$ be the subset of $(L_s)_{\mathcal{U}}^{\mathbb{N}}$ of all (φ_n) such that $\varphi_n \in E_n$ for all n. We may define an isometric embedding $J : L_s \to \Pi_{\mathcal{U}}(E_n)$ by mapping φ to the coset of any sequence (φ_n) where $\varphi_n \in E_n$ and $\varphi_n \to \varphi$.

Now define $S : \Pi_{\mathcal{U}}(E_n) \to Y_{\mathcal{U}}^{\mathbb{N}}$ by

$$S\varphi_n = (T_n\varphi_n).$$

Then $SJ : L_s \to Y_{\mathcal{U}}^{\mathbb{N}}$ and $\|SJ\| \le 2MC$. We observe that

$$\|SJ\chi_{(0,1)}\| = 1,$$

and that if $\varphi \in L_\infty(0,1)$ then

$$\|SJ\varphi\psi\| \le M\|\varphi\|_\infty \|SJ\psi\|.$$

Indeed suppose $\psi_n \in E_n$, $\psi_n \to \psi$ in L_s and $\varphi_n \in E_n$, $\varphi_n \to \varphi$ in L_s, and $\|\varphi_n\|_\infty \le \|\varphi\|_\infty$. Then

$$\|T_n(\varphi_n\psi_n)\| \le M\|\varphi\|_\infty \|T_n\psi_n\|$$

and the result follows.

It now follows that there exists a Borel subset B of $(0,1)$ of maximal measure such that $SJ\chi_B = 0$; let $A = (0,1)\setminus B$. Since $SJ \neq 0$, we have $\lambda(A) > 0$. Suppose $\psi_n \in L_s(A)$ and $SJ\psi_n \to 0$; we shall show that $\psi_n \to 0$ in $L_0(A)$. We may suppose $\|SJ\psi_n\| \le 2^{-n}$.

For $\varepsilon > 0$, let

$$A_n^\varepsilon = \{\xi : |\psi_n(\xi)| \geq \varepsilon\} .$$

Then

$$\|SJ\chi_{A_n^\varepsilon}\| \leq \frac{M}{\varepsilon} 2^{-n}$$

and hence

$$\left\|SJ\left(\sum_{m+1}^{m+\ell} \chi_{A_n^\varepsilon}\right)\right\|^r \leq \left(\frac{M}{\varepsilon}\right)^r \left(\sum_{m+1}^{m+\ell} 2^{-nr}\right)$$

i.e.

$$\left\|SJ\left(\sum_{m+1}^{m+\ell} \chi_{A_n^\varepsilon}\right)\right\| \leq \frac{M}{\varepsilon}(1 - 2^{-r})^{-1/r}\, 2^{-(m+1)} .$$

Hence

$$\left\|SJ\left(\chi\left(\bigcup_{m+1}^{m+\ell} A_n^\varepsilon\right)\right)\right\| \leq \frac{M^2}{\varepsilon}(1 - 2^{-r})^{-1/r}\, 2^{-(m+1)}$$

Letting $\ell \to \infty$

$$\left\|SJ\chi\left(\bigcup_{m+1}^{\infty} A_n^\varepsilon\right)\right\| \leq \frac{M^2}{\varepsilon}(1 - 2^{-r})^{-1/r}\, 2^{-(m+1)} .$$

Hence if

$$\limsup A_n^\varepsilon = \bigcap_{m=1}^{\infty} \bigcup_{m+1}^{\infty} A_n^\varepsilon ,$$

$$\|SJ\chi(\limsup A_n^\varepsilon)\| \leq \frac{M^3}{\varepsilon}(1 - 2^{-r})^{-1/r}\, 2^{-(m+1)}$$

for all m. Hence as $A_n^\varepsilon \subset A$, we have

$$SJ(\chi(B \cup \limsup A_n^\varepsilon)) = 0$$

and

$$\lambda(\limsup A_m^\varepsilon) = 0$$

i.e.

$$\lambda(A_n^\varepsilon) \to 0 .$$

We conclude that if Z is any subspace of $L_s(A)$ on which the L_s-topology and the L_0-topology coincide, then $SJ|Z$ is an isomorphism. In particular we conclude that Y_U has a subspace isomorphic to ℓ_2. From Corollary 3.7, we conclude that ℓ_2 is finitely representable in X, and it follows that $p(X) \leq 2$.

If $s < 2$ then Y_U has a subspace isomorphic to ℓ_q for any $s < q \leq 2$, and ℓ_q is finitely representable in X for $s < q \leq 2$. As $s > p$ is arbitrary we conclude that ℓ_q is finitely representable in X for $p < q \leq 2$. Clearly this implies also that ℓ_p is finitely representable in X.

In order to state our results for general quasi-normed spaces, let us say that ℓ_p is <u>weakly finitely representable</u> in a quasi-normed space X if given $n \in \mathbb{N}$ and $\delta > 0$ there exists a subspace L of X with $\dim L = n$ and a linear isomorphism $T: \ell_p^{(n)} \to L$ such that if B is the unit ball of $\ell_p^{(n)}$ and U is the unit ball of X,

$$L_n \cap U \subset T(B) \subset L_n \cap (U + \delta U).$$

We can now apply Lemma 2.1 to deduce

Theorem 4.7 *Suppose X is an infinite-dimensional quasi-Banach space. Then for $0 < q \leq 2$, ℓ_q is weakly finitely representable in X if and only if $q \geq p$, where p is the convexity type of X.*

Corollary 4.8 *(Dvoretzky's theorem for locally bounded spaces) ℓ_2 is weakly finitely representable in any infinite-dimensional quasi-Banach space.*

Corollary 4.9 *ℓ_1 is weakly finitely representable in any non-locally convex quasi-Banach space.*

Corollary 4.10 *If $0 < q < 1$, then a quasi-Banach space X is r-convex for some $r > q$ if and only if ℓ_q is not weakly finitely representable in X.*

Proof. X is r-convex for some $r > q$ if and only if $a_n = O(n^{\frac{1}{q}})$ (see [3]), or equivalently

$$\lim_{n \to \infty} \frac{\log a_n}{\log n} < \frac{1}{q}.$$

Now apply Proposition 4.1.

A quasi-Banach space X is called <u>strictly</u> p-<u>convex</u> if it is p-convex and not q-convex for any $q > p$. The <u>galb</u> $G(X)$ of X is the space of sequences (a_n) such that $\|x_n\| \leq 1$ implies that $\left(\sum_{n=1}^{N} a_n x_n\right)_{N=1}^{\infty}$ is bounded. The galb $G(X)$ is a quasi-Banach sequence space when quasi-normed by

$$\|(a_n)\| = \sup_{\|x_n\|\le 1} \sup_N \|\sum_{n=1}^{N} a_n x_n\| .$$

The following theorem answers a question of Turpin [8] p.77.

Theorem 4.12 A quasi-Banach space X is strictly p-convex if and only if $G(X) = \ell_p$ $(0 < p < 1)$.

Proof. If $G(X) = \ell_p$, then X is strictly p-convex ([8] p.57). Conversely if X is strictly p-convex, then ℓ_p is weakly finitely representable in X. This implies that for each n, there exist $e_1,\ldots,e_n \in X$ such that if $\sum_{i=1}^{n} |t_i|^p = 1$

$$1 \le \|\Sigma\, t_i\, e_i\| \le 2 .$$

Hence if $(a_n) \in G(X)$

$$\|\sum_{i=1}^{n} a_i\, e_i\| \ge (\Sigma |a_i|^p)^{1/p}$$

and hence

$$\|(a_i)\| \ge \frac{1}{2}\left(\sum_{i=1}^{n} |a_i|^p\right)^{1/p}$$

It follows that $\Sigma |a_n|^p < \infty$ i.e. $G(X) \subset \ell_p$. However $G(X) \supset \ell_p$ since X is p-convex.

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