CHAPTER 26

Interpolation of Banach Spaces

Nigel Kalton* and Stephen Montgomery-Smith†

Department of Mathematics, University of Missouri, Columbia, MO 65211, USA
E-mail: nigel@math.missouri.edu; stephen@math.missouri.edu

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1. Introduction

There are several excellent books now available treating the general theory of interpolation from various points of view (see, for example, Bergh and Löfström [8], Bennett and Sharpley [5], Krein, Petunin and Semenov [69] and Brudnyi and Kruglyak [12]). The aim of this chapter is to consider the interaction between interpolation theory and the geometry of Banach spaces, and so we will not treat many topics that can be found elsewhere.

Historically, interpolation theory as an abstract concept was developed by Lions, Peetre and Calderón in the 1960s. The first real application in Banach space theory seems to be the celebrated Davis–Figiel–Johnson–Pełczyński factorization theorem [35] from 1974, although at the time the language of interpolation was not used; we can now see in retrospect that this result belongs to interpolation theory. This result will be discussed below (Theorem 3.4). In the 1980's Pisier played a pioneering role in bringing interpolation techniques into the mainstream of Banach space theory. Interpolation played a role (at least implicitly) in the development of Pisier's work on the Grothendieck program [93] and in the local theory of Banach spaces [94]. More recently Pisier, Kislyakov and Xu have studied interpolation of Hardy spaces and non-commutative analogues: we refer to [65,67,68,97,95] and [96]. Some of these ideas are covered in [66]. See also [58].

In this article we will treat some rather different topics. We will concentrate on the real (θ, p)-method and the complex method. We first introduce these and discuss the Davis–Figiel–Johnson–Pełczyński factorization theorem. This leads us naturally to consider the general problem of interpolation of Banach space properties and properties of operators by these methods. In this area Theorem 5.2 is very useful; it gives a general construction to give a counterexample to many possible conjectures. We also draw attention to the Cwikel problem: is compactness of an operator preserved by complex interpolation? This problem, we believe is quite challenging for Banach space theorists.

Next we discuss Calderón couples. The characterization of pairs of r.i. space which form Calderón couples curiously involves conditions (shift properties) which have natural meaning in the context of Banach space theory.

We then devote much of the remainder of the article to developing differential methods in interpolation theory. This theory was initiated by Rochberg and Weiss in 1983 [101] and is very relevant to the construction of twisted sums or extensions of Banach spaces (see also [60]). We develop this theory specializing to the case of interpolation of Banach sequence spaces and relate it to the theory of entropy functions [42,55] and [87]. We discuss applications in harmonic analysis and in operator theory.

2. The basics

In this section, we will introduce the basic ideas of interpolation. First we define the notion of a *Banach couple*. Let W be a Hausdorff topological vector space (the *ambient space*). Suppose X_0 and X_1 are two Banach spaces which are continuously embedded into W. We refer to $\overline{X} = (X_0, X_1)$ as a Banach couple. We can then define the sum and intersection spaces. The sum space $\Sigma(\overline{X}) = X_0 + X_1$ is equipped with the norm:

$$\|x\|_{\Sigma\overline{X}} = \inf\{\|x_0\|_{X_0} + \|x_1\|_{X_1}\colon x = x_0 + x_1,\ x_0 \in X_0,\ x_1 \in X_1\}.$$

The intersection space $\Delta(\overline{X}) = X_0 \cap X_1$ is a Banach space under the norm:

$$\|x\|_{\Delta(\overline{X})} = \max\{\|x\|_{X_0}, \|x\|_{X_1}\}.$$

Once we have defined these spaces, note that it is always possible to replace the ambient space W by the sum space $\Sigma(\overline{X})$.

If $\overline{X}$ and $\overline{Y}$ are two Banach couples, then a linear operator $T:\overline{X} \to \overline{Y}$ (we write $T \in \mathcal{L}(\overline{X}, \overline{Y})$) is a bounded linear map $T:\Sigma(\overline{X}) \to \Sigma(\overline{Y})$ such that $T(X_0) \subset Y_0$ and $T(X_1) \subset Y_1$. From our assumptions, it then follows that T is bounded from X_j to Y_j for $j = 1, 2$. We define

$$\|T\|_{\overline{X}\to\overline{Y}} = \max\{\|T\|_{X_0\to Y_0}, \|T\|_{X_1\to Y_1}\}.$$

If $\overline{X} = \overline{Y}$ we simply write $\|T\|_{\overline{X}}$.

For any Banach couple $\overline{X}$, an *intermediate space* Z is a Banach space such that $\Delta(\overline{X}) \subset Z \subset \Sigma(\overline{X})$. Z is then called an *interpolation* space if for every $T \in \mathcal{L}(\overline{X})$ we have $T(Z) \subset Z$. We then have from the Closed Graph Theorem that $\|T\|_Z \leqslant C\|T\|_{\overline{X}}$ for some C. If $C = 1$ we say that Z is an exact interpolation space. Notice that $\Delta(\overline{X})$ and $\Sigma(\overline{X})$ immediately give two exact interpolation spaces.

A motivating example for this set-up, and indeed the original Banach couple, is obtained by taking an arbitrary σ-finite measure space (Ω, μ), and letting $X_0 = L_\infty(\mu)$, and $X_1 = L_1(\mu)$. The ambient space can be either the space L_0 of all measurable functions (with the topology of convergence in measure on subsets of finite measure) or $L_\infty + L_1$.

It is sometimes convenient to impose mild extra conditions on a Banach couple.

The simplest such requirement is that the intersection space $\Delta(\overline{X})$ is dense in both X_0 and X_1 in their respective topologies. This allows to form a dual Banach couple. If we take $\Delta(\overline{X})^*$ as an ambient space then X_0^*, X_1^* can be considered as continuously embedded into $\Delta(\overline{X})^*$. Thus $(\overline{X})^* = (X_0^*, X_1^*)$ is also a Banach couple with $\Sigma(\overline{X}^*) = \Delta(\overline{X})^*$ and $\Delta(\overline{X}^*) = (\Sigma\overline{X})^*$ in a natural way. Note that it does not then necessarily follow that the dual couple $\overline{X}^*$ has a dense intersection space, although this does follow if, for example, X_0 and X_1 are reflexive Banach spaces.

A second natural assumption is that of *Gagliardo completeness*. This requires that the sets $B_{X_j} = \{x\colon \|x\|_{X_j} \leqslant 1\}$ are closed in the topology of $\Sigma(\overline{X})$ for $j = 1, 2$. The weaker assumption that the closure of each B_{X_j} is contained in X_j allows us from the Closed Graph Theorem to obtain Gagliardo completeness for certain equivalent norms on each X_j. An easier example of a non-Gagliardo complete pair is the pair $(C[0,1], L_1[0,1])$ where it is easily seen that the closure of B_{X_0} in the topology of the sum space is the ball of $L_\infty[0,1]$.

Interpolation theory has its origins in the classical Riesz–Thorin and Marcinkiewicz theorems. Both these theorems lead to the idea of an *interpolation method*. This is a functor $\mathbf{F}$ which assigns to any Banach couple $\overline{X}$ an interpolation space $\mathbf{F}(\overline{X})$ in such a way that if $\overline{X}, \overline{Y}$ are two interpolation couples and $T \in \mathcal{L}(\overline{X}, \overline{Y})$ then T maps $\mathbf{F}(\overline{X})$ to $\mathbf{F}(\overline{Y})$ boundedly. The Riesz–Thorin theorem abstracts to the *complex methods* and the Marcinkiewicz theorem is abstracted in the *real methods*.

One may isolate two basic types of questions implicit in interpolation theory. One can study a particular method in order to develop applications. In this case one needs to take specific practical examples of Banach couples and then calculate the effect of the corresponding method. The second type of question is to identify all interpolation spaces for a given couple.

In Banach space theory, interpolation has played a pivotal role in several areas (particularly in the local theory and in problems related to Grothendieck's theorem). It is probably fair to say that in Banach space theory one is mainly interested in knowing about the preservation of properties of spaces or operators under interpolation.

3. The K-functional and the (θ, p)-methods

The fundamental notion of real interpolation theory is the K-functional. Suppose $\overline{X}$ is a Banach couple. We define the K-functional by

$$K(x,t) = K\big(x,t;\overline{X}\big) = \inf\big\{\|x_0\|_{X_0} + t\|x_1\|_{X_1}\colon x = x_0 + x_1\big\}, \quad x \in \Sigma(X).$$

Thus $K(x,1)$ is simply the usual norm of $\Sigma(\overline{X})$ and each $x \to K(x,t)$ gives an equivalent norm on the sum space for which it becomes an exact interpolation space. It is easy to check that for fixed x the function $t \to K(x,t)$ is increasing and concave for $0 < t < \infty$. If $x \in \Delta(X)$ one has an estimate

$$K(x,t) \leqslant \min\big\{\|x\|_{X_0}, t\|x\|_{X_1}\big\} \leqslant \|x\|_{\Delta}\min(1,t).$$

There is a dual construct known as the J-functional:

$$J(x,t) = J\big(x,t,\overline{X}\big) = \max\big\{\|x\|_{X_0}, t\|x\|_{X_1}\big\}, \quad x \in \Delta\big(\overline{X}\big).$$

These form a family of exact interpolation norms on $\Delta(\overline{X})$.

If $0 < \theta < 1$ and $1 \leqslant p < \infty$ we define the real interpolation spaces $X_{\theta,p} = (X_0, X_1)_{\theta,p}$ by $x \in X_{\theta,p}$ if and only if

$$\|x\|_{\theta,p} = \left(\int_0^\infty t^{-\theta p} K(x,t)^p \frac{\mathrm{d}t}{t}\right)^{1/p} < \infty. \tag{3.1}$$

If $p = \infty$ we define $X_{\theta,\infty}$ as the space of x such that

$$\|x\|_{\theta,\infty} = \sup_{t>0} t^{-\theta} K(x,t) < \infty.$$

It is easily seen that both these definitions can be given in discrete form, e.g.,

$$\|x\|_{\theta,p} \approx \left(\sum_{n\in\mathbb{Z}} 2^{-\theta p n} K\big(2^n,t\big)^p\right)^{1/p}. \tag{3.2}$$

The functor which takes the couple $\overline{X}$ to $X_{\theta,p}$ is the (θ, p)-method; this clearly provides an example of an exact interpolation method. The theory of this method is well-developed and understood and we can refer to [5] and [8] for a full discussion of such topics as reiteration and duality. For our purposes it is useful to point out an equivalent definition in terms of the J-functional, first obtained in the fundamental paper of Lions and Peetre [76]. Consider the case $1 \leqslant p < \infty$. Define for $x \in X_0 + X_1$,

$$\|x\|'_{\theta,p} = \inf\left\{\left(\sum_{k\in\mathbb{Z}} \max\{\|x_k\|_0, 2^k\|x_k\|_1\}^p\right)^{1/p} : x = \sum_{k\in\mathbb{Z}} 2^{\theta k} x_k\right\}, \tag{3.3}$$

where the series converges in $X_0 + X_1$. Then $x \in X_{\theta,p}$ if and only if $\|x\|'_{\theta,p} < \infty$ and the norms $\|x\|'_{\theta,p}$ and $\|x\|_{\theta,p}$ are equivalent. In (3.3) we have formulated the J-method discretely; it is more usual to use a continuous version. The equivalence of the J-method and the K-method of definition can be obtained from the Fundamental Lemma, which we discuss later (Theorem 6.1). Note that we must have that $\Delta(\overline{X})$ is dense in the spaces $X_{\theta,p}$ provided $1 \leqslant p < \infty$. Using this, one can show a duality theorem [74]:

THEOREM 3.1. *Suppose $\Delta(\overline{X})$ is dense in both X_0 and X_1. Then if $1 \leqslant p < \infty$ and $0 < \theta < 1$ the dual of $(X_0, X_1)_{\theta,p}$ can be identified naturally with $(X_0^*, X_1^*)_{\theta,q}$ where $1/p + 1/q = 1$.*

The (θ, p)-methods have proved extremely useful in many branches of analysis including Banach spaces. We conclude this section by discussing the first major application of interpolation in Banach spaces theory, the Davis–Figiel–Johnson–Pełczyński factorization theorem [35]. The basic idea of this theorem is to establish conditions under which certain interpolation spaces are reflexive, although in the initial paper the language of interpolation was not used. Later Beauzamy [3] established the general result. Consider the spaces $Z_n = (\Delta(\overline{X}), J_{2^n})$ where $J_t(x) = J(x,t)$ is a norm on $\Delta(\overline{X})$. Now we can use (3.3) to define a quotient mapping $Q : \ell_p(Z_n)_{n\in\mathbb{Z}} \to X_{\theta,p}$ by

$$Q\big((a_k)_{k\in\mathbb{Z}}\big) = \sum_{k\in\mathbb{Z}} 2^{\theta k} a_k.$$

The following lemma is an easy gliding hump argument:

LEMMA 3.2. *Suppose $1 < p < \infty$ and $0 < \theta < 1$. Suppose $\mathbf{a}_n = (a_{nk})_{k\in\mathbb{Z}}$ is a bounded sequence in $\ell_p(Z_n)$ such that for each k we have $\lim_{n\to\infty} a_{nk} = 0$ weakly in $X_0 + X_1$. Then $Q\mathbf{a}_n$ converges to zero weakly in $X_{\theta,p}$.*

PROOF. It is enough to construct a sequence of convex combinations of $(Q(\mathbf{a}_k))_{k\geqslant n}$ which is weakly null. First by Mazur's theorem we can take convex combinations and assume $\lim_{n\to\infty} \|a_{nk}\|_{X_0+X_1} = 0$ for each k. It follows quickly that $\lim_{n\to\infty} \|a_{nk}\|_{\theta,p} = 0$ for each k. Indeed we can split $a_{nk} = b_{nk} + c_{nk}$ where b_{nk} is bounded in X_0 and converges to zero in X_1 while c_{nk} is bounded in X_1 and converges to zero in X_0. Then we use the estimate

$$\|a_{nk}\|_{\theta,p} \leqslant C\big(\|b_{nk}\|_{X_0}^{1-\theta}\|b_{nk}\|_{X_1}^{\theta} + \|c_{nk}\|_{X_0}^{1-\theta}\|c_{nk}\|_{X_1}^{\theta}\big).$$

It follows that we can find a sequence $N_n \to \infty$ so that

$$\Big\| Q(\mathbf{a}_n) - \sum_{|k|\geqslant N_n} 2^{\theta k} a_{nk}\Big\|_{\theta,p} \to 0.$$

Let $b_{nk} = a_{nk}$ if $|k| \geqslant N_n$ and 0 otherwise. Standard gliding hump arguments show that $\mathbf{b}_n$ is weakly null. It is then easy to conclude that $\mathbf{a}_n$ is also weakly null. □

An immediate consequence due to Beauzamy [3] is that:

THEOREM 3.3. *Suppose $1 < p < \infty$ and $0 < \theta < 1$. Then $(X_0, X_1)_{\theta,p}$ is reflexive if and only if $B_{\Delta(\overline{X})}$ is relatively weakly compact in $\Sigma(\overline{X})$.*

This follows from the preceding lemma and the Eberlein–Smulian theorem. Now the Factorization Theorem of Davis, Figiel, Johnson and Pełczyński is given by:

THEOREM 3.4 ([35]). *Suppose X and Y are Banach spaces and $T: X \to Y$ is weakly compact. Then there is a reflexive space R and a factorization of $T = BA$ where $A: X \to R$ and $B: R \to Y$ are bounded.*

PROOF. We use the following typical trick. Let K be the closure of $T(B_X)$ and let Y_0 be the Banach space generated by taking K as it's unit ball. Let $Y_1 = Y$ and then take $R = Y_{\theta,p}$ for some choice of $0 < \theta < 1$ and $1 < p < \infty$. For A we treat T as an operator into R and for B we take the inclusion of R into Y. □

There is a sense in which the (θ, p)-methods give rise to Banach spaces with relatively simple structure. This is the content of a theorem of Levy [71].

THEOREM 3.5. *Suppose $\Delta(\overline{X})$ is not closed in $\Sigma(\overline{X})$. Then for $0 < \theta < 1$, $1 \leqslant p < \infty$ the spaces $(X_0, X_1)_{\theta,p}$ contain a complemented copy of ℓ_p.*

In order to prove this theorem, we first prove a preliminary lemma:

LEMMA 3.6. *Suppose $\Delta(\overline{X})$ is not closed in $\Sigma(\overline{X})$. Then either*

(1) *For every $t > s > 0$ and $\varepsilon > 0$ there exists $x \in \Delta(X)$ with*

$$\tau^{-1}K(\tau, x) \geqslant (1-\varepsilon)s^{-1}K(s, x), \quad s < \tau < t,$$

or

(2) *For every* $t > s > 0$ *and* $\varepsilon > 0$ *there exists* $x \in \Delta(X)$ *with*

$$K(\tau, x) \geqslant (1-\varepsilon)K(t, x), \quad s < \tau < t.$$

PROOF. Note first that $t^{-1}K(t,x)$ is decreasing and $K(t,x)$ is increasing. Assume (1) fails. Then there exists $\varepsilon > 0$ and $t > s > 0$ such that $t^{-1}K(t,x) < (1-\varepsilon)s^{-1}K(s,x)$. Then for any $x \in \Delta(\overline{X})$ we have $K(t,x) \leqslant (1-\varepsilon)t\|x\|_{X_1}$. Thus, putting $\delta = 1 - \varepsilon/2$ we can find $u_1 \in X_0$, $v_1 \in X_1$ so that $x = u_1 + v_1$ and $\|u_1\|_{X_0} \leqslant t\|x\|_{X_1}$ and $\|v_1\|_{X_1} \leqslant \delta\|x\|_{X_1}$. Now we can iterate the argument as in the Open Mapping Theorem and write $v_1 = u_2 + v_2$ where $\|u_2\|_{X_0} \leqslant \delta t\|x\|_{X_1}$ and $\|v_2\|_{X_1} \leqslant \delta^2 t\|x\|_{X_1}$. Continuing in this way we construct $(u_n)_{n=1}^{\infty}$ in X_0 and $(v_n)_{n=1}^{\infty}$ in X_1 such that

$$\|u_n\|_{X_0} \leqslant t\delta^{n-1}\|x\|_{X_1}, \qquad \|v_n\|_{X_1} \leqslant t\delta^n\|x\|_{X_1}$$

and

$$x = u_1 + \cdots + u_n + v_n.$$

Clearly $\sum_{n=1}^{\infty} u_n$ converges in X_0 and its sum must be x (by computing in $X_0 + X_1$). Hence $\|x\|_{X_0} \leqslant t(1-\delta)^{-1}\|x\|_{X_1}$. Similarly the failure of (1) implies that $\|x\|_{X_1} \leqslant C\|x\|_{X_0}$ whenever $x \in \Delta(\overline{X})$ for a suitable constant C. Thus if both (1) and (2) fail then the two norms $\|\cdot\|_{X_0}$ and $\|\cdot\|_{X_1}$ are equivalent on the intersection, and this implies the intersection is closed in $X_0 + X_1$. □

We now turn to the proof of Theorem 3.5:

PROOF. Note that (3.2) the space $X_{\theta,p}$ can be regarded as a subspace of the ℓ_p-sum of the space $X_0 + X_1$ with the norms $2^{-\theta n}K(2^n, x)$ for $n \in \mathbb{Z}$. To show it has a complemented subspace isomorphic to ℓ_p requires only the existence of a normalized sequence $(x_m)_{m=1}^{\infty}$ in $X_{\theta,p}$ so that for each n we have $\lim_{m\to\infty} K(2^n, x_m) = 0$ (i.e., x_m converges to 0 in $X_0 + X_1$). This follows by standard gliding hump techniques.

If such a sequence does not exist then there is a constant C so that

$$\left(\sum_{n\in\mathbb{Z}} 2^{-n\theta p}K\left(2^n, x\right)^p\right)^{1/p} \leqslant CK(1, x).$$

That this is impossible follows from the preceding lemma. □

4. The complex method

We first define the complex method for a Banach couple $\overline{X}$ which we now assume consists of complex Banach spaces. We introduce a Banach space $\mathcal{F}$ of analytic functions as follows. Let $\mathcal{S} = \{z\colon 0 < \Re z < 1\}$ and let $\mathcal{F}$ be the space of analytic functions $F : \mathcal{S} \to \Sigma(\overline{X})$

such that F extends continuously to the closure $\overline{\mathcal{S}}$ and the functions $t \to F(j+\mathrm{i}t)$ are continuous and bounded in X_j for $j=0,1$. We norm $\mathcal{F}$ by

$$\|F\|_{\mathcal{F}} = \max_{j=0,1} \sup_{-\infty<t<\infty} \left\|F(j+\mathrm{i}t)\right\|_{X_j}.$$

We then define the interpolation spaces $X_\theta = [X_0, X_1]_\theta$ by $x \in X_\theta$ if and only if there exists $F \in \mathcal{F}$ with $F(\theta)=x$ and then we set

$$\|x\|_{X_\theta} = \inf\left\{\|F\|_{\mathcal{F}}\colon\ F(\theta)=x\right\}.$$

This method is known as Calderón's first method or lower method, and is usually called simply the complex method. It was introduced independently by Lions [73] and Calderón [14]; most of the basic theory was developed by Calderón in [14].

Calderón's second method or upper method ([14]) is described similarly but taking a different family $\widetilde{\mathcal{F}}$ of analytic functions. Suppose $F\colon \mathcal{S} \to \Sigma(\overline{X})$ is a bounded analytic function. Then any anti-derivative $F^\#$ is Lipschitz and extends continuously to $\overline{\mathcal{S}}$. We say $F \in \widetilde{\mathcal{F}}$ if $t \to F^\#(\mathrm{i}t)$ is Lipschitz into X_0 and $t \to F^\#(1+\mathrm{i}t)$ is Lipschitz into X_1. We then put

$$\|F\|_{\widetilde{\mathcal{F}}} = \max_{j=1,2} \left\{ \sup_{s<t\in\mathbb{R}} \frac{\|F^\#(\mathrm{i}t) - F^\#(\mathrm{i}s)\|_{X_0}}{|t-s|}, \right.$$
$$\left. \sup_{s<t\in\mathbb{R}} \frac{\|F^\#(1+\mathrm{i}t) - F^\#(1+\mathrm{i}s)\|_{X_1}}{|t-s|} \right\}.$$

Then we define the spaces $X_{[\theta]} = [X_0, X_1]^\theta$ by $x \in X_{[\theta]}$ if and only if there exists $F \in \widetilde{\mathcal{F}}$ with $F(\theta)=x$ and we define

$$\|x\|_{X_{[\theta]}} = \inf\left\{\|F\|_{\widetilde{\mathcal{F}}}\colon\ F(\theta)=x\right\}.$$

It is clear that the complex interpolation spaces X_θ and $X_{[\theta]}$ are further examples of exact interpolation spaces. It is also clear that in general $X_\theta \subset X_{[\theta]}$ and the injection is of norm one. The fundamental difference between the upper and lower methods is that $\Delta(\overline{X})$ is always dense in X_θ, but not necessarily in $X_{[\theta]}$. In fact it is shown in [7] that X_θ is simply the closure of $\Delta(\overline{X})$ in $X_{[\theta]}$.

Let us discuss conditions under which the two methods coincide. We will need the Poisson kernel for the strip. These are maps $P\colon \partial\mathcal{S} \times \mathcal{S} \to (0,\infty)$ such that if u is harmonic and bounded on $\mathcal{S}$ and extends continuously to $\overline{\mathcal{S}}$ then

$$\begin{aligned} u(z) &= \int_{\partial\mathcal{S}} P(w,z)u(w)\,|\mathrm{d}w| \\ &= \int_{-\infty}^{\infty} P(\mathrm{i}t,z)u(\mathrm{i}t)\,\mathrm{d}t + \int_{-\infty}^{\infty} P(1+\mathrm{i}t,z)u(1+\mathrm{i}t)\,\mathrm{d}t. \end{aligned}$$

LEMMA 4.1. *Suppose $F \in \mathcal{F}$. Then*

$$\|F(\theta)\|_{X_\theta} \leqslant \exp\left(\int_{\partial\mathcal{S}} P(w,\theta)\log\|F(w)\|_{X_{\Re w}}|\mathrm{d}w|\right).$$

This lemma is proved very simply using the existence of appropriate outer functions (it is perhaps most easily seen by noting that the strip is conformally equivalent to the unit disk).

Based on this we can quickly see the connection between the two methods described above:

PROPOSITION 4.2. *Suppose $F \in \widetilde{\mathcal{F}}$. If $F^{\#}(j+\mathrm{i}t)$ is differentiable in X_j on a set of positive measure for either $j=0$ or $j=1$ then $F(\theta) \in X_\theta$ for $0<\theta<1$ and $\|x\|_{X_\theta} \leqslant \|F\|_{\widetilde{\mathcal{F}}}$.*

The following corollary was proved first with the hypothesis that one space is reflexive in [14]; see [91].

COROLLARY 4.3. *If either X_0 or X_1 has the Radon–Nikodym property then the spaces $[X_0, X_1]_\theta$ and $[X_0, X_1]_{[\theta]}$ coincide isometrically.*

PROOF OF PROPOSITION 4.2. For each $h>0$ let $F_h(z) = h^{-1}(F^{\#}(z+\mathrm{i}h) - F^{\#}(z))$. Then $F_h \in \mathcal{F}$ and $\|F_h\|_{\mathcal{F}} \leqslant \|F\|_{\widetilde{\mathcal{F}}}$. For fixed θ we have that $F_h(\theta) \to F(\theta)$ in $\Sigma(\overline{X})$. Assume $F^{\#}(\mathrm{i}t)$ is differentiable on a set E of positive measure. Then $F_h(\mathrm{i}t)$ converges in X_0 a.e. on E. Note that

$$\|F_{h_1}(\theta) - F_{h_2}(\theta)\|_{X_\theta} \leqslant \exp\left(\int_{\partial\mathcal{S}} P(w,\theta)\log\|F_{h_1}(w) - F_{h_2}(w)\|_{X_{\Re w}}|\mathrm{d}w|\right)$$

and this inequality implies that $\|F_{h_1}(\theta) - F_{h_2}(\theta)\|_{X_\theta}$ converges to zero as $h_1, h_2 \to 0$, i.e., that $\lim_{h\to 0} F_h(\theta)$ exists in X_θ. The proposition follows. □

The following is the duality theorem for complex interpolation due to Calderón [14].

THEOREM 4.4. *Suppose $\overline{X}$ is a Banach couple such that $\Delta(\overline{X})$ is dense in both X_0 and X_1. Then the dual space of $[X_0, X_1]_\theta$ can be identified isometrically with $[X_0^*, X_1^*]^\theta$. In particular if one of the spaces X_0 or X_1 is an Asplund space (i.e., either X_0^* or X_1^* has the Radon–Nikodym property) then the dual of $[X_0, X_1]_\theta$ can be identified with $[X_0^*, X_1^*]_\theta$.*

There is a sense in which real interpolation scales can be regarded as special cases of complex scales. Let us quote the Re-iteration theorem (see [14,75,64,22]):

THEOREM 4.5. *Suppose $\overline{X}$ is a complex Banach couple. Then:*

(1) *If $0 \leqslant \phi_1 < \phi_2 \leqslant 1$ then $[X_{\phi_1}, X_{\phi_2}]_\theta$ coincides isometrically with $[X_0, X_1]_{(1-\theta)\phi_1+\theta\phi_2}$.*
(2) *If $1 \leqslant p < \infty$ and $0 < \phi_1 < \phi_2 < 1$ the space $[X_{\phi_1,p}, X_{\phi_2,p}]_\theta$ coincides with (up to equivalence of norm) $X_{\phi_1(1-\theta)+\phi_2\theta,p}$.*

See also [33] for some endpoint results and [30] for extensions to the quasi-Banach setting.

One of the drawbacks of the complex method is that in general it seems relatively difficult to calculate complex interpolation spaces. There is one exception to this rule, which is the case when one has a pair of Banach lattices. The following theorem follows from the work of Calderón [14].

THEOREM 4.6. *Suppose that $\overline{X} = [X_0, X_1]$ is a couple of Banach function spaces on some measure space (K, μ). Assume that either X_0 or X_1 has the Radon–Nikodym property. Then the space $[X_0, X_1]_\theta$ is isometric to the space $X_0^{1-\theta} X_1^\theta$, where*

$$\|f\|_{X_0^{1-\theta} X_1^\theta} = \inf\bigl\{\|g\|_{X_0}^{1-\theta} \|h\|_{X_1}^\theta \colon g \in X_0,\ h \in X_1,\ |f| = |g|^{1-\theta}|h|^\theta\bigr\}.$$

Let us note that this theorem can be applied when one can has a Banach couple $\overline{X}$ where X_0 and X_1 have a simultaneous unconditional basis; in particular it can be to study many types of function spaces (Besov spaces, Hardy spaces, Triebel–Lizorkin spaces, etc.) where one can find such a basis using wavelets. Essentially this approach was used by Frazier and Jawerth [40] (using instead the essentially equivalent idea of the ϕ-transform). It seems however to be a general rule that the only cases where complex interpolation spaces are calculable are those when Theorem 4.6 can be used. For example, the interpolation of Schatten ideals by the complex method is possible only because it can be reduced to the interpolation of symmetric sequence spaces.

5. Properties preserved by interpolation

There is vast literature on preservation of properties under interpolation. We consider properties of the underlying Banach spaces or of operators. Let us first discuss the underlying spaces. Suppose $\mathcal{P}$ is a property of Banach spaces: we ask for conditions so that if one or both of the spaces X_0, X_1 has the property $\mathcal{P}$ then the intermediate spaces $(X_0, X_1)_{\theta,p}$ or $[X_0, X_1]_\theta$ obtained by real or complex interpolation inherit the property. It is also possible to discuss other methods of interpolation of course.

Let us give an example. If X_0 or X_1 is reflexive so are the spaces $(X_0, X_1)_{\theta,p}$ and $[X_0, X_1]_\theta$ for $0 < \theta < 1$ and $1 < p < \infty$; the former is implied by Theorem 3.3 and the latter is due to Calderón [14].

In fact there is a very simple technique to see that certain types of properties interpolate for the real or complex methods.

PROPOSITION 5.1. *Let $\overline{X}$ be a Banach couple. Then:*

(1) *If $0 < \theta < 1$ and $1 < p < \infty$ then $(X_0, X_1)_{\theta,p}$ is isomorphic to a quotient of a subspace of the Banach space $\ell_p(X_0 \oplus X_1)$.*

(2) *If $0 < \theta < 1$ then $[X_0, X_1]_\theta$ is isomorphic to a subspace of a quotient of $L_p(X_0 \oplus X_1)$ for any choice of $1 \leqslant p < \infty$.*

Case (i) of Proposition 5.1 is essentially proved in Section 3. Case (ii) follows from some alternative formulations of the complex method. From this proposition one can see immediately that if, say, X_0 and X_1 have non-trivial type then so do $(X_0, X_1)_{\theta,p}$ and $[X_0, X_1]_\theta$ when $0 < \theta < 1$ and $1 < p < \infty$.

Suppose $\overline{X}$ is a Banach couple. We shall say that an interpolation space X is θ-exponential where $0 < \theta < 1$ if whenever $T \in \mathcal{L}(\overline{X}, \overline{X})$ then

$$\|T\|_{X \to X} \leqslant C\|T\|^{1-\theta}_{X_0 \to X_0}\|T\|^{\theta}_{X_1 \to X_1}.$$

The real interpolation spaces $(X_0, X_1)_{\theta,p}$ and the complex interpolation spaces $[X_0, X_1]_\theta$ are examples of θ-exponential interpolation spaces.

The following theorem is due to Garling and Montgomery-Smith [41]. It provides a strong converse to Proposition 5.1.

THEOREM 5.2. *Let Z be the quotient of a subspace of a separable Banach space Y. Then there exists a Banach couple $\overline{X}$ such that both the end-point spaces X_0 and Y_0 are isomorphic to $Y \oplus Y \oplus Y$, but such for every $0 < \theta < 1$ and every θ-exponential interpolation space X we have that X contains a complemented subspace isomorphic to Z.*

So, for example, for any separable Banach space Z, there is a Banach couple $\overline{X}$ so that the end point spaces are isomorphic to ℓ_1, but such that for every $0 < \theta < 1$ and every θ-exponential interpolation space X, we have that X contains a complemented subspace isomorphic to Z. Thus we see that many properties cannot be inherited by any interpolation method which is θ-exponential for some $0 < \theta < 1$. For example, the Radon–Nikodym property and non-trivial cotype can never be preserved by such a method.

Indeed, Dilworth and Sobecki [36] showed that any Banach space property that is passed from the end point spaces to the spaces created by the real or complex method must also pass from any Banach space to any subspace of any quotient of that space. They remark that the only property of ℓ_1 preserved under either the real or complex methods is separability.

Now suppose $\overline{X}$ and $\overline{Y}$ are two Banach couples and that $T \in \mathcal{L}(\overline{X}, \overline{Y})$. We now discuss properties of the operator T which can be interpolated. Let us first note that if $T : X_0 \to Y_0$ is weakly compact then $T : (X_0, X_1)_{\theta,p} \to (Y_0, Y_1)_{\theta,p}$ is weakly compact for any $0 < \theta < 1$ and $1 < p < \infty$. This is due to Aizenstein (see [12] for a full discussion of interpolation of weak compactness by general real methods). In fact, in this case it can be seen to follow from the corresponding result for the property of reflexivity and the Factorization Theorem 3.4. The same argument establishes a similar result for complex interpolation.

We shall now consider the question of interpolating compactness. If both $T : X_0 \to Y_0$ and $T : X_1 \to Y_1$ are compact then, as early as 1969, Hayakawa [45] showed that $T : (X_0, X_1)_{\theta,p} \to (Y_0, Y_1)_{\theta,p}$ is compact for $0 < \theta < 1$ and $1 \leqslant p < \infty$. The stronger one-sided result was proved, in full generality, only in 1992 by Cwikel [25]:

THEOREM 5.3. *Suppose $\overline{X}$ and $\overline{Y}$ are two Banach couples and that $T \in \mathcal{L}(\overline{X}, \overline{Y})$ is such that $T : X_0 \to Y_0$ is compact. Then $T : (X_0, X_1)_{\theta,p} \to (Y_0, Y_1)_{\theta,p}$ is compact for $0 < \theta < 1$ and $1 \leqslant p < \infty$.*

Curiously the same problem for the complex method is unsolved. It was first considered by Calderón [14] in 1964. Surprisingly, it is not even known under two-sided conditions.

PROBLEM 5.4. Suppose $\overline{X}$ and $\overline{Y}$ are Banach couples and $T \in \mathcal{L}(\overline{X}, \overline{Y})$ is such that $T : X_j \to X_j$ is compact for $j = 0, 1$. Is $T : [X_0, X_1]_\theta \to [X_0, X_1]_\theta$ compact for $0 < \theta < 1$?

This problem appears challenging for Banach space theorists, as the discussion in [29] shows. In [27] a partial result was given:

THEOREM 5.5. *Suppose $\overline{X}$ and $\overline{Y}$ are complex Banach couples and that $T \in \mathcal{L}(\overline{X}, \overline{Y})$ is such that $T : X_0 \to Y_0$ is compact. Suppose X_0 is a UMD-space. Then $T : [X_0, X_1]_\theta \to [Y_0, Y_1]_\theta$ is compact for $0 < \theta < 1$.*

Some other conditions on X_0 are also considered in [27]. For example, it suffices (in place of assuming X_0 is a UMD-space) that X_0 is itself an interpolation space $X_0 = [E, X_1]_\theta$ for some $0 < \theta < 1$ and some E. In this form the result can be regarded as an improvement (for complex interpolation) of the following earlier extrapolation result of [25]:

THEOREM 5.6. *Suppose $\overline{X}$ and $\overline{Y}$ are Banach couples and that $T \in \mathcal{L}(\overline{X}, \overline{Y})$. Suppose $1 \leqslant p < \infty$ and that for some $0 < \theta < 1$ we have $T : [X_0, X_1]_\theta \to [Y_0, Y_1]_\theta$ (respectively $T : (X_0, X_1)_{\theta,p} \to (Y_0, Y_1)_{\theta,p}$) is compact. Then $T : [X_0, X_1]_\theta \to [Y_0, Y_1]_\theta$ (respectively $T : (X_0, X_1)_{\theta,p} \to (Y_0, Y_1)_{\theta,p}$) is compact for every $0 < \theta < 1$.*

Extrapolation theorems of this type can also be proved for certain properties of Banach spaces. Let us fix our attention on complex interpolation. Suppose $X_\theta = [X_0, X_1]_\theta$. It is clear from Theorem 4.5, for example, that if X_θ is reflexive for some $0 < \theta < 1$ it is reflexive for every $0 < \theta < 1$. There is an abstract approach to these ideas via the notion of the Kadets distance (see [63]).

Let X and Y be two subspaces of a Banach space Z. We define $\Lambda(X, Y)$ to be the Hausdorff distance between B_X and B_Y, i.e.,

$$\Lambda(X, Y) = \max\Big\{\sup_{x \in B_X} \inf_{y \in B_Y} \|x - y\|, \sup_{y \in B_Y} \inf_{x \in B_X} \|x - y\|\Big\}.$$

Now suppose X and Y are any two Banach spaces. We define the Kadets distance $d_K(X, Y)$ to be the infimum of $\Lambda(\widetilde{X}, \widetilde{Y})$ over all spaces Z containing isometric copies of $\widetilde{X}, \widetilde{Y}$. It may be shown that d_K is a pseudo-metric on any set of Banach spaces; unfortunately there are non-isomorphic Banach spaces X, Y so that $d_K(X, Y) = 0$. The following theorem now has the content that the map $\theta \to [X_0, X_1]_\theta$ is continuous for the Kadets metric.

THEOREM 5.7. *Suppose $\overline{X}$ is a Banach couple and $X_\theta = [X_0, X_1]_\theta$ for $0 < \theta < 1$. Then if $0 < \theta < \phi < 1$,*

$$d_K(X_\theta, X_\phi) \leqslant 2\frac{\sin(\pi(\phi - \theta))/2}{\sin(\pi(\phi + \theta))/2}.$$

Theorem 5.7 allows us to prove extrapolation theorems by showing that certain properties of Banach spaces define sets which are either open and closed or merely open. If the property is an isomorphic invariant then one can use Theorem 4.5 to give also the corresponding extrapolation result for the real method.

Let us say a property $\mathcal{P}$ is *stable* if there exists $\alpha > 0$ so that if X has $\mathcal{P}$ and $d_K(X, Y) < \alpha$ then Y has $\mathcal{P}$. Stable properties define open and closed sets. Typical stable properties are separability, reflexivity, containing ℓ_1, super-reflexivity and having non-trivial type.

THEOREM 5.8. *Suppose $\overline{X}$ is a Banach couple and $X_\theta = [X_0, X_1]_\theta$ for $0 < \theta < 1$. Then if $\mathcal{P}$ is a stable property and X_θ has $\mathcal{P}$ for some $0 < \theta < 1$ then X_θ has $\mathcal{P}$ for every $0 < \theta < 1$.*

Some other properties are merely open, i.e., they define open sets. For example (see [63]), $X \approx c_0$ and $X \approx \ell_1$ are both open for the Kadets metric.

THEOREM 5.9. *Suppose $\overline{X}$ is a Banach couple and $X_\theta = [X_0, X_1]_\theta$ for $0 < \theta < 1$. Then if $\mathcal{P}$ is an open property and X_θ has $\mathcal{P}$ for some $0 < \theta < 1$ then there exists $\delta > 0$ so that X_ϕ has $\mathcal{P}$ for every $|\phi - \theta| < \delta$.*

There are many unresolved questions about the Kadets metric. For example, we can consider the set of separable Banach spaces under the Kadets metric. This pseudo-metric space is not connected (for example, the super-reflexive spaces form an open and closed subset). It is not difficult to show that the component of any Banach space X contains all isomorphic copies of X. One can ask to identify the component containing ℓ_2. It is not clear if this contains all super-reflexive spaces. An old extrapolation result of Pisier [92] implies it contains all super-reflexive Banach lattices. Another intriguing question is to identify the component of c_0. It is tempting to conjecture this consists of all spaces isomorphic to a subspace of c_0.

We refer to [63] for a fuller discussion of these ideas and of the relationship to the Gromov–Hausdorff distance.

6. Calderón couples

We now turn to the interpolation theory question of determining all intermediate spaces for a given couple. Let us first note that the construction of the (θ, p)-spaces in (3.1) and (3.3) can be generalized in an obvious way by replacing the a weighted L_p-space by an arbitrary Banach function space.

To make this precise we define our notion of a Banach function space over a σ-finite measure space (Ω, μ). We say that a Banach space $E, \|\cdot\|_E$ continuously embedded in the space $\mathcal{M}$ of all measurable functions (with the topology of convergence in measure on subsets of finite measure) is a Banach function space if whenever $g \in E$ and $|f| \leqslant |g|$ a.e. then $f \in E$ and $\|f\|_E \leqslant \|g\|_E$.

Let us suppose E is a Banach function space on the space $(0, \infty)$ (with Lebesgue measure) with the property that $\min(1, t) \in E$. Then given a Banach couple we can define an interpolation space X_E as the space of all x such that $K(t, x) \in E$ and we can put

$$\|x\|_{X_E} = \|K(t, x)\|_E.$$

Each such function space E then induces an interpolation method. The interpolation method associated to E is called a K-method, with parameter E.

We say that an interpolation space X for the Banach couple (X_0, X_1) is *K-monotone* if there is a constant C so that whenever $y \in X$ and $x \in X_0 + X_1$ with

$$K(t, x) \leqslant K(t, y), \quad 0 < t < \infty,$$

then $x \in X$ and $\|x\|_X \leqslant C\|y\|_X$. It is clear that each K-method yields a K-monotone interpolation space. It is a deep result of Brudnyi and Kruglyak that for Gagliardo complete couples the converse is true, i.e., that every K-monotone interpolation space can be obtained by a K-method.

The so-called *Fundamental Lemma* and *K-divisibility Principle* are the key ingredients of this result. The Fundamental Lemma appears in [24]. A forerunner appeared in Cwikel and Peetre [32].

THEOREM 6.1. *There is an absolute constant C with the following property. Let (X_0, X_1) be a Gagliardo complete couple and suppose $x \in \Sigma(\overline{X})$. Then there exists a sequence $(u_j)_{j\in\mathbb{Z}}$ so that $u_j \in \Delta(\overline{X})$ except for at most 2 values of j, $x = \sum_{j\in\mathbb{Z}} u_j$ in $\Sigma(\overline{X})$ and:*

$$K(t, x) \leqslant \sum_{j\in\mathbb{Z}} \min\bigl(\|u_j\|_{X_0}, t\|u_j\|_{X_1}\bigr) \leqslant CK(t, x), \quad 0 < t < \infty. \tag{6.1}$$

REMARK. Here if for some j, we have $u_j \notin X_0 \cap X_1$ we interpret

$$\min\bigl(\|u_j\|_{X_0}, t\|u_j\|_{X_1}\bigr) < \infty$$

to imply that $u_j \in X_0 \cup X_1$. The precise value of the constant C has been investigated further in [26].

The principle of K-divisibility of Brudnyi and Kruglyak [12] was announced in [11]. The Fundamental Lemma was used by Cwikel [24] to give an independent proof of K-divisibility.

THEOREM 6.2. *There is an absolute constant C so that if (X_0, X_1) is a Gagliardo complete couple, $x \in \Sigma(\overline{X})$ and φ_j is a sequence of concave functions such that*

$$\sum_{j=1}^{\infty} \varphi_j(1) < \infty$$

and

$$K(t,x) \leqslant \sum_{j=1}^{\infty} \varphi_j(t), \quad 0<t<\infty,$$

then there exist $u_j \in (\overline{X})$ *such that* $x=\sum_{j=1}^{\infty} u_j$ *in* $\Sigma(\overline{X})$ *and* $K(t,u_j) \leqslant C\varphi_j(t)$ *for* $0<t<\infty$ *and* $j \in \mathbb{N}$.

We now discuss the proofs of Theorems 6.1 and 6.2. Let us first remark that they are equivalent. If we assume Theorem 6.2 then we can obtain Theorem 6.1 by observing that since $t \to K(t,x)$ is concave it may be represented (uniquely) in the form

$$K(t,x)=a+bt+\int_0^{\infty} \min(s,t)\,\mathrm{d}\mu(s), \tag{6.2}$$

where μ is a Borel measure on $(0,\infty)$ such that $\int \min(s,1)\,\mathrm{d}\mu(s)<\infty$. By approximation, if $\varepsilon>0$ one can find a sequences $(s_n)_{n\in\mathbb{Z}}^{\infty}$, $(c_n)_{n\in\mathbb{Z}}$ in $(0,\infty)$ such that

$$K(t,x) \leqslant a+bt+\sum_{n\in\mathbb{Z}} c_n \min(s_n,t) \leqslant (1+\varepsilon)K(t,x), \quad 0<t<\infty. \tag{6.3}$$

Note that if $K(t,u) \leqslant a$ for all t then by Gagliardo completeness $\|u\|_{X_0} \leqslant a$ and similarly if $K(t,u) \leqslant bt$ for all t then $\|u\|_{X_1} \leqslant b$. If $K(t,u) \leqslant c\min(s,t)$ for all t then $\|u\|_{X_0} \leqslant cs$ and $\|u\|_{X_1} \leqslant c$. Thus if apply Theorem 6.2 to (6.3) we obtain Theorem 6.1 with constant $C(1+\varepsilon)$.

Next we consider the converse direction. In this case we find (u_n) as in Theorem 6.1 and let

$$\psi(t)=\sum_{n\in\mathbb{Z}} \min\bigl(\|u_n\|_{X_0}, t\|u_n\|_{X_1}\bigr).$$

Thus

$$K(t,x) \leqslant \psi(t) \leqslant CK(t,x), \quad 0<t<\infty.$$

Now consider the set Γ of continuous maps $\theta:(0,\infty)\to \ell_1$ of the form $\theta(t)=(\theta_j(t))_{j\in\mathbb{N}}$ where each θ_j is non-negative and concave and we have $\theta_j(t) \leqslant \varphi_j(t)$ but $\sum_{n\in\mathbb{Z}} \theta_j(t) \geqslant C^{-1}\psi(t)$ for $0<t<\infty$). It is not difficult to see by the Ascoli–Arzela theorem that Γ is compact for the topology of uniform convergence on compacta, and so has a minimal element $\sigma(t)=(\sigma_j(t))_{n\in\mathbb{Z}}$. It then follows without difficulty that in fact we have

$$\sum_{j\in\mathbb{Z}} \sigma_j(t)=C^{-1}\psi(t), \quad 0<t<\infty.$$

Indeed if $\sum \sigma_j > C^{-1}\psi(t)$ on some maximal open interval I then it is easy to see that $\sum \sigma_n$ must be affine on I; if $I = (\alpha, \beta)$ where $0 < \alpha < \beta < \infty$ then the fact that ψ is concave leads to a contradiction, while the other cases when $\alpha = 0$ and/or $\beta = \infty$ can be treated similarly.

Now using the definition of ψ and the fact that each σ_j is concave we see that

$$\sigma_j(t) = C^{-1} \sum_{k\in\mathbb{Z}} \delta_{jk} \min\bigl(\|u_k\|_{X_0}, t\|u_k\|_{X_1}\bigr),$$

where $\sum_{k\in\mathbb{Z}} \delta_{jk} = 1$ and $\delta_{jk} \geqslant 0$. (One way to see this is to use the representation 6.2 for ψ and σ_j.) Let

$$v_j = \sum_{k\in\mathbb{Z}} \delta_{jk} u_k.$$

Then $\sum_{j=1}^{\infty} v_j$ converges absolutely in $X_0 + X_1$ to x and

$$K(t, v_j) \leqslant C\sigma_j(t) \leqslant C\varphi_j(t).$$

It is clear from the foregoing discussion that if we define γ_1 as the infimum of all constants C for which Theorem 6.1 holds and γ_2 as the infimum of all constants C for which Theorem 6.2 holds then $\gamma_1 = \gamma_2$. Their common value, γ is called the K-divisibility constant. Its exact value is unknown and seems to be a challenging problem. The best estimate from above was obtained by Cwikel, Jawerth and Milman [26], $\gamma \leqslant 3 + 2\sqrt{2}$. On the other hand, an example of Kruglyak [70] gives a lower estimate $\gamma > 1.6$.

Let us now sketch the ideas in the proof of Theorem 6.1, but without attempting to give the most delicate estimates (following [5]); we refer the reader to [26] for these. For fixed $x \in X_0 + X_1$ let us define $t_0 = 1$ and then construct a sequence $(t_j)_{j\in\mathbb{Z}}$ by two-sided induction such that for any j we have that one of the three mutually exclusive possibilities holds:

(1) $$\min\left(\frac{K(t_j, x)}{K(t_{j-1}, x)}, \frac{t_j K(t_{j-1}, x)}{t_{j-1} K(t_j, x)}\right) = 2$$

or

(2) $t_{j-1} = 0$ and

$$\min\left(\frac{K(t_j, x)}{K(t, x)}, \frac{t_j K(t, x)}{t K(t_j, x)}\right) < 2, \quad 0 < t < t_j,$$

or

(3) $t_j = \infty$ and

$$\min\left(\frac{K(t, x)}{K(t_{j-1}, x)}, \frac{t K(t_{j-1}, x)}{t_{j-1} K(t, x)}\right) < 2, \quad t_{j-1} < t < \infty.$$

For each $j \in \mathbb{Z}$ such that $0 < t_j < \infty$ pick $v_j \in X_0$ and $w_j \in X_1$ so that $x = v_j + w_j$ and $\|v_j\|_{X_0} + t\|w_j\|_{X_1} < 2K(t_j, x)$. If $t_j = 0$ let $v_j = 0$ and $w_j = x$. If $t_j = \infty$, let $v_j = x$ and $w_j = 0$.

Next let $u_j = v_j - v_{j-1}$. If $0 < t_{j-1} < t_j < \infty$ then

$$\|u_j\|_{X_0} \leqslant 4K(t_j, x), \qquad \|u_j\|_{X_1} \leqslant 2t_{j-1}^{-1} K(t_{j-1}, X).$$

It follows that if $t_{j-1} \leqslant t \leqslant t_j$ then

$$\min\big(\|u_j\|_{X_0}, t\|u_j\|_{X_1}\big) \leqslant 8K(t, x).$$

If $0 = t_{j-1} < t_j$ then $u_j = v_j$ and

$$\|u_j\|_{X_0} \leqslant 2K(t_j, x).$$

In this case either

$$\lim_{t\to 0} K(t, x) \geqslant \frac{1}{2} K(t_j, x)$$

or

$$t_j^{-1} K(t_j, x) \leqslant \lim_{t\to 0} t^{-1} K(t, x) \leqslant 2t_j^{-1} K(t_j, x).$$

In the latter case we use Gagliardo completeness to deduce that $\|x\|_{X_1} \leqslant 2t_j^{-1} K(t_j, x)$ and hence

$$\|u_j\|_{X_1} \leqslant \|x\|_{X_1} + \|w_j\|_{X_1} \leqslant 4t_j^{-1} K(t_j, x).$$

In either case we have

$$\min\big(\|u_j\|_{X_0}, t\|u_j\|_{X_1}\big) \leqslant 8K(t, x), \quad 0 < t \leqslant t_j.$$

Similarly if $t_{j-1} < t_j = \infty$ we obtain

$$\min\big(\|u_j\|_{X_0}, t\|u_j\|_{X_1}\big) \leqslant 8K(t, x), \quad t_{j-1} \leqslant t < \infty.$$

Now if $t_{j-1+r} \leqslant t \leqslant t_{j+r}$ where $r \in \mathbb{Z} \setminus \{0\}$ then we may see that

$$K(t, x) \geqslant 2^{r-1} K(t_j, x), \quad r > 0,$$

and

$$t^{-1} K(t, x) \geqslant 2^{1-r} t_{j-1}^{-1} K(t_{j-1}, x), \quad r < 0.$$

Hence

$$\min\left(\|u_j\|_{X_0}, t\|u_j\|_{X_1}\right) \leqslant 8 \cdot 2^{-|r|} K(t,x).$$

Hence

$$\sum_{j\in\mathbb{Z}} \min\left(\|u_j\|_{X_0}, t\|u_j\|_{X_1}\right) \leqslant 24K(t,x), \quad 0 < t < \infty.$$

It is clear that $\sum u_j$ converges absolutely in $X_0 + X_1$ and it is not difficult to check that its sum must be x in all possible cases.

The above argument is clearly quite crude with regards to constants. In [26] a somewhat more delicate analysis is performed, keeping track of the intercepts on the axes of the tangents to the concave function $t \to K(t,x)$. With this analysis one can achieve the constant $3 + 2\sqrt{2} + \varepsilon$ for any $\varepsilon > 0$.

The main conclusion from the principle of K-divisibility is that K-monotone interpolation spaces coincide exactly with interpolation spaces obtained by the K-method:

THEOREM 6.3 (Brudnyi and Kruglyak [11]). *If $\overline{X}$ is a Gagliardo complete couple then any K-monotone interpolation space is given by a K-method.*

Suppose Y is a K-monotone interpolation space. The idea of Theorem 6.3 is that one can define a Banach function space E on $(0,\infty)$ by

$$\|f\|_E = \inf\left\{\sum_{j=1}^{\infty} \|y_j\|_Y \colon \left|f(t)\right| \leqslant \sum_{j=1}^{\infty} K(t,y_j),\ 0 < t < \infty\right\}.$$

Now if $x \in X_0 + X_1$ and $K(t,x) \in E$ we can find $y_j \in Y$ so that

$$K(t,x) \leqslant \sum_{j=1}^{\infty} K(t,y_j), \quad 0 < t < \infty,$$

and

$$\sum_{j=1}^{\infty} \|y_j\|_Y \leqslant 2\left\|K(t,x)\right\|_E.$$

Then by K-divisibility (Theorem 6.2) we can decompose $x = \sum_{j=1}^{\infty} u_j$ in $X_0 + X_1$ so that

$$K(t,u_j) \leqslant CK(t,y_j),$$

where C is a universal constant. If Y is K-monotone this implies that $u_j \in Y$ and that we have an estimate $\|u_j\|_Y \leqslant C_1\|y_j\|_Y$ for some constant C_1 depending on Y. Hence $x = \sum_{j=1}^{\infty} u_j \in Y$ and $\|x\|_Y \leqslant C_2\|K(t,x)\|_E$.

We say that a Banach couple is a *Calderón couple* if every interpolation space is K-monotone (or, by Theorem 6.3, every interpolation space is given by a K-method). This terminology is based on the classical Calderón–Mitjagin theorem on interpolation spaces for (L_1, L_∞). This theorem is in some sense already classical, but we will discuss it below as motivation.

Let us first make an equivalent formulation of the problem. Suppose $0 \neq x \in \Sigma(\overline{X})$. We can define an orbit space for x, O_x namely the space $\{Tx\colon\ T \in \mathcal{L}(\overline{X})\}$ with the norm

$$\|y\|_{O_x} = \inf\bigl\{\|T\|_{\overline{X}}\colon\ Tx = y\bigr\}.$$

Then O_x is an interpolation space. If O_x is monotone then there is a constant C so that for any y satisfying $K(t,y) \leqslant K(t,x)$ we have $T \in \mathcal{L}(\overline{X})$ with $Tx = y$ and $\|T\|_{\overline{X}} \leqslant C$. We thus have:

PROPOSITION 6.4. *$\overline{X}$ is a Calderón couple if and only if for every $x \in \Sigma(X)$ there is a constant $C = C(x)$ so that if $y \in \Sigma(\overline{X})$ and $K(t,y) \leqslant K(t,x)$ for $0 < t < \infty$ then there exists $T \in \mathcal{L}(\overline{X})$ with $Tx = y$.*

Somewhat surprisingly it appears to be unknown if the constant C can be chosen independent of x. If this is the case we call $\overline{X}$ a *uniform* Calderón couple.

We now consider the special case of the pair $\overline{L} = (L_1(\mathbb{R}), L_\infty(\mathbb{R}))$ where $\mathbb{R}$ is equipped with standard Lebesgue measure. In this case the K-functional is computable and is given by the formula:

$$K(t,f) = \int_0^t f^*(s)\,\mathrm{d}s,$$

where f^* is the decreasing rearrangement of $|f|$, i.e., the function on $(0,\infty)$ given by

$$f^*(s) = \sup_{\lambda(F)=s} \inf_{u\in F} \bigl|f(u)\bigr|.$$

If we introduce f^{**} as usual by setting

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\,\mathrm{d}s$$

then of course $K(t,f) = tf^{**}(t)$.

We recall a Banach function space X is symmetric (or a symmetric lattice ideal) if it satisfies the condition that if $f \in X$ and g is any measurable function with $g^* \leqslant f^*$ it follows that $g \in X$ and $\|g\|_X \leqslant \|f\|_X$. It is not difficult to show any interpolation space for $\overline{L}$ is a symmetric space.

Now for $f, g \in L_\infty + L_1$ let us say $f \prec g$ if $f^{**}(t) \leqslant g^{**}(t)$ for $0 < t < \infty$. In many symmetric Banach function spaces X the property $f \in X$ and $g \prec f$ implies $g \in X$ and $\|g\|_X \leqslant \|f\|_X$; for example, this holds if X is separable. However it does not hold in general, so we shall use the term *rearrangement-invariant* or r.i. space to mean a symmetric space with this additional property.

The following theorem is due to Ryff [103] and Calderón [15]:

PROPOSITION 6.5. *If $f \prec g$ then there is an operator $T \in \mathcal{L}(\overline{L})$ with $\|T\|_{\overline{L}} = 1$ and $Tg = f$.*

THEOREM 6.6. *The interpolation spaces for the couple (L_1, L_∞) coincide with the r.i. spaces on $\mathbb{R}$ and (L_1, L_∞) is a Calderón couple.*

Let us remark that this theorem is equally valid for the couple $(L_1, L_\infty[0, 1])$.

It is natural then to try to extend this Theorem 6.6 to other function spaces. A major advance was made in this direction by Sparr [106] and [107] (see also [1]):

THEOREM 6.7. *The Banach couple $(L_{p_0}(w_0), L_{p_1}(w_1))$ is a Calderón couple for any choice of $1 \leqslant p_j \leqslant \infty$ and any pair of weight functions w_j.*

On the other hand, Ovchinnikov shows that the pair $(L_1 + L_\infty, L_1 \cap L_\infty)$ is not a Calderon couple [88]. Indeed Maligranda and Ovchinnikov [78] showed that the $L_p \cap L_{p'}$ and $L_p + L_{p'}$ are not K-monotone with respect to this couple when $1 < p < \infty$, $p \neq p'$ and $1/p + 1/p' = 1$. However these spaces are complex interpolation spaces for this couple.

This raises the general question of classifying pairs of r.i. spaces (X_0, X_1) on either $(0, 1)$ or $(0, \infty)$ which form Calderón couples. For special examples (certain types of Lorentz spaces and Marcinkiewicz spaces) positive results were obtained by Cwikel [23] and Merucci [81]. In [56] a full study of this problem was undertaken and although the results are not complete, a good description was obtained for sufficiently "separated" pairs of spaces, in a sense to be described. Curiously enough some of the properties which surface in the characterization have a flavor suggestive of Banach space theory.

We first need to introduce some standard ideas (see [72], for example). If X is an r.i. space on $[0, 1]$ or $[0, \infty)$ then the dilation operators D_a on X are then defined by $D_a f(t) = f(t/a)$ (where we regard f as vanishing outside $[0, 1]$ in the former case). We can then define the Boyd indices p_X and q_X of X by

$$p_X = \lim_{a \to \infty} \frac{\log a}{\log \|D_a\|_X}$$

and

$$q_X = \lim_{a \to 0} \frac{\log a}{\log \|D_a\|_X}.$$

In many texts, the reciprocals of p_X and q_X are used for the Boyd indices following the original convention of Boyd [10]. The Boyd indices are of course extremely useful in interpolation theory because of the classical Boyd interpolation theorem [10], which we will discuss in Section 7.

For convenience we restrict our discussion to the case of r.i. spaces over $(0, \infty)$. Let $e_n = \chi_{(2^n, 2^{n+1})}$. Associated to each r.i. space X we can introduce a Banach sequence space S_X modelled on $\mathbb{Z}$ defined by $\xi = (\xi_n)_{n \in \mathbb{Z}}$ if and only if $\sum_{n \in \mathbb{Z}} \xi_n e_n \in X$ and

$$\|\xi\|_{S_X} = \left\| \sum_{n \in \mathbb{Z}} \xi_n e_n \right\|_X .$$

Now it is clear from consideration of averaging projections that (X, Y) is a Calderón couple if and only if (S_X, S_Y) is also a Calderón couple. It turns out we can answer this question under separation conditions on the Boyd indices in terms of some conditions with a Banach space flavor. Let us suppose that E is a Banach sequence space modelled on $\mathbb{Z}$. We shall say that E has the *right shift-property* (*RSP*) if whenever $(x_n)_{n=1}^N$, $(y_n)_{n=1}^N$ are two sequences satisfying

(1) $\operatorname{supp} x_1 < \operatorname{supp} y_1 < \operatorname{supp} x_2 < \cdots < \operatorname{supp} x_n < \operatorname{supp} y_n$,
(2) $\|y_n\|_E \leqslant \|x_n\|_E$, $n = 1, 2, \ldots, N$,

then

$$\left\| \sum_{n=1}^N y_n \right\| \leqslant C \left\| \sum_{n=1}^N x_n \right\| .$$

Similarly we say E has the *left-shift property* (*LSP*) if whenever $(x_n)_{n=1}^N$, $(y_n)_{n=1}^N$ are two sequences satisfying

(1) $\operatorname{supp} x_1 > \operatorname{supp} y_1 > \operatorname{supp} x_2 > \cdots > \operatorname{supp} x_n > \operatorname{supp} y_n$,
(2) $\|y_n\|_E \leqslant \|x_n\|_E$, $n = 1, 2, \ldots, N$,

then

$$\left\| \sum_{n=1}^N y_n \right\| \leqslant C \left\| \sum_{n=1}^N x_n \right\| .$$

We then say that an r.i. space X on $(0, \infty)$ is *stretchable* if S_X has (RSP) and *compressible* if S_X has (LSP). If X is both stretchable and compressible then X is *elastic*.

The main theorems of [56] then assert the following:

THEOREM 6.8. *Let (X, Y) be a pair of r.i. spaces on $(0, \infty)$ such that $p_Y > q_X$. Then (X, Y) is a (uniform) Calderón couple if and only if X is stretchable and Y is compressible.*

THEOREM 6.9. *Let X be an r.i. space on $(0, \infty)$; then (X, L_∞) is a Calderón couple if and only if X is stretchable.*

Of course the condition $p_Y > q_X$ is a quite strong separation condition on the Boyd indices; it asserts that the intervals $[p_X, q_X]$ and $[p_Y, q_Y]$ do not intersect. A remarkable feature of the conclusion of Theorem 6.8 is that the condition that X is stretchable (or Y is compressible) is independent of the normalization of e_n. To illustrate this note that if X is a Lorentz space with norm

$$\|f\|_X = \left(\int_0^\infty |f^*(t)|^p w(t)\,\mathrm{d}t\right)^{1/p}$$

for a suitable decreasing weight function w then X is always elastic because S_X up to normalization is simply ℓ_p.

The above Theorems 6.8 and 6.9 have similar statements when $(0, \infty)$ is replaced by $[0, 1]$ and for sequence spaces. It is necessary simply to formulate the shift properties on sequence spaces modelled on $\mathbb{N}$ or $\mathbb{Z} \setminus \mathbb{N}$. We refer to [56] for details.

It is possible to give a rather complicated characterization of stretchable and compressible Orlicz spaces. In fact for Orlicz spaces the conditions are equivalent and any such space is elastic. The following theorem is given in [56] (we specialize to $[0, 1]$ for definiteness).

THEOREM 6.10. *Let F be an Orlicz function. Then the following conditions on F are equivalent*:

(1) $L_F[0, 1]$ *is elastic* (*respectively, stretchable, respectively, compressible*).
(2) $(L_\infty[0, 1], L_F[0, 1])$ *is a Calderón couple.*
(3) *There is a bounded monotone increasing function* $w : [1, \infty) \to \mathbb{R}$ *and a constant* C *so that if* $1 \leqslant s \leqslant t$ *and* $0 < x \leqslant 1$ *we have*

$$\frac{F(tx)}{F(t)} \leqslant C\frac{F(sx)}{F(s)} + w(t) - w(s).$$

(4) *There is a bounded monotone increasing function* $w : [1, \infty) \to \mathbb{R}$ *and a constant* C *so that if* $1 \leqslant s \leqslant t$ *and* $0 < x \leqslant 1$ *we have*

$$\frac{F(sx)}{F(s)} \leqslant C\frac{F(tx)}{F(t)} + w(t) - w(s).$$

These conditions are a little difficult to check. They are related to somewhat similar criteria for Orlicz spaces to coincide with Lorentz spaces [84]. Perhaps the simplest practical condition which follows is the following.

THEOREM 6.11. *Let X be an Orlicz space on* $[0, 1]$. *Then if* $(X, L_\infty[0, 1])$ *is a Calderón couple we have* $p_X = q_X$.

This allows the construction of some very easy counter-examples to the conjecture that $(L_F[0, 1], L_\infty[0, 1])$ is a Calderón couple for every Orlicz function F.

In spite of the difficulty in classifying Calderón couples, there is a form of converse to the theorem of Sparr, obtained by Cwikel and Nilsson [31]. Here we consider all possible changes of density. If X is a Banach function space we define $X(w) = \{f\colon\ fw \in X\}$ with $\|f\|_{X(w)} = \|fw\|_X$, where w is a weight function (a strictly positive measurable function).

THEOREM 6.12. *Let (X_0, X_1) be a pair of Banach function spaces on $[0,1]$ or $[0,\infty)$. Suppose that for every pair of weight functions the pair $(X_0(w_0), X_1(w_1))$ is a Calderón couple. Then there exist $1 \leqslant p_0, p_1 \leqslant \infty$ and weight functions v_0, v_1 so that $X_0 = L_{p_0}(v_0)$ and $X_1 = L_{p_1}(v_1)$ up to equivalence of norm.*

Finally let us note a problem raised by Cwikel which was solved in [80]. Cwikel asked if a pair (X, Y) of complex Banach spaces is a Calderón couple if and only if every complex interpolation space is K-monotone. In [80] counter-examples are exhibited even for pairs of r.i. spaces.

7. Interpolation spaces for (L_p, L_q)

Throughout this section we will suppose that our rearrangement invariant spaces are over $[0,1]$ or $[0,\infty)$. Let us start by noting that the K-functional for (L_p, L_q) can be approximated in terms of the Hardy operators. To this end, we have the following formula of Holmstedt [47]:

$$t^{-1/p} K\big(f, t^{1/p-1/q}\big) \approx \left(\frac{1}{t}\int_0^t f^*(s)^p \,\mathrm{d}s\right)^{1/p} + \left(\frac{1}{t}\int_t^\infty f^*(s)^q \,\mathrm{d}s\right)^{1/q}.$$

This formula, combined with the fact that (L_p, L_q) is a Calderón couple, can be used to obtain useful results. For example, it is now easy to prove the following interpolation result [46]. Given an Orlicz function Φ, we will say that Φ is p-convex if the map $t \mapsto \Phi(t^{1/p})$ is convex, and q-concave if the map $t \mapsto \Phi(t^{1/q})$ is concave (we will say that all functions are ∞-concave).

THEOREM 7.1. *Let $1 \leqslant p < q \leqslant \infty$, and let X be an interpolation space for (L_p, L_q). Then there is a positive constant c such that the following holds. If f, g are functions such that $\|g\|_\Phi \leqslant \|f\|_\Phi$ for every function Φ that is p-convex and q-concave, and if $f \in X$, then $g \in X$ with $\|g\|_X \leqslant c\,\|f\|_X$.*

In [61], the authors were able to obtain the following characterization of interpolation spaces for (L_p, L_q). In order to state this result, we need the notion of conditional expectation. On $[0,1]$, this is standard. On $[0,\infty)$ the same construction works, as long as the σ-field's atoms all have finite measure.

THEOREM 7.2. *Let $1 \leqslant p < q \leqslant \infty$. A rearrangement invariant space X is an interpolation space for (L_p, L_q) if and only if there is a positive constant c such that for any function f, and any sub-σ-field $\mathcal{M}$ whose atoms have finite measure, we have that*

$$\big\|\big(\mathbb{E}\big(|f|^p|M\big)\big)^{1/p}\big\|_X \leqslant c\,\|f\|_X$$

and if $q < \infty$

$$\|f\|_X \leqslant c\big\|\big(\mathbb{E}\big(|f|^q|M\big)\big)^{1/q}\big\|_X.$$

This gives the following collection of sufficient conditions, the fourth of which is the classical Boyd interpolation theorem [10], which is also proved in [72] (see also [86]).

THEOREM 7.3. *Let X be a rearrangement invariant space, and $1 \leqslant p < q \leqslant \infty$ Suppose that any of the following hold:*
- *X is p-convex and q-concave;*
- *X is p-convex and has upper Boyd index less than q;*
- *X is q-concave and has lower Boyd index greater than p;*
- *X has Boyd indices strictly between p and q.*

Then X is an interpolation space for (L_p, L_q).

Finally we end this section with some results about the span of the Rademacher series in rearrangement invariant spaces. That is, given a rearrangement invariant space L on $[0,1]$, we can form a sequence space R_L which is the space of sequences (a_n) whose norm $\|\sum_{n=1}^{\infty} a_n r_n\|_L$ is finite. (Here (r_n) denotes the sequence of Rademacher functions on $[0,1]$.)

It was shown by Rodin and Semenov [102] that R_L is isomorphic to the space ℓ_2 if and only if L contains the space G, where G is the closure of the simple functions in the Orlicz space derived from an Orlicz function equivalent to $\exp(x^2)$. They went on to calculate the space R_L for some lattices that do not contain G.

More recently this work was extended by Astashkin [2]. One of the main results of this paper can be summarized as follows.

THEOREM 7.4. *A symmetric sequence space S is naturally isomorphic to R_L for some rearrangement invariant space L on $[0,1]$ if and only if S is an interpolation space for the couple (ℓ_1, ℓ_2).*

8. Extensions

Let us briefly describe the elements of the theory of *extensions* (or *twisted sums*). This discussion overlaps the discussion in [60] but here our emphasis is slightly different. We note that a good reference for the general theory of twisted sums in the context of Banach space theory is [18]. Let X and Y be a Banach spaces (or more generally quasi-Banach spaces). An extension of X by Y is (formally) a short exact sequence

$$0 \to Y \to Z \to X \to 0,$$

where Z is a quasi-Banach space. Less formally we regard the space Z is an extension of X by Y if $Z \supset Y$ and Z/Y is isomorphic to X. One can of course restrict extensions to lie in the category of Banach spaces.

There is a general construction of extensions via *quasi-linear maps*. Let V be any vector space containing Y (we may take $Y = V$ but some flexibility is useful here.) A map $\Omega : X \to V$ is called quasi-linear if

- $\Omega(\lambda x) = \lambda\Omega(x)$, $x \in X$, $\lambda \in \mathbb{K}$.
- There is a constant C so that if $x_1, x_2 \in X$ then $\Omega(x_1+x_2) - \Omega(x_1) - \Omega(x_2) \in Y$ and

$$\left\|\Omega(x_1+x_2) - \Omega(x_1) - \Omega(x_2)\right\| \leqslant C\big(\|x_1\| + \|x_2\|\big). \tag{8.1}$$

We then can define an extension $X \oplus_\Omega Y$ to be the subspace of $X \oplus V$ of all (x, v) such that $v - \Omega x \in Y$, and equipped with the quasi-norm

$$\left\|(x, v)\right\| = \|x\| + \|v - \Omega x\|.$$

In general this is not a norm, but it will be equivalent to a norm if it satisfies an estimate of the form

$$\left\|\left(\sum_{j=1}^n x_j, \sum_{j=1}^n v_j\right)\right\| \leqslant C\sum_{j=1}^n \left\|(x_j, v_j)\right\|, \quad x_1, \dots, x_n \in X, v_1, \dots, v_n \in V.$$

It follows that $X \oplus_\Omega Y$ is isomorphic to a Banach space if (and only if) (8.1) is replaced by the stronger inequality:

$$\left\|\sum_{k=1}^n \Omega x_k - \Omega\left(\sum_{k=1}^n x_k\right)\right\| \leqslant C\sum_{k=1}^n \|x_k\|. \tag{8.2}$$

In fact, in the above construction, it is only necessary that Ω be defined on a dense linear subspace; the construction above yields a space whose completion is an extension.

Now it is a key fact that every extension can be represented in this form. Indeed if Z is an extension of X we can define two maps $F : X \to Z$ and $L : X \to Z$ such that $qF = qL = I_X$ where q is the quotient map. F is defined to be homogeneous (not necessarily linear) and satisfy $\|F(x)\| \leqslant 2\|x\|$, while L is required to be linear (but not necessarily bounded). If we set $\Omega(x) = F(x) - L(x)$ then $\Omega : X \to Y$ is quasilinear and one can easily set up a natural isomorphism between Z and $X \oplus_\Omega Y$. Notice, however, that the choice of Ω depends heavily on certain arbitrary choices (e.g., of the linear map L).

We refer to an extension Z of X as *trivial* if there is a bounded projection of Z onto X. In this case Z splits as a direct sum $X \oplus Y$. It is easy to show that $X \oplus_\Omega Y$ splits if and only if there is a linear map $L : X \to V$ so that $\Omega x - Lx \in Y$ for all x and

$$\|\Omega x - Lx\| \leqslant C\|x\|, \quad x \in X.$$

In [60] we discussed the case of *minimal extensions*. A minimal extension is an extension by the scalar field $\mathbb{K}$. In this case all Banach extensions are trivial, by the Hahn–Banach theorem. A Banach space X is called a $\mathcal{K}$-space if all minimal extensions of X are trivial. The following proposition is then very useful:

PROPOSITION 8.1 ([51]). *X is a $\mathcal{K}$-space if and only if any extension of X by a Banach space is* (*isomorphic to*) *a Banach space.*

It is conjectured (see [60]) that X is a $\mathcal{K}$-space if and only if X^* has non-trivial cotype. It is known that any space with non-trivial type is a $\mathcal{K}$-space.

An extension of X by X is called a *self-extension*; in this case we introduce the notation $d_\Omega X = X \oplus_\Omega X$. We will see shortly that these are intimately related with interpolation theory, but first let us discuss the historical origins of the study of self-extensions.

9. Self-extensions of Hilbert spaces

A self-extension of a Hilbert space is called a *twisted Hilbert space*. The basic question of the existence of a non-trivial twisted Hilbert space was apparently first raised by Palais. It was solved in 1975 by Enflo, Lindenstrauss and Pisier [39] who produced the first non-trivial example of a twisted Hilbert space. A few years later in [62] an alternative example was constructed based on the ideas of Ribe's construction of a non-trivial minimal extension of ℓ_1 (see [99] and [60]). We will discuss this example and some variants in this section. It is interesting that the link between minimal extensions of ℓ_1 and self-extensions of ℓ_2 is now much better understood than it was in 1979, and we will explain this connection later.

Let us recall that Ribe's space is associated to the quasilinear map $\Omega : c_{00} \to \mathbb{R}$ given by

$$\Omega\xi = \sum_{n=1}^{\infty} \xi_n \log|\xi_n| - \left(\sum_{n=1}^{\infty} \xi_n\right) \log\left|\sum_{n=1}^{\infty} \xi_n\right|. \tag{9.1}$$

We define a corresponding self-extension of ℓ_2, denoted by Z_2 by taking $\Omega : \ell_2 \to \omega$ (ω is the space of all sequences) as

$$\Omega\xi = \left(\xi_n \log \frac{|\xi_n|}{\|\xi\|_2}\right)_{n=1}^{\infty}.$$

Here we interpret $0 \log 0$ and $0 \log 0/0$ as 0. Thus Z_2 is the space of pairs of sequences $((\xi_n)_{n=1}^{\infty}, (\eta_n)_{n=1}^{\infty})$ such that

$$\|(\xi,\eta)\| = \left(\sum_{n=1}^{\infty} |\xi_n|^2\right)^{1/2} + \left(\sum_{n=1}^{\infty} \left|\eta_n - \xi_n \log \frac{|\xi_n|}{\|\xi\|_2}\right|^2\right)^{1/2} < \infty.$$

This equation defines a quasi-norm; the fact that it is equivalent to a norm (and thus Z_2 is a genuine Banach self-extension) follows from Proposition 8.1.

The Banach space properties of the space Z_2 are of some interest. It is immediate that there is a natural unconditional Schauder decomposition into two-dimensional spaces and it is shown in [62] that it has no unconditional basis; in [50] it is shown to fail local unconditional structure as well. In fact in [17] it is shown that any space with a two-dimensional UFDD (E_n) (or even a UFDD with bounded dimensions) with local unconditional structure has an unconditional basis which can be chosen from the subspaces. The main unresolved problems concerning Z_2 are:

- Is Z_2 prime?
- Is Z_2 isomorphic to its hyperplanes.

It has been widely conjectured that Z_2 is not isomorphic to its hyperplanes (in some sense, Z_2 is even-dimensional and its hyperplanes are odd-dimensional!). Of course, since the celebrated example of Gowers [43], this problem is less pressing.

Let us notice (as in [62]) that this construction can be generalized quite a bit. Let $F:\mathbb{R}\to\mathbb{R}$ be any Lipschitz map, and let E be any Banach sequence space and define

$$\Omega_F(\xi)=\left(\xi_n F\left(\log\frac{|\xi_n|}{\|\xi\|_E}\right)\right)_{n=1}^{\infty}.$$

Then Ω_F induces a self-extension $d_{\Omega_F}E$ of E. In fact one can go further and consider complex sequence spaces and then allow $F:\mathbb{R}\to\mathbb{C}$. In [57] this idea was exploited taking $E=\ell_2$ and $F(t)=t^{1+i\alpha}$. This produces a complex Banach space $Z_2(\alpha)$ which is not isomorphic to its conjugate space. The conjugate space of a complex Banach space X is the space $\overline{X}$ on which multiplication is defined by $\lambda\times x=\bar{\lambda}x$. In this case it is not difficult to see that the conjugate space of $Z_2(\alpha)$ is isomorphic to $Z_2(-\alpha)$ and then it can be shown without undue difficulty that $Z_2(\alpha)$ and $Z_2(-\alpha)$ are not isomorphic as complex Banach spaces; see [57] or [6]. Earlier examples had been constructed by probabilistic methods by Bourgain [9] and Szarek [108].

10. Analytic families of Banach spaces

In this section we sketch the origins of the theory of non-linear commutators and analytic families and how it relates to the preceding examples. Let us introduce the idea of an *analytic family* of Banach spaces. To do this we will abstract the ideas of complex interpolation introduced in Section 4; this has the added convenience of incorporating the description of interpolating families of spaces by Coifman, Cwikel, Rochberg, Sagher and Weiss [20].

Let us suppose $\mathcal{U}$ is an open subset of the complex plane conformally equivalent to the open unit disk $\mathcal{D}$; in fact we need only consider the case $\mathcal{U}=\mathcal{D}$ and $\mathcal{U}=\mathcal{S}:=\{z\colon 0<\Re z<1\}$. Next let W be some complex Banach space (the ambient space) and let $\mathcal{F}$ be a Banach space of analytic functions $F:\mathcal{U}\to W$. We assume that $\mathcal{F}$ is equipped with a norm $\|\cdot\|_{\mathcal{F}}$ such that:

- The evaluation map $F\to F(z)$ $(\mathcal{F}\to W)$ is bounded for each $z\in\mathcal{U}$.
- If $\varphi:\mathcal{U}\to\mathcal{D}$ is a conformal equivalence then $F\in\mathcal{F}$ if and only if $\varphi F\in\mathcal{F}$ and $\|F\|_{\mathcal{F}}=\|\varphi F\|_{\mathcal{F}}$.

We will call such a space $\mathcal{F}$ admissible. Then for $z\in\mathcal{U}$ and $x\in W$ we define

$$\|x\|_z=\inf\bigl\{\|F\|_{\mathcal{F}}\colon\ F(z)=x\bigr\}$$

and let

$$X_z=\bigl\{x\in W;\ \|x\|_z<\infty\bigr\}.$$

The family of spaces $(X_z)_{z\in\mathcal{U}}$ is then called an analytic family of Banach spaces. If W_0 is the linear span of the spaces $\{X_z\colon z\in\mathcal{U}\}$ then a linear map $T: W_0\to W_0$ will be called *interpolating* if $F\to T\circ F$ is defined and bounded on $\mathcal{F}$. It then follows that $T(X_z)\subset X_z$ for each $z\in\mathcal{U}$ and

$$\|T\|_{X_z\to X_z}\leqslant\|T\circ F\|_{\mathcal{F}\to\mathcal{F}}.$$

If we take a Banach couple $\overline{X}=(X_0,X_1)$ and define $\mathcal{F}$ as in Section 4 then one may see that $\{X_z\colon z\in\mathcal{U}\}$ is an analytic family and $X_z=X_{\Re z}$ where X_θ is the complex interpolation space between X_0 and X_1. Thus our definition abstracts the ideas of complex interpolation. Under these assumptions any $T\in\mathcal{L}(\overline{X},\overline{Y})$ is interpolating. The upper method also yields an analytic family at least when $\overline{X}$ is Gagliardo complete.

Let us note that it is possible to describe the ideas of this section in much more generality, by relaxing our assumptions on $\mathcal{F}$, so that real and other methods may be included; we refer to [28] for a fuller discussion, using an annulus in place of the disk. To keep our discussion reasonably crisp we will retain our much stronger conditions.

We now invoke ideas of Rochberg and Weiss [101] (which in embryonic form appear in work of Schechter [104]). For each z we define a derived space $dX_z\subset W\times W$ by $dX_z=\{(x_1,x_2)\colon \|(x_1,x_2)\|_{dX_z}<\infty\}$ where

$$\big\|(x_1,x_2)\big\|_{dX_z}=\inf\big\{\|F\|_{\mathcal{F}};\ F(z)=x_1,\ F'(z)=x_2\big\}. \tag{10.1}$$

Let Y be the subspace of dX_z defined by $x_1=0$. We claim that Y is an isometric copy of X_z. Indeed let φ be a conformal map of $\mathcal{U}$ onto $\mathcal{D}$ with $\varphi(z)=0$. Then if $F(z)=0$ we can write $F=\varphi G$ where $\|G\|_{\mathcal{F}}=\|F\|_{\mathcal{F}}$. Then $F'(z)=\varphi'(z)G(z)$ and so

$$\big\|(0,x_2)\big\|_{dX_z}=\big|\varphi'(z)\big|^{-1}\|x_2\|_{X_z}.$$

On the other hand dX_z/Y is trivially isometric to X_z so that we have a short exact sequence

$$0\to X_z\to dX_z\to X_z\to 0$$

and dX_z is a self-extension of X_z.

Thus we can use the ideas from Section 8 to give a representation of dX_z in the form $d_\Omega X_z$ where $\Omega: X_z\to W$ is a quasilinear map. It is easy enough to see that an appropriate Ω is given by $\Omega(x)=F'(z)$ where F is any choice of $F\in\mathcal{F}$ with $F(z)=x$ and $\|F\|_{\mathcal{F}}\leqslant C\|x\|_z$. In many circumstances there is a unique optimal choice of F with $\|F\|_{\mathcal{F}}=\|x\|_z$ and in this case one can define Ω in a very natural way. In general, there is some arbitrariness in the definition of Ω but any two such choices differ by a bounded function. Thus we have

$$\big\|(x_1,x_2)\big\|_{dX_z}\approx\|x_1\|_z+\|x_2-\Omega x_1\|_z.$$

Rochberg and Weiss used this construction to obtain *commutator estimates*. If T is an interpolating operator then $(x_1,x_2)\to(Tx_1,Tx_2)$ is bounded on dX_z and this implies

PROPOSITION 10.1. *If T is an interpolating operator then there is a constant C so that if $x \in X_z$ then $[T, \Omega]x = T\Omega x - \Omega T x \in X_z$ and*

$$\big\|[T, \Omega]x\big\|_z \leqslant C\|x\|_z.$$

To conclude this section, let consider the case of interpolation of ℓ_p-spaces. Suppose $1 \leqslant p_1 < p_0 \leqslant \infty$. We will take $W = \omega$, the space of all complex sequences, as our ambient space. Let us consider the space $\mathcal{G}$ of analytic functions on the strip $\mathcal{S}$ of the form $F(z) = (f_k(z))_{k=1}^{\infty}$ where $(f_k)_{k=1}^{\infty}$ are bounded analytic functions. We can then extend each f_k a.e. to the boundary of the strip by taking non-tangential limits, i.e.,

$$f_k(j + \mathrm{i}t) = \lim_{x \to j} f_k(x + \mathrm{i}t), \quad j = 0, 1.$$

Now define $\mathcal{F}$ to be the space of all $F \in \mathcal{G}$ so that $F(j + \mathrm{i}t) \in \ell_{p_j}$ a.e. for $j = 0, 1$ and

$$\|F\|_{\mathcal{F}} = \max_{j=0,1}\Big(\operatorname*{ess\,sup}_{-\infty<t<\infty}\big\|F(j + \mathrm{i}t)\big\|_{\ell_{p_j}}\Big) < \infty.$$

It may be shown that this method coincides with the upper Calderón method, and hence by Corollary 4.3 yields the same interpolation spaces as complex interpolation, i.e., $X_z = \ell_{p_z}$ where if $\theta = \Re z$ we have

$$\frac{1}{p_z} = \frac{1-\theta}{p_1} + \frac{\theta}{p_0}.$$

The advantage of using this method, however, is that we can write down explicit extremals when computing norms in the interpolation spaces.

Now suppose $0 < \theta < 1$ and $x = (\xi_k)_{k=1}^{\infty} \in \ell_p$ where $p = p_\theta$. In this case we can construct an optimal F so that $F(\theta) = x$ and $\|F\|_{\mathcal{F}} = \|x\|_{\ell_p}$. Let us suppose first that $\|x\|_{\ell_p} = 1$. Then F is given by

$$f_k(z) = x_k|x_k|^{(\frac{p}{p_1} - \frac{p}{p_0})(z-\theta)}$$

with $f_k(z) = 0$ if $x_k = 0$. It follows that we can define $\Omega : \ell_p \to \omega$ by the formula:

$$\Omega(x) = \left(\left(\frac{p}{p_1} - \frac{p}{p_0}\right)x_k \log|x_k|\right)_{k=1}^{\infty}.$$

In general by homogeneity we have

$$\Omega(x) = \left(\left(\frac{p}{p_1} - \frac{p}{p_0}\right)x_k \log\left(\frac{|x_k|}{\|x\|_{\ell_p}}\right)\right)_{k=1}^{\infty}.$$

If we take $p_0 = 1$, $p_1 = \infty$ and $\theta = 1/2$ we have

$$\Omega(x) = 2\left(x_k \log\left(\frac{|x_k|}{\|x\|_{\ell_2}}\right)\right)_{k=1}^{\infty}$$

so that (except for a normalization factor of 2) one sees that Z_2 is really nothing other than $dX_{1/2}$ for this interpolation process.

Of course exactly the same calculations are possible for function spaces and this was originally done in that context by Rochberg and Weiss [101]. In particular they noticed that if one applies this and Proposition 10.1 to the Hilbert transform H on the space $L_p(\mathbb{T})$ where $1 < p < \infty$ one obtains inequalities of the form

$$\left\| H\big(f \log|f|\big) - Hf\big(\log|Hf|\big)\right\|_{L_p} \leqslant C_p \|f\|_p.$$

Let us also notice that if one use the couple $(L_2(\ell_p), L_2(\ell_q))$ where $1/p + 1/q = 1$ it is not difficult to use these ideas to see that Z_2 is a (UMD)-space; this result was originally shown directly in [52].

Let us remark that it is also possible to develop a theory of commutator estimates for real interpolation spaces ([49]). However as remarked in Section 4 one can treat real interpolation scales as complex interpolation scales. For a unified treatment see [28]. We also remark that nothing prohibits us from generalizing our ideas to higher-order results by considering the map $F \to (F(z), F'(z), F''(z), \ldots, F^{(n)}(z))$ for $n \geqslant 1$. There is quite a substantial literature on higher-order estimates (see, e.g., [16,82] and [100]).

11. Entropy functions and extensions

In this section we will extend the ideas developed in the preceding sections to more general spaces. Our discussion will overlap with ideas in [60]. We will be discussing interpolation of lattices, but it will be convenient to discuss the special case of sequence spaces; function spaces can be treated almost identically but certain irritating (but fundamentally unimportant) complications arise.

As in [60] we will use the term *Banach sequence space* to mean a Banach sequence space X equipped with a norm $\|\cdot\|_X$ such that

- The basis vectors $e_n \in X$.
- If $x \in X$ and $|\eta_k| \leqslant |x_k|$ for every k then $\eta \in X$ and $\|\eta\|_X \leqslant \|x\|_X$.
- For every $n \in \mathbb{N}$ the linear functional $\eta \to \eta_n$ is continuous.
- If x is a sequence such that $n \in \mathbb{N}$ $S_n x = (x_1, \ldots, x_n, 0, \ldots) \in X$ and $\sup \|S_n x\|_X < \infty$ then $x \in X$ and $\|x\|_X = \sup_{n\in\mathbb{N}} \|S_n x\|_X$.

Thus a key assumption is that a Banach sequence space will always be assumed to have the Fatou property. This is essentially the statement that $B_X = \{x\colon \|x\|_X \leqslant 1\}$ is closed for the topology of pointwise convergence. Of course if we consider only reflexive spaces this is immediate; the main significance is that we consider ℓ_∞ and not c_0 to be Banach sequence spaces. The analogous assumption for function spaces is that B_X should be closed under convergence almost everywhere.

We will use X^* for the Köthe dual of X, i.e., $x^* \in X^*$ if and only if

$$\|x^*\|_{X^*} := \sup\left\{\sum_{k=1}^{\infty} |x_k^*||x_k|\colon \|x\|_X \leqslant 1\right\} < \infty.$$

We need a fundamental result of Lozanovsky [77]:

PROPOSITION 11.1. *Suppose X is a Banach sequence space. Suppose $u \in \ell_1$ with $u \geqslant 0$ and $\|u\|_1 = 1$. Then there exist $x \in B_X, x^* \in B_{X^*}$ with $x \geqslant 0, x^* \geqslant 0$ and $u = xx^*$, i.e., $u_k = x_k x_k^*$ for $1 \leqslant k < \infty$. Furthermore x, x^* are uniquely determined if we insist that $x_k^* = x_k = 0$ when $u_k = 0$.*

This result was originally obtained by Lozanovsky by interpolation techniques; it is also valid in function spaces. Several subsequent proofs has appeared (e.g., [42]). The factorization $u = xx^*$ is called the *Lozanovsky factorization* of u.

Let us also introduce the notion of the *entropy function* of X:

$$\Phi_X(u) = \sup_{\|x\|_X \leqslant 1} \sum_{k=1}^{\infty} u_k \log|x_k|. \tag{11.1}$$

The study of this functional goes back to Gillespie [42] in 1981. The term *indicator function of X* was used in [55] and the name *entropy function of X* in [87] where it plays an important role in the solution of the distortion problem. As explained in [60] for any such Banach sequence space X the entropy function Φ_X extends to a quasi-linear map $\Phi_X : c_{00} \to \mathbb{R}$ which induces a minimal extension of ℓ_1.

The entropy function was used by Gillespie [42] to give a simple proof of the Lozanovsky factorization (Proposition 11.1). We treat the case of sequence spaces; with some extra technical work, the argument can be extended to function spaces. It suffices to prove the proposition if $u \in c_{00}$ (as then standard limiting arguments can be used, exploiting the Fatou property). For fixed u pick $x \in B_X$ with $x \geqslant 0$ to maximize the expression $\sum_{k=1}^{\infty} u_k \log|x_k|$. Now if $\xi \in B_X$ with $\xi \geqslant 0$ we have

$$\sum_{k=1}^{\infty} u_k \log\bigl(x_k + t(\xi_k - x_k)\bigr) \leqslant \sum_{k=1}^{\infty} u_k \log x_k, \quad 0 \leqslant t \leqslant 1.$$

Note that if $u_k > 0$ then we must have $x_k > 0$. Hence differentiating

$$\sum_{u_k \neq 0} u_k \frac{\xi_k - x_k}{x_k} \leqslant 0.$$

Let $x_k^* = u_k / x_k$ if $u_k \neq 0$ and 0 otherwise. Then

$$\sum_{k=1}^{\infty} \xi_k x_k^* \leqslant \sum_{k=1}^{\infty} u_k = 1$$

so that $\|x^*\|_{X^*} \leqslant 1$.

Uniqueness is immediate since if $u = xx^* = yy^*$ where $\|x\|_X = \|y\|_X = \|x^*\|_{X^*} = \|y^*\|_{Y^*}$ and $x, x^*, y, y^* \geqslant 0$ then

$$\sum_{k=1}^{\infty} \frac{1}{4}(x_k + y_k)\left(x_k^* + y_k^*\right) \leqslant 1$$

and this can only happen if $x_k = y_k$ and $x_k^* = y_k^*$ whenever $u_k \neq 0$.

Note that this implies that if $u \in c_{00}^+$,

$$\Phi_X(u) = \sum_{k=1}^{\infty} u_k \log|x_k|,$$

where $x \in B_X$ is determined by the Lozanovsky factorization of $u/\|u\|_1$. We can exploit to canonically extend Φ_X to c_{00} by *defining*

$$\Phi_X(u) = \sum_{k=1}^{\infty} u_k \log|x_k|,$$

where x is given by the Lozanovsky factorization of $|u|/\|u\|_1$. This definition of Φ_X is also quasi-linear on c_{00}.

Next we turn to complex interpolation of Banach sequence spaces, using the upper method which may be formulated as described at the end of Section 10. If X_0 and X_1 are two Banach sequence spaces we define $\mathcal{F}$ to be the subset of $\mathcal{G}$ of functions such that

$$\|F\|_{\mathcal{F}} := \max_{j=0,1} \operatorname*{ess\,sup}_{-\infty<t<\infty} \left\|F(j+\mathrm{i}t)\right\|_{X_j} < \infty$$

and then we obtain an analytic family X_z for $z \in \mathcal{S}$. In this case X_θ for $0 < \theta < 1$ is obtained by the *Calderón formula* $X_\theta = X_0^{1-\theta} X_1^\theta$, i.e., $x \in X_\theta$ if and only if

$$\|x\|_{X_\theta} := \inf\left\{\|x_0\|_{X_0}^{1-\theta} \|x_1\|_{X_1}^\theta \colon |x| = |x_0|^{1-\theta} |x_1|^\theta\right\} < \infty.$$

As we have seen in Theorem 4.6 above the spaces X_θ are simply the usual interpolation spaces for complex interpolation when either X_0 or X_1 has the Radon–Nikodym property (essentially this means one is separable under our restrictions). Note that the entropy functions linearize this interpolation method, i.e.,

$$\Phi_{X_\theta} = (1-\theta)\Phi_{X_0} + \theta\Phi_{X_1}.$$

The Lozanovsky factorization theorem gives the formula

$$\Phi_X + \Phi_{X^*} = \Lambda,$$

where $\Lambda = \Phi_{\ell_1}$ is given by

$$\Lambda(x) = \sum_{n=1}^{\infty} x_n \log \frac{|x_n|}{\|x\|_1}.$$

Thus Λ coincides on the positive cone with Ribe functional given by (9.1). Note all also that $\Phi_{\ell_p} = \frac{1}{p}\Lambda$ for $1 \leqslant p < \infty$ and in particular that $\Phi_{\ell_\infty} = 0$.

The following result is the specialization to our situation of Theorem 5.2 of [55], giving a full characterization of entropy functions:

THEOREM 11.2 ([55]). *Let $\Phi : c_{00}^{+} \to \mathbb{R}$ be any functional. Then in order that there exists a Banach sequence space X such that $\Phi_X = \Phi$ it is necessary and sufficient that:*
(1) *Φ is positive homogeneous, i.e., $\Phi(\alpha u) = \alpha\Phi(u)$ if $\alpha \geqslant 0$.*
(2) *Φ is convex.*
(3) *$\Lambda - \Phi$ is convex.*

Note that $(X, \|\cdot\|_X)$ is unique: in fact we can characterize B_X by

$$B_X = \left\{x\colon \sum_{k=1}^{\infty} u_k \log|x_k| \leqslant \Phi(u)\ \forall u \geqslant 0\right\}.$$

Moreover, the inequality

$$\left|\Phi_Y(u) - \Phi_X(u)\right| \leqslant C\|u\|_1, \quad u \geqslant 0,$$

is equivalent to the statement that $Y = X$ and

$$\mathrm{e}^{-C}\|x\|_Y \leqslant \|x\|_X \leqslant \mathrm{e}^{C}\|x\|_Y, \quad x \in X.$$

It is amusing to point out that this identification gives a "calculus" for Banach sequence spaces which can be used to prove an old extrapolation result of Pisier [92]:

COROLLARY 11.3 ([92]). *Let X be a Banach sequence space which is p-convex and q-concave where $1 < p < 2$ and $1/p + 1/q = 1$. Then there is a Banach sequence space Y such that $X = Y^{1-\theta}\ell_2^{\theta}$ where $\theta = 2/p - 1$.*

To prove Corollary 11.3 it suffices to note that X is p-convex (with constant one) if and only if there is a sequence space Y with $X = Y^{1/p}$ so that Φ_X satisfies that $\frac{1}{p}\Lambda - \Phi_X$ is convex. Similarly q-concavity (with constant one) means that $\frac{1}{p}\Lambda - \Phi_{X^*}$ is convex. If we now solve the equation $\Phi_X = (1-\theta)\Phi + \frac{1}{2}\theta\Lambda$ it can be shown that Φ and $\Lambda - \Phi$ are convex so that we can determine Y with $\Phi = \Phi_Y$.

Now fixing X_0 and X_1 let us describe the derived spaces dX_θ. Assume $x \in X_\theta$ and $\|x\|_{X_\theta} = 1$. Then there is an optimal factorization $|x| = |x_0|^{1-\theta}|x_1|^{\theta}$ where $x_j \in B_{X_j}$. It can be assumed that x_0, x_1 have the same support as X. The optimal choice of $F \in \mathcal{F}$ with $F(\theta) = x$ is given by $F(z) = x|x_1|^{z-\theta}|x_0|^{\theta-z}$. Thus dX_θ can be identified with $d_\Omega X_\theta$ where

$$\Omega x = x\big(\log|x_1| - \log|x_0|\big).$$

However if we similarly interpolate X_0^* and X_1^* then it follows from the calculus developed above (or in most cases from the basic duality theorem of [14]) that the corresponding intermediate spaces are given by X_θ^*. Now assume x can be normed by some $x^* \in X_\theta^*$ (e.g., if x^* is in the closure of c_{00}). Then x^* has a similar factorization $|x^*| = |x_0^*||x_1^*|$ and the corresponding optimal F^* is given by $F^*(z) = x^*|x_1^*|^{z-\theta}|x_0^*|^{\theta-z}$. Let us suppose $F(z) = (f_k(z))$ and $G(z) = (g_k(z))$. Then $\sum_{k=1}^{\infty} f_k(z)g_k(z)$ assumes its maximum value 1 at the interior point $z = \theta$ and hence is constant. It then follows that for each k the functions $f_k(z)g_k(z)$ can only take real values and thus individually constant. Thus $|x||x^*| = |x_0^*||x_1^*| = u$ say. Thus x_1 and x_0 are determined by the Lozanovsky factorization of $|x||x^*|$ where x^* norms x. Note also that we have

$$\langle \Omega x, x^*\rangle = \Phi_{X_1}(xx^*) - \Phi_{X_0}(xx^*)$$

if $x \in c_{00}$. This last formula can be extended to a larger domain but not to all $x \in X$. It may also be shown that arbitrary $x \in X$ and $x^* \in X^*$ (not necessarily norming x we have

$$\big|\langle \Omega(x), x^*\rangle - \Phi(xx^*)\big| \leqslant C\|x\|_X\|x^*\|_{X^*}, \quad x \in X,\ x^* \in X^*, \tag{11.2}$$

where $\Phi = \Phi_{X_1} - \Phi_{X_0}$, for a suitable constant C. There is thus an intimate relation between entropy functions and the derived spaces dX_θ. To illustrate the meaning of this statement suppose X_1 is obtain from X_0 by a change of weight, i.e., $X_1 = \{x\colon xw \in X_0\}$ for some positive weight sequence $w = (w_n)$ and $\|x\|_{X_1} = \|xw\|_{X_0}$. Then $\|x\|_{X_\theta} = \|xw^\theta\|_{X_0}$. In this case Φ is linear and

$$\Phi(u) = -\sum_{n=1}^{\infty} u_n \log w_n.$$

The spaces dX_θ which arise in this way are special self-extensions of X_θ, in the sense that the multiplication operators $M_a x = ax$ for $a \in \ell_\infty$ must naturally extend to dX_θ, since they are interpolating operators. In terms of Ω this means an estimate of the form

$$\big\|\Omega(ax) - a\Omega(x)\big\|_{X_\theta} \leqslant C\|a\|_\infty\|x\|_{X_\theta}.$$

We may say that the spaces dX_θ are *lattice self-extensions* (in [53] the term *lattice twisted sum* was used). This notion can be made precise but we will not do so here for lack of space.

Let us now consider the problem from the opposite direction. Suppose we fix a Banach sequence space X; we wish to characterize all lattice self-extensions. It can be shown that this reduces to looking at the notion of a *centralizer* (cf. [53]). A map $\Omega : X \to \omega$ is called a centralizer if it is homogeneous (i.e., $\Omega(\alpha x) = \alpha\Omega(x)$ for $\alpha \in \mathbb{C}$ and $x \in X$) and there is a constant C so that:

$$\left\|a\Omega x - \Omega(ax)\right\|_X \leqslant C\|a\|_\infty\|x\|_X, \quad a \in \ell_\infty,\ x \in X. \tag{11.3}$$

It is easy to show that a centralizer is quasilinear and so induces a self-extension $d_\Omega X$ which is a lattice self-extension.

Let us say that Ω is real if $\Omega(x)$ is real-valued whenever x is real-valued (in particular, this holds if we form a lattice self-extension of the real space X and then complexify). In general Ω is equivalent to $\Omega_1 + \mathrm{i}\Omega_2$ for suitable real centralizer Ω_1, Ω_2.

Now a key idea (and a rather simple calculation) is that every centralizer on X lifts to a centralizer on ℓ_1:

PROPOSITION 11.4 ([53]). *Let Ω be a real centralizer on X. Then there is real centralizer Ω' on ℓ_1 such that for a suitable constant C we have*

$$\left\|\Omega'(xx^*) - x^*\Omega(x)\right\|_1 \leqslant C\|x\|_X\left\|x^*\right\|_{X^*}, \quad x \in X,\ x^* \in X^*.$$

For $u \in c_{00}$ we define $\Phi(u) = \sum_{k=1}^\infty (\Omega'(u))_k$. From the fact that Ω' is a centralizer it follows that the series must converge and that $\Phi : c_{00} \to \mathbb{C}$ is quasilinear on ℓ_1 and takes real sequences to $\mathbb{R}$. Furthermore we have:

$$\left|\Phi(xx^*) - \langle\Omega(x), x^*\rangle\right| \leqslant C\|x\|\left\|x^*\right\|, \quad x \in X,\ x^* \in X^*. \tag{11.4}$$

We thus are led back to the problem of characterizing all quasi-linear maps on ℓ_1 or alternatively all minimal extensions of ℓ_1 (see [60].) If we compare (11.4) and (11.2) it is apparent that we can represent $d_\Omega X$ in the form dX_θ for suitable X_0, X_1 and $0 < \theta < 1$ if we show that Φ is equivalent to $\Phi_{X_1} - \Phi_{X_0}$. The main result we need is as follows:

THEOREM 11.5. *Suppose $0 < \varepsilon < 1$; then there is a constant C so that if $\Phi : c_{00}^+ \to \mathbb{R}$ is a map satisfying*:

(1) $\Phi(\alpha u) = \alpha\Phi(u)$, $\alpha \geqslant 0$, $u \in c_{00}^+$.

(2) $|\Phi(u+v) - \Phi(u) - \Phi(v)| \leqslant (1-\varepsilon)\log 2(\|u\|_1 + \|v\|_1)$, $u, v \in c_{00}^+$, *then there exists a Banach sequence space X so that*

$$\left|\Phi(u) - \left(\Phi_X(u) - \Phi_{X^*}(u)\right)\right| \leqslant C\|u\|_1, \quad u \in c_{00}^+.$$

Now if X is super-reflexive Theorem 11.5 can be applied to show (see [55]):

THEOREM 11.6. *Let X be a super-reflexive Banach sequence space and suppose Ω is a real centralizer on X. Then for some $c > 0$ there exist super-reflexive Banach sequence spaces X_0, X_1 so that $X = X_{1/2} = X_0^{1/2}X_1^{1/2}$ and the derived space $dX_{1/2}$ is induced by a centralizer Ω' equivalent to $c\Omega$, i.e.,*

$$\left\|\Omega'(x) - c\Omega(x)\right\|_X \leqslant C\|x\|_X, \quad x \in X.$$

In particular the lattice self-extension $d_\Omega X$ is isomorphic to the derived space $dX_{1/2}$.

We stress that this entire program can also be carried for function spaces and it is in this form that it is developed in [53] and [55].

12. Commutator estimates and their applications

We now revisit Proposition 10.1, which so far we have not exploited. Initially we continue to treat sequence spaces for technical reasons, but in fact our main interest lies in similar calculations for function spaces.

Let us suppose X_0 and X_1 are two reflexive Banach sequence spaces, and that $T : X_j \to X_j$ is a bounded operator for $j = 0, 1$. In this case the interpolation procedure described in the preceding section coincides with complex interpolation. Let $0 < \theta < 1$ we have the estimate for $X_\theta = X_0^{1-\theta} X_1^\theta$.

$$\big\|[T, \Omega]x\big\|_{X_\theta} \leqslant C\|x\|_{X_\theta}.$$

Suppose $x, x^* \in c_{00}$; then we have

$$\big|\big\langle \Omega(x), T^*x^*\big\rangle - \big\langle \Omega(Tx), x^*\big\rangle\big| \leqslant C\|x\|_{X_\theta}\big\|x^*\big\|_{X_\theta^*}.$$

Noting that $\Omega x \in c_{00}$ as well, we can apply 11.2 to obtain

$$\big|\Phi\big(x.T^*x^*\big) - \Phi\big(Tx.x^*\big)\big| \leqslant C\|x\|_{X_\theta}\big\|x^*\big\|_{X_\theta^*}, \quad x \in X,\ x^* \in X^*,$$

where $\Phi = \Phi_{X_1} - \Phi_{X_0}$. Noting that Φ is quasilinear we obtain the following estimate:

$$\big|\Phi\big(x.T^*x^* - Tx.x^*\big)\big| \leqslant C\|x\|_{X_\theta}\big\|x^*\big\|_{X_\theta^*}.$$

We can do slightly better than this by exploiting the Krivine theorem (see [72]) that T actually maps $X_j(\ell_2)$ to $X_j(\ell_2)$. We finally obtain:

LEMMA 12.1. *Suppose X_0, X_1 are reflexive Banach sequence spaces and $T \in \mathcal{L}(\overline{X}, \overline{X})$. If $0 < \theta < 1$ then there is a constant C so that if $x_1, \ldots, x_n, x_1^*, \ldots, x_n^* \in c_{00}$ then if $\Phi = \Phi_{X_1} - \Phi_{X_0}$.*

$$\left|\Phi\left(\sum_{n=1}^N x_n.T^*x_n^* - Tx_n.x_n^*\right)\right| \leqslant C\sum_{n=1}^N \|x_n\|_{X_\theta}\big\|x_n^*\big\|_{X_\theta}. \tag{12.1}$$

These calculations are carried out in [53] for function spaces, although the strengthening by using Krivine's theorem has not been observed before.

In order to push these estimates further we now suppose T is bounded on ℓ_{p_0} and ℓ_{p_1} where $p_0 < p_1$. If we fix $p_0 < p < p_1$ we can obtain (12.1) with

$$\Phi(u) = \big(p_0^{-1} - p_1^{-1}\big)\Lambda(u) = \big(p_0^{-1} - p_1^{-1}\big)\sum_{n=1}^\infty u_n \log|u_n|.$$

However more can be achieved if we apply Boyd's theorem for sequence spaces (Theorem 7.3) and note that T is bounded on any r.i. sequence space whose Boyd indices obey $p_0 < p_X < q_X < p_1$. Thus we can consider any pair X_1, X_0 of r.i. sequence spaces satisfying these conditions and $X_0^{1-\theta} X_1^{\theta} = \ell_p$. We then get a family of estimates. In fact it suffices to consider Orlicz sequence spaces and we must consider the family of quasilinear maps given by

$$\Phi(F; u) = \sum_{n=1}^{\infty} u_n F(\log|u_n|), \qquad F \in \mathrm{Lip}_1,$$

where Lip_1 is the family of Lipschitz maps $F : \mathbb{R} \to \mathbb{R}$ with Lipschitz constant one. We then get a *uniform* estimate.

Let us replace Lip_1 by the space of all $\mathrm{Lip}_{1,c}$ of all $F \in \mathrm{Lip}_1$ so that F' is compactly supported (i.e., F is constant on $(-\infty, -a]$ and on $[a, \infty)$ for some $a > 0$. For such F we have the advantage that $\Phi(F; u)$ is defined for all $u \in \ell_1$. Then we can define a quasi-Banach space h_1^{sym} (the *symmetric Hardy sequence space*) as the space of all sequence $\xi = (\xi_n)_{n=1}^{\infty}$ such that

$$\|\xi\|_{h^1_{\mathrm{sym}}} = \sum_{n=1}^{\infty} |\xi_k| + \sup_{F \in \mathrm{Lip}_{1,c}} \Phi(F; \xi) < \infty.$$

It is clear that $\xi \in h^1_{\mathrm{sym}}$ implies $\xi \in \ell_1$ and that $\sum_{n=1}^{\infty} \xi_n = 0$. A convenient description of h^1_{sym} is as follows. Suppose $\xi \in \ell_1$. Let $(\lambda_n)_{n=1}^{\infty}$ be a sequence such that $|\lambda_n|$ is monotonically decreasing and such that if $\alpha \neq 0$ the sets $\{n\colon \lambda_n = \alpha\}$ and $\{n\colon \xi_n = \alpha\}$ have the same cardinality. Then $\xi \in h^1_{\mathrm{sym}}$ if and only if

$$\sum_{n=1}^{\infty} \frac{|\lambda_1 + \cdots + \lambda_n|}{n} < \infty. \tag{12.2}$$

We refer to [53] and [54] for details. This equivalence actually hinges on replacing Orlicz spaces by Lorentz-type spaces in the above argument.

One can then show that the above ideas yield the following theorem:

THEOREM 12.2. *Suppose $1 < p_0 < p < p_1 < \infty$ and $1/p + 1/q = 1$. Suppose that $T : \ell_{p_j} \to \ell_{p_j}$ is bounded for $j = 0, 1$. Then the bilinear form $B_T(x, x^*) = x.T^*x^* - Tx.x^*$ is bounded from $\ell_p \times \ell_q$ to h^1_{sym}.*

Although we have given our exposition in terms of sequence spaces, the most interesting applications of these ideas are found with function spaces, where essentially the same steps can be made (with some annoying technicalities). Let us consider a measure space (K, μ) where K is a Polish space and μ is a σ-finite non-atomic Borel measure with either

$\mu(K) = 1$ or $\mu(K) = \infty$. We define the *symmetric Hardy function space* $H^1_{\mathrm{sym}}(K, \mu)$ to be the space of $f \in L_1(K, \mu)$ such that

$$\|f\|_{H^1_{\mathrm{sym}}} = \int_K |f| \, \mathrm{d}\mu + \sup_{F \in \mathrm{Lip}_{1,c}} \int_K |f| F\big(\log|f|\big) \, \mathrm{d}\mu < \infty.$$

As before H^1_{sym} is a quasi-Banach space of functions. A description such as (12.2) can be obtained. Let us note that $f \in H^1_{\mathrm{sym}}$ if and only if $\Re f, \Im f \in H^1_{\mathrm{sym}}$. If $f \in L_0(K_1, \mu_1)$ and $g \in L_0(K_2, \mu_2)$ are two measurable functions we write $f \sim g$ if whenever B is any Borel subset of $\mathbb{C} \setminus \{0\}$ we have $\mu_1(f_1 \in B) = \mu_2(f_2 \in B)$. Let us note that $f \in H^1_{\mathrm{sym}}$ if and only if $\Re f, \Im f \in H^1_{\mathrm{sym}}$. Therefore it suffices to consider real functions. If $f \in L_1(K, \mu)$ is real then we may find a function $f_d : \mathbb{R} \to \mathbb{R}$ so that f_d is decreasing and non-positive on $(-\infty, 0)$, decreasing and non-negative on $(0, \infty)$, and such $f_d \sim f$. That is, if B is any Borel subset of $\mathbb{R} \setminus \{0\}$ we have $\mu(f \in B) = m(f_d \in B)$ where m denotes either Lebesgue measure on $\mathbb{R}$. Now let

$$M(t) = \int_{-t}^{t} f_d(s) \, \mathrm{d}s, \quad t > 0.$$

Then $f \in H^1_{\mathrm{sym}}$ is equivalent to:

$$\int_0^\infty \frac{|M(t)|}{t} \, \mathrm{d}t < \infty. \tag{12.3}$$

The continuous analogue of Theorem 12.2 is given by:

THEOREM 12.3. *Suppose* $1 < p_0 < p < p_1 < \infty$ *and* $1/p + 1/q = 1$. *Suppose that* $T : L_{p_j}(K, \mu) \to L_{p_j}(K, \mu)$ *is bounded for* $j = 0, 1$. *Then the bilinear form* $B_T(f, g) = f.T^*g - Tf.g$ *is bounded from* $L_p \times L_q$ *to* $H^1_{\mathrm{sym}}(K, \mu)$.

Let us interpret this result by considering $K = \mathbb{T}$ and $\mathrm{d}\mu = (2\pi)^{-1} \mathrm{d}\theta$. It is natural to consider the Riesz projection (or equivalently the Hilbert transform) given by

$$Rf(\theta) \sim \sum_{n \geqslant 0} \hat{f}(n) \mathrm{e}^{\mathrm{i}n\theta},$$

where

$$\hat{f}(n) = \int_{\mathbb{T}} f\big(\mathrm{e}^{\mathrm{i}\theta}\big) \mathrm{e}^{-\mathrm{i}n\theta} \frac{\mathrm{d}\theta}{2\pi}.$$

Since R is bounded on L_p for $1 < p < \infty$ we can apply Theorem 12.3 when $p = q = 2$. Note that R^* is the Banach space adjoint of R (not the Hilbert space adjoint). Suppose

$f \in H_2$ and $g \in H_{2,0}$ (i.e., $g \in H_2$ with $g(0) = 0$). Then $R^*g = 0$ and $Rf = f$. Hence we obtain an estimate $\|fg\|_{H^1_{\mathrm{sym}}} \leqslant C\|f\|_2\|g\|_2$. But by standard factorization this reduces to

$$\|f\|_{H^1_{\mathrm{sym}}} \leqslant C\|f\|_1, \quad f \in H_{1,0}.$$

In particular if $f \in H_{1,0}$ then $\Re f$ satisfies (12.3). This result is, in fact, the main part of a theorem of Ceretelli [19], proved independently somewhat later by Davis [34].

THEOREM 12.4. *Let $f \in L_1(\mathbb{T})$ be real-valued. Then there exists $g \in H_{1,0}(\mathbb{T})$ with $\Re g \sim f$ if and only if $f \in H^1_{\mathrm{sym}}$.*

Next we consider non-commutative analogues of the ideas developed above. Suppose $\mathcal{H}$ is a separable Hilbert space. If X is a symmetric Banach sequence then we denote by $\mathcal{C}_X$ the space of all operators T whose singular numbers $s_n(T)$ satisfy

$$\|T\|_{\mathcal{C}_X} = \left\|\left(s_n(T)\right)_{n=1}^{\infty}\right\|_X < \infty.$$

In the case $X = \ell_1$ we obtain the trace-class $\mathcal{C}_1$.

Much of the foregoing theory can be generalized using the fact that $[\mathcal{C}_{X_0}, \mathcal{C}_{X_1}] = \mathcal{C}_{X_0^{1-\theta}X_1^{\theta}}$ if X_0, X_1 are reflexive Banach sequence spaces. Probably the most interesting application is to theory of commutators (or traces). A trace τ on a two-sided ideal of compact operators $\mathcal{J}$ is any linear map such that $\tau(AB) = \tau(BA)$ for all $a \in \mathcal{J}$ and $B \in \mathcal{B}(H)$. Let us define the commutator subspace $\operatorname{Comm}\mathcal{J}$ to be the space of all $A \in \mathcal{J}$ so that $\tau(A) = 0$ for all traces τ on $\mathcal{J}$. Then $\operatorname{Comm}\mathcal{J}$ is the linear span of all commutators $[A, B] = AB - BA$ for $a \in \mathcal{J}$ and $B \in \mathcal{B}(H)$. The problem of identifying $\operatorname{Comm}\mathcal{C}_1$ goes back to [89]. It was shown by Gary Weiss in 1980 [109] (see also [110]) that $\operatorname{Comm}\mathcal{C}_1$ does not coincide with $\{T \in \mathcal{C}_1\colon \operatorname{tr} T = 0\}$ or, equivalently, that there exist discontinuous traces on $\mathcal{C}_1$. The precise identification, however, requires the non-commutative analogue of h^1_{sym} and this was done by interpolation-style arguments in [54]. We define the eigenvalue sequence $(\lambda_n(T))_{n=1}^{\infty}$ for a compact operator T to be the sequence of non-zero eigenvalues of T repeated according to algebraic multiplicity, completed by zeros if there are only finitely many, and arranged so that $(|\lambda_n(T)|)_{n=1}^{\infty}$ is decreasing. There is some possible ambiguity here if T has two different eigenvalues with the same absolute value, but this does not cause problems.

THEOREM 12.5. *Suppose $T \in \mathcal{C}_1$. Then $T \in \operatorname{Comm}\mathcal{C}_1$ if and only its eigenvalue sequence $(\lambda_n(T))_{n=1}^{\infty} \in h^1_{\mathrm{sym}}$ or equivalently*

$$\sum_{n=1}^{\infty} \frac{|\lambda_1(T) + \cdots + \lambda_n(T)|}{n} < \infty.$$

It was shown in [54] that if $T \in \operatorname{Comm}\mathcal{J}$ then T is the sum of at most six commutators. This theorem has recently been improved dramatically following work of Dykema, Figiel, Wodzicki and Weiss [37] (see also [59] and [38]) where the case of general ideals is treated; in particular it is shown in [37] that three commutators suffice in Theorem 12.5.

One can introduce a quasi-normed analogue of h^1_{sym} denoted by $\mathcal{C}h_1 := \operatorname{Comm}\mathcal{C}_1$. It can be shown that this is made into a quasi-Banach space by the quasi-norm:

$$\|T\|_{\mathcal{C}h_1} := \|T\|_{\mathcal{C}_1} + \left\|\left(\lambda_n(T)\right)_{n=1}^{\infty}\right\|_{h_1^{\mathrm{sym}}}.$$

We remark that the quasi-Banach spaces h^1_{sym}, H^1_{sym} and $\mathcal{C}h_1$ are all examples of *logconvex* spaces. A quasi-Banach space X is logconvex if it satisfies an estimate of the type

$$\left\|\sum_{k=1}^{n} x_k\right\| \leqslant C\left(\sum_{k=1}^{n}\|x_k\|\log\frac{1}{\|x_k\|}\right)$$

whenever $\sum_{k=1}^{n}\|x_k\| = 1$. One can apply this to prove results of the following type:

THEOREM 12.6 ([54]). *Suppose (A_n) is a sequence of trace-class operators and B_n is a sequence of bounded operators. Suppose*

$$\sum_{n=1}^{\infty}\|A_n\|_{\mathcal{C}_1}\|B_n\|\left(1+\log_+\frac{1}{\|A_n\|_{\mathcal{C}_1}\|B_n\|}\right) < \infty.$$

Then $\sum_{n=1}^{\infty}[A_n, B_n] \in \operatorname{Comm}\mathcal{C}_1$.

Now suppose X is any symmetric sequence space. Then the entropy function Φ_X obeys an estimate

$$\left|\Phi_X(u)\right| \leqslant C\|u\|_{h^1_{\mathrm{sym}}}, \quad u \in h^1_{\mathrm{sym}}.$$

We can then define $\Phi_X(T) := \Phi_X((\lambda_n(T))_{n=1}^{\infty})$ for $T \in \mathcal{C}_1$. Suppose $S, T \in \mathcal{C}_1$. Consider the operator:

$$A = \begin{pmatrix} S+T & 0 & 0 \\ 0 & -S & 0 \\ 0 & 0 & -T \end{pmatrix}.$$

Then A is the sum of two commutators and it may shown that we have an estimate:

$$\|A\|_{\mathcal{C}h_1} \leqslant C\big(\|S\|_{\mathcal{C}_1} + \|T\|_{\mathcal{C}_1}\big).$$

This leads quickly to an estimate:

$$\left|\Phi_X(S+T) - \Phi_X(S) - \Phi_X(T)\right| \leqslant C\big(\|S\|_{\mathcal{C}_1} + \|T\|_{\mathcal{C}_1}\big).$$

Thus Φ_X is quasi-linear on $\mathcal{C}_1$ and induces a minimal extension of $\mathcal{C}_1$.

We close by remarking that it would be natural to attempt to classify all minimal extensions of $\mathcal{C}_1$ in a similar way to Theorem 11.5 (for ℓ_1). However it is not clear how to do this.

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Edited by

W.B. JOHNSON
Texas A&M University, College Station, Texas, USA

J. LINDENSTRAUSS
The Hebrew University of Jerusalem, Jerusalem, Israel

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