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Uniqueness of Unconditional Bases in Quasi-Banach Spaces

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ABSTRACT. We present an overview of the problem of uniqueness of unconditional basis up to permutation in quasi-Banach spaces, showing the latest results as well as the techniques (different from the locally convex case) used in the proofs.

1. Definitions and Notation

A (real) *quasi-Banach space* X is a complete metrizable vector space whose topology is given by a quasi-norm on X. That is, a map $\|\cdot\| : X \to \mathbb{R}$, $x \mapsto \|x\|$, satisfying

i) $\|x\| > 0 \quad (x \in X, x \neq 0)$
ii) $\|\alpha x\| = |\alpha|\|x\| \quad (\alpha \in \mathbb{R}, x \in X)$
iii) $\|x_1 + x_2\| \leq C(\|x_1\| + \|x_2\|) \quad (x_1, x_2 \in X)$,

where C is a constant independent of x_1 and x_2. If $C = 1$, X is a Banach space.

If, in addition, X is a vector lattice and $\|x\| \leq \|y\|$ whenever $|x| \leq |y|$ we say that X is a *quasi-Banach lattice.*

We recall that a quasi-Banach lattice X is said to be *p-convex*, where $0 < p < \infty$, if there is a constant $M > 0$ such that for any $n \in \mathbb{N}$ and any $x_1, \dots, x_n \in X$ we have

$$\Big\|\Big(\sum_{i=1}^{n} |x_i|^p\Big)^{1/p}\Big\| \leq M\Big(\sum_{i=1}^{n} \|x_i\|^p\Big)^{1/p}.$$

The procedure to define the element $(\sum_{i=1}^{n} |x_i|^p)^{1/p} \in X$ is exactly the same as in Banach lattices (see [**19**], pp. 40-41).

A quasi-Banach space X is called *natural* if it is isomorphic to a closed subspace of a quasi-Banach lattice which is p-convex for some $p > 0$ (see [**11**]).

When dealing with a quasi-Banach space X, it is often convenient to know which is the "smallest" Banach space containing X. To define it formally, if $(X, \|\cdot\|)$ is a quasi-Banach space whose dual separates the points of X, the *Banach envelope* of X, denoted by $\hat{X}$, is the completion of the normed space $(X, \|\cdot\|_c)$, where $\|\cdot\|_c$

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is the Minkowski functional of the convex hull of the unit ball of X. For instance, the Banach envelope of the sequence spaces ℓ_p $(0 < p < 1)$ is ℓ_1. It is easy to see that any unconditional basis of a quasi-Banach space X is also an unconditional basis of its Banach envelope $\hat{X}$, equivalent to its normalization and with the same unconditional constant. Our basic references for Banach envelopes are [**12**] and [**13**].

An unconditional basic sequence $(u_n)_{n=1}^{\infty}$ in X is *complemented* if there is a bounded projection $P : X \longrightarrow [u_n]$.

If $(e_n)_{n=1}^{\infty}$ is an unconditional basis of X and $(u_n)_{n=1}^{\infty}$ is an unconditional basic sequence of the form $u_n = \sum_{k\in\mathcal{A}_n} e_k^*(u_n)e_k$, where the sets $(\mathcal{A}_n)_n$ are disjoint subsets of $\mathbb{N}$, we say that $(u_n)_{n=1}^{\infty}$ is *disjoint* with respect to $(e_n)_{n=1}^{\infty}$.

Let $(u_n)_{n=1}^{\infty}$ be a set of disjointly supported vectors in the unconditional basis $(e_n)_{n\in\mathbb{N}}$ of X. Then, $(u_n)_{n=1}^{\infty}$ is an unconditional basic sequence which is complemented if and only if there exists a biorthogonal sequence $(u_n^*)_{n=1}^{\infty} \in X^*$, $u_n^*(u_m) = \delta_{nm}$, and such that the projection

$$P(x) = \sum_{n=1}^{\infty} u_n^*(x)u_n$$

is well defined and bounded.

2. The problem of uniqueness of unconditional basis in quasi-Banach spaces

If $(X, \|.\|)$ is a quasi-Banach space with a normalized unconditional basis $(e_n)_{n=1}^{\infty}$ (i.e. $\|e_n\| = 1$ for all $n \in \mathbb{N}$), we say that X has *unique unconditional basis* if whenever $(x_n)_{n=1}^{\infty}$ is another normalized unconditional basis of X, then $(x_n)_{n=1}^{\infty}$ is *equivalent* to $(e_n)_{n=1}^{\infty}$. That is, there is a constant D so that

$$D^{-1}\Big\|\sum_{i=1}^{n} a_i x_i\Big\| \leq \Big\|\sum_{i=1}^{n} a_i e_i\Big\| \leq D\Big\|\sum_{i=1}^{n} a_i x_i\Big\|,$$

for any choice of scalars $(a_i)_{i=1}^{n}$ and every $n \in \mathbb{N}$. For notation, we write $(x_n)_{n=1}^{\infty} \sim (e_n)_{n=1}^{\infty}$ when $(x_n)_{n=1}^{\infty}$ and $(e_n)_{n=1}^{\infty}$ are equivalent.

The problem of the uniqueness of unconditional basis is classical. The earliest results were obtained by Lindenstrauss and Pelczynski, who proved in 1968 that c_0 and ℓ_1 have a unique unconditional basis ([**18**]). On the other hand, it was well known that the Hilbert space ℓ_2 had a unique unconditional basis too ([**15**]). Lindenstrauss and Zippin proved in 1969 ([**20**]) that c_0, ℓ_1 and ℓ_2 are the only Banach spaces with this property. Therefore, the uniqueness of unconditional basis is a rare property for Banach spaces.

The situation for quasi-Banach spaces which are not Banach spaces is quite different. Indeed, since the structure of a locally bounded space which is not locally convex is more rigid, it is more difficult for such a space to have a basis. Because of that, when a space has an unconditional basis, it is more likely to be unique. For example, in 1977 it was shown in [**10**] that a wide class of non-locally convex Orlicz sequence spaces, including the spaces ℓ_p for $0 < p < 1$, have a unique unconditional basis. Nawrocki and Ortynski ([**21**]) investigated the Lorentz sequence spaces $d(w,p)$ for $0 < p < 1$, described their Banach envelopes and proved that if the sequence $\omega = (\omega_n)_{n=1}^\infty$ verifies

$$\inf_n \frac{(\omega_1 + \cdots + \omega_n)^{1/p}}{n} = 0$$

then $d(\omega, p)$ has uncountably many non-equivalent unconditional basis. Answering a question raised in [**21**], the authors proved in [**14**] that if $0 < p < 1$ and

$$\lim_{n\to\infty} \frac{(\omega_1 + \cdots + \omega_n)^{1/p}}{n} = \infty$$

then $d(w, p)$ has a unique unconditional basis.

If an unconditional basis is unique, then it must be equivalent to all its permutations and hence must be symmetric. Consequently, for spaces without symmetric basis it is more natural to consider the property of *uniqueness of unconditional basis up to permutation*: whenever $(x_n)_{n=1}^\infty$ and $(e_n)_{n=1}^\infty$ are two normalized unconditional bases of X, there is a permutation $\pi : \mathbb{N} \longrightarrow \mathbb{N}$ such that $(x_n)_{n\in\mathbb{N}} \sim (e_{\pi(n)})_{n\in\mathbb{N}}$.

In [**8**], Edelstein and Wojtaszczyk proved that any finite direct sum of the spaces c_0, ℓ_1 and ℓ_2 has a unique unconditional basis, up to permutation.

Attempting a complete classification of the Banach spaces with unique unconditional basis up to permutation, Bourgain, Casazza, Lindenstrauss and Tzafriri studied in [**4**] infinite direct sums of these spaces. They showed that $c_0(\ell_1)$, $c_0(\ell_2)$, $\ell_1(c_0)$ and $\ell_1(\ell_2)$ all have unique unconditional basis up to permutation, while the result is not true for $\ell_2(c_0)$ and $\ell_2(\ell_1)$. They also found an unexpected additional space, the 2-convexified Tsirelson space $T^{(2)}$ (see [**7**]) for the definition), with a unique unconditional basis up to permutation. Their results indicate that a complete classification of Banach spaces with this property is very unlikely to be achieved. More recently, further examples of "pathological" spaces with unique unconditional basis up to permutation have been provided in [**5**] and in [**9**].

It seemed only natural to translate the question of uniqueness of unconditional basis (up to permutation) into the setting of non-locally convex spaces that are infinite direct sums of the classical quasi-Banach spaces with a unique unconditional basis, which are the following:

$$\begin{aligned}
\ell_p(\ell_q) &= (\ell_q \oplus \ell_q \oplus \cdots \oplus \ell_q \oplus \cdots)_p \\
\ell_p(c_0) &= (c_0 \oplus c_0 \oplus \cdots \oplus c_0 \oplus \cdots)_p \\
\ell_p(\ell_1) &= (\ell_1 \oplus \ell_1 \oplus \cdots \oplus \ell_1 \oplus \cdots)_p \\
\ell_p(\ell_2) &= (\ell_2 \oplus \ell_2 \oplus \cdots \oplus \ell_2 \oplus \cdots)_p \\
c_0(\ell_p) &= (\ell_p \oplus \ell_p \oplus \cdots \oplus \ell_p \oplus \cdots)_0 \\
\ell_1(\ell_p) &= (\ell_p \oplus \ell_p \oplus \cdots \oplus \ell_p \oplus \cdots)_1 \\
\ell_2(\ell_p) &= (\ell_p \oplus \ell_p \oplus \cdots \oplus \ell_p \oplus \cdots)_2,
\end{aligned}$$

where $0 < p, q < 1$.

All the above can be seen as spaces of infinite matrices: if $(E, \|.\|_E)$ and $(F, \|.\|_F)$ are one of the quasi-Banach sequence spaces, $(\ell_p, \|.\|_p)$ $(p \in (0, 1] \cup \{2\})$ or $(c_0, \|.\|_\infty)$, then the space $E(F)$ consists of all infinite matrices $(x_{lk})_{l,k=1}^\infty$ satisfying $\mathbf{x}_l = (x_{lk})_{k=1}^\infty \in F$ for all $l \in \mathbb{N}$ and $(\|\mathbf{x}_l\|_F)_{l=1}^\infty \in E$. The quasi-norm of an element

$(x_{lk})_{l,k=1}^{\infty} \in E(F)$ is defined as

$$\|(x_{lk})_{l,k}\|_{E(F)} = \|(\|(x_{lk})_{k=1}^{\infty}\|_F)_{l=1}^{\infty}\|_E.$$

The space $E(F)$ is a quasi-Banach space whose Banach envelope is $\hat{E}(\hat{F})$ and whose dual can be identified with $E^*(F^*)$.

These spaces have a canonical unconditional basis that will be denoted by $(e_{lk})_{l,k=1}^{\infty}$. The (l,k) coordinate of $e_{l_0k_0}$ is 1 if $l = l_0$ and $k = k_0$, and 0 otherwise.

The lattice structure induced by the canonical basis on $E(F)$ is clearly p-convex if $(E, \|.\|_E)$ and $(F, \|.\|_F)$ are p-convex.

While the case of the space $\ell_2(\ell_p)$ $(0 < p < 1)$ remains open, the uniqueness of unconditional basis up to permutation has been proved for complemented subspaces with unconditional basis of all the other spaces. We can summarize the results in a theorem:

THEOREM 2.1. *Let X be one of the following quasi-Banach spaces: $\ell_p(\ell_q)$, $\ell_p(c_0)$, $\ell_p(\ell_1)$, $\ell_p(\ell_2)$, $c_0(\ell_p)$, and $\ell_1(\ell_p)$, $0 < p, q < 1$; and let Q be a bounded linear proyection from X onto a subspace Z which has a normalized unconditional basis $(x_n)_{n=1}^{\eta}$ with unconditional constant $K \geq 1$. Then, $(x_n)_{n=1}^{\eta}$ is equivalent to a permutation of a subbasis of the canonical basis of X. The equivalence constant only depends on K and $\|Q\|$. As a consequence, the following quasi-Banach spaces have unique unconditional basis up to permutation: $\ell_p \oplus \ell_q$, $\ell_p(\ell_q^n)_{n=1}^{\infty}$, $\ell_q \oplus \ell_p(\ell_q^n)_{n=1}^{\infty}$, $\ell_p(\ell_q)$, $\ell_p \oplus c_0$, $\ell_p(\ell_\infty^n)_{n=1}^{\infty}$, $c_0 \oplus \ell_p(\ell_\infty^n)_{n=1}^{\infty}$, $\ell_p(c_0)$, $\ell_1 \oplus \ell_p$, $\ell_1(\ell_p^n)_{n=1}^{\infty}$, $\ell_p \oplus \ell_1(\ell_p^n)_{n=1}^{\infty}$, $\ell_1(\ell_p)$, $c_0(\ell_p^n)_{n=1}^{\infty}$, $\ell_p \oplus c_0(\ell_p^n)_{n=1}^{\infty}$, $c_0(\ell_p)$, with $0 < p, q < 1$.*

The proof of the above theorem is quite different depending on the space X. Next, we sketch the proofs. For the sake of simplicity, we will take $Z = X$ and $(x_n)_{n=1}^{\infty}$ a normalized unconditional basis of X. The general case is analogous.

3. Uniqueness of the unconditional basis of $\ell_p(\ell_q)$ and $\ell_p(X)$, $0 < p, q < 1$.

The main references for this section are **[14]** and **[16]** .

For $0 < p, q \leq 1$ fixed, $\ell_p(\ell_q)$ is the space of infinite matrices $(x_{lk})_{l,k=1}^{\infty}$ satisfying

$$\|(x_{lk})_{l,k}\|_{p,q} = \Big(\sum_{l=1}^{\infty}\big(\sum_{k=1}^{\infty}|x_{lk}|^q\big)^{p/q}\Big)^{1/p} < \infty.$$

For each $0 < p, q < 1$, $(\ell_p(\ell_q), \|\cdot\|_{p,q})$ is a quasi-Banach space whose Banach envelope is $(\ell_1, \|\cdot\|_1)$ and whose dual space can be identified with ℓ_∞.

The lattice structure induced by the canonical basis on $\ell_p(\ell_q)$ is $\min\{p,q\}$-convex.

The proof of the uniqueness of unconditional basis in $\ell_p(\ell_q)$ relies on a technique, specific to the non-locally convex case, that has been used in all the proofs of uniqueness of unconditional basis so far. It was introduced in **[10]** to prove the uniqueness of unconditional basis in non-locally convex Orlicz sequence spaces (cf. Theorem 2.3 of **[14]**):

LEMMA 3.1 ("Large coefficients" technique). *Let X be a natural quasi-Banach space with normalized unconditional bases $(e_{lk})_{l,k=1}^{\infty}$ and $(x_n)_{n=1}^{\infty}$. Suppose there is a constant $\beta > 0$ (independent of n) and an injective map $\pi : S \subseteq \mathbb{N} \longrightarrow \mathbb{N}\times\mathbb{N}$ so that*

$$|e_{\pi(n)}^*(x_n)| \geq \beta \quad \text{and} \quad |x_n^*(e_{\pi(n)})| \geq \beta$$

for all $n \in S$. Then, there exist positive constants ρ, ρ' so that

$$\rho\Big\| \sum_{n\in S} \alpha_n e_{\pi(n)}\Big\| \le \Big\| \sum_{n\in S} \alpha_n x_n\Big\| \le \rho'\Big\| \sum_{n\in S} \alpha_n e_{\pi(n)}\Big\|$$

for any sequence of scalars (α_n) finitely non zero.

A normalized unconditional basis $(x_n)_{n=1}^\infty$ of a quasi-Banach space is called *strongly absolute* if for any $\varepsilon > 0$ there exist $C_\varepsilon > 0$ so that

$$\sum_{n=1}^N |\alpha_n| \le C_\varepsilon \sup_n |\alpha_n| + \varepsilon \|\sum_{n=1}^N \alpha_n x_n\|$$

for any choice of scalars $(\alpha_n)_{n=1}^N$ and $N \in \mathbb{N}$.

Intuitively, if a quasi-Banach space has a strongly absolute unconditional basis, it is far from being a Banach space.

LEMMA 3.2. *For $0 < p < 1$, the canonical basis of ℓ_p is strongly absolute.*

Lemma 3.2 is fundamental when we want to apply Lemma 3.1 to those quasi-Banach spaces in which ℓ_p $(0 < p < 1)$ is involved. It means that, if for a sequence $(a_n)_{n\in\mathbb{N}} \in \ell_p$ $(0 < p < 1)$, its ℓ_1-norm and its ℓ_p-quasi-norm are comparable, then so is its ℓ_∞ norm i.e if for some constants C_1, C_2, we have

$$C_1 \le \|(a_n)_n\|_1 \le \|(a_n)_n\|_p \le C_2,$$

then for $\varepsilon = \dfrac{C_1}{2C_2}$, we get

$$\|(a_n)_n\|_\infty \ge \frac{C_1}{2C_\varepsilon}.$$

So we can ensure the existence of a large coordinate in the sequence $(a_n)_n$.

Lemma 3.1 and Lemma 3.2 imply that any normalized unconditional basis $(x_n)_{n=1}^\infty$ of $\ell_p(\ell_q)$ is equivalent to a subset of the canonical basis $(e_{lk})_{l,k=1}^\infty$ and, therefore, equivalent to it (Theorem 2.7 of [**14**]).

This result is actually true for any space $\ell_p(X)$ $(0 < p < 1)$ of sequences of elements of a natural quasi-Banach space X such that

$$\|(y_n)_{n=1}^\infty\|_{\ell_p(X)} = \Big(\sum_{n=1}^\infty \|y_n\|_X^p\Big)^{1/p} < \infty,$$

whenever X has a strongly absolute normalized unconditional basis.

If $(e_n)_{n=1}^\infty$ is an unconditional basis of X then by repeating $(e_n)_{n=1}^\infty$ in each coordinate we get a strongly absolute unconditional basis of the natural quasi-Banach space $\ell_p(X)$ that we denote by $\ell_p(e_n)$. Using Lemma 3.1 and Lemma 3.2 the authors proved:

THEOREM 3.3. (Proposition 3.1 of [**14**]) *Let X be a natural quasi-Banach space with a strongly absolute normalized unconditional basis $(e_n)_{n=1}^\infty$, and assume $0 < p < 1$. Then, if $(x_n)_{n=1}^\infty$ is another normalized unconditional basis of X, the bases $\ell_p(e_n)$ and $\ell_p(x_n)$ of $\ell_p(X)$ are equivalent up to permutation.*

We improved the previous result by utilizing the following result of Wojtaszczyk:

LEMMA 3.4. (Theorem 2.12 of [**22**]) *Let Y be a natural quasi-Banach space with strongly absolute unconditional basis. Assume also that Y is isomorphic to some of its cartesian powers Y^s, $s = 2, 3, \ldots$ Then, all normalized unconditional bases in Y are permutatively equivalent.*

Now, as an easy consequence we obtain:

THEOREM 3.5. *Let X be a natural quasi-Banach space with a strongly absolute normalized unconditional basis $(e_n)_{n=1}^{\infty}$ (not necessarily unique up to permutation), and let $0 < p < 1$. Then, $\ell_p(X)$ has a unique unconditional basis up to permutation.*

4. Uniqueness of the unconditional basis of $c_0(\ell_p)$, $0 < p < 1$.

The main references for this section are [**16**] and [**17**].

For $0 < p \leq 1$ fixed, $c_0(\ell_p)$ is the space of infinite matrices $(x_{lk})_{l,k=1}^{\infty}$ such that $(x_{lk})_{k=1}^{\infty} \in \ell_p$ for all $l \in \mathbb{N}$, $\sum_{k=1}^{\infty} |x_{lk}|^p \overset{l\to\infty}{\longrightarrow} 0$, and

$$\|(x_{lk})_{l,k}\|_p = \sup_l \Big(\sum_{k=1}^{\infty} |x_{lk}|^p\Big)^{1/p} < \infty.$$

For each $0 < p < 1$, $(c_0(\ell_p), \|\cdot\|_p)$ is a quasi-Banach space whose Banach envelope is $(c_0(\ell_1), \|\cdot\|_1)$ and whose dual space can be identified with $\ell_1(\ell_\infty)$. That is, the Banach space of infinite matrices $a = (a_{lk})_{l,k=1}^{\infty}$ satisfying

$$\|a\| = \sum_{l=1}^{\infty} \sup_k |a_{lk}| < \infty.$$

The lattice structure induced by the canonical basis on $c_0(\ell_p)$ is p-convex.

Bourgain, Casazza, Lindenstrauss and Tzafriri proved in [**4**]:

THEOREM 4.1 (Theorem of uniqueness in $c_0(\ell_1)$). *Let $(x_n)_{n=1}^{\infty}$ be a normalized unconditional basis of $c_0(\ell_1)$. Then, there exist a constant Δ, and a partition of the integers into mutually disjoint subsets $(B_j)_{j=1}^{\infty}$, such that*

$$\Delta^{-1} \sup_j \sum_{n \in B_j} |a_n| \leq \Big\| \sum_{n=1}^{\infty} a_n x_n \Big\|_c \leq \Delta \sup_j \sum_{n \in B_j} |a_n| \tag{4.1}$$

for any finitely non-zero sequence of scalars $(a_n)_n$.

Our aim is to show:

THEOREM 4.2. (Theorem 3.2 of [**16**], Theorem 2.1 of [**17**]) *Let $(x_n)_{n=1}^{\infty}$ be a normalized unconditional basis of $c_0(\ell_p)$ $(0 < p < 1)$. Then, $(x_n)_{n=1}^{\infty}$ is equivalent to a permutation of a subbasis of the canonical basis of $c_0(\ell_p)$. That is, there exist constants Δ_1, Δ_2 and a partition of $\mathbb{N}$ into mutually disjoint subsets $(B_j)_{j=1}^{\infty}$, such that*

$$\Delta_1 \sup_j \Big(\sum_{n \in B_j} |a_n|^p\Big)^{1/p} \leq \Big\| \sum_{n=1}^{\infty} a_n x_n \Big\| \leq \Delta_2 \sup_j \Big(\sum_{n \in B_j} |a_n|^p\Big)^{1/p},$$

for any finitely non-zero sequence of scalars $(a_n)_n$.

Despite the fact that the duality techniques the authors used in Theorem 4.1 cannot be translated to the non-locally convex case, it is essential to our arguments that the property we want to prove for $c_0(\ell_p)$ is shared by its Banach envelope.

PROOF. From the fact that $(x_n)_{n=1}^{\infty}$ is a normalized unconditional basis of $c_0(\ell_p)$, it follows that $(x_n)_{n=1}^{\infty}$ is an unconditional basis of its Banach envelope $c_0(\ell_1)$, where Theorem 4.1 occurs.

First we see that for each j, $(x_n)_{n\in B_j}$ is equivalent to an ℓ_q-basis in the q-Banach envelope $c_0(\ell_q)$ for any $0 < p < q \le 1$, where $(B_j)_{j=1}^{\infty}$ is the partition in (4.1). To prove that, we use "large coefficients" techniques.

Then, we study the behaviour of the sequence $(x_n^*)_{n=1}^{\infty} \subset \ell_1(\ell_\infty)$, and prove that, except for a uniformly finite number of n's in each B_j, the norm $\|x_n^*\| = \sum_{l=1}^{\infty} \sup_k |x_n^*(e_{lk})|$ is concentrated in a uniformly finite number of l's. Hence, there are large coefficients among the coordinates of each of those x_n^*, and we use Lemma 3.1 to prove that the corresponding x_n are equivalent to a subbasis of the canonical basis of $c_0(\ell_p)$. The subbasis of $(x_n)_{n\in\mathbb{N}}$ of elements for which the norm $\|x_n^*\|$ is not concentrated in a uniformly finite number of l's is equivalent in $c_0(\ell_p)$ to a c_0-basis. □

5. Uniqueness of the unconditional basis of $\ell_p(c_0)$ and $\ell_p(\ell_2)$, $0<p<1$.

The main references for this section are [**1**] and [**2**].

For $0 < p \le 1$ fixed, $\ell_p(c_0)$ is the space of infinite matrices $(x_{lk})_{l,k=1}^{\infty}$ satisfying $x_{lk} \overset{k\to\infty}{\longrightarrow} 0$ for all $l \in \mathbb{N}$ and

$$\|(x_{lk})_{l,k}\|_p = \Big(\sum_{l=1}^{\infty} \sup_k |x_{lk}|^p\Big)^{1/p} < \infty.$$

For each $0 < p < 1$, $(\ell_p(c_0), \|\cdot\|_p)$ is a quasi-Banach space whose Banach envelope is $(\ell_1(c_0), \|\cdot\|_1)$ and whose dual space can be identified with $\ell_\infty(\ell_1)$, that is, the Banach space of infinite matrices $a = (a_{lk})_{l,k=1}^{\infty}$ satisfying

$$\|a\| = \sup_l \sum_{k=1}^{\infty} |a_{lk}| < \infty.$$

The lattice structure induced by the canonical basis on $\ell_p(c_0)$ is clearly p-convex.

THEOREM 5.1. (Theorem 2.1 of [**1**], Theorem 2.1 of [**2**]) *Let $(x_n)_{n=1}^{\infty}$ be a normalized unconditional basis for $\ell_p(c_0)$ $(0 < p < 1)$. Then, $(x_n)_{n=1}^{\infty}$ is equivalent to a permutation of a subbasis of the canonical basis of $\ell_p(c_0)$. That is, there exist constants Δ_1, Δ_2 and a partition of $\mathbb{N}$ into mutually disjoint subsets $(B_j)_{j=1}^{\infty}$, such that*

$$\Delta_1\Big(\sum_{j=1}^{\infty} \sup_{n\in B_j} |a_n|^p\Big)^{1/p} \overset{(2)}{\le} \Big\|\sum_{n=1}^{\infty} a_n x_n\Big\| \overset{(1)}{\le} \Delta_2\Big(\sum_{j=1}^{\infty} \sup_{n\in B_j} |a_n|^p\Big)^{1/p}, \tag{5.1}$$

for any finitely non-zero sequence of scalars $(a_n)_n$.

Bourgain, Casazza, Lindenstrauss and Tzafriri proved in [**4**]:

THEOREM 5.2 (Theorem of uniqueness in $\ell_1(c_0)$). *Let $(x_n)_{n=1}^{\infty}$ be a normalized unconditional basis for $\ell_1(c_0)$. Then, there exist a constant Δ, and a partition of the integers into mutually disjoint subsets $(B_j)_{j=1}^{\infty}$, such that*

$$\Delta^{-1}\sum_{j=1}^{\infty} \sup_{n\in B_j} |a_n| \le \Big\|\sum_{n=1}^{\infty} a_n x_n\Big\|_c \le \Delta\sum_{j=1}^{\infty} \sup_{n\in B_j} |a_n| \tag{5.2}$$

for any finitely non-zero sequence of scalars $(a_n)_n$.

As we saw in the $c_0(\ell_p)$-case, although the techniques used in Theorem 5.2 cannot be translated to the non-locally convex case, the result itself is essential to our arguments, and the partition given by that result is the one that works for our Theorem.

PROOF. We prove Theorem 5.1 in two parts corresponding to each of the inequalities (1) and (2) in the equation (5.1). Apart from Theorem 5.2, the proof of inequality (1) is based on the following result:

LEMMA 5.3. (Theorem 3.3 of [**13**]) *Let Y be a natural quasi-Banach space with unconditional basis such that Y^* has finite cotype. Then, there exists a constant A, depending only on p and the cotype constant of Y^*, such that*

$$\|y\|_Y \leq A\|y\|_c$$

for every $y \in Y$. (That is, Y is isomorphic to its Banach envelope.)

From the fact that $(x_n)_{n=1}^{\infty}$ is a normalized unconditional basis of $\ell_p(c_0)$, it follows that $(x_n)_{n=1}^{\infty}$ is an unconditional basis of its Banach envelope $\ell_1(c_0)$, where Theorem 5.2 applies.

For each j let us consider $X_j = \overline{\{x_n; n \in B_j\}}^{\ell_p(c_0)}$, where $(B_j)_{j=1}^{\infty}$ is the partition in (5.2). Then,

$$\hat{X}_j = \overline{\{x_n; n \in B_j\}}^{\ell_1(c_0)} \simeq c_0.$$

By Lemma 5.3, $X_j \simeq c_0$; or equivalently

$$\Big\| \sum_{n \in B_j} a_n x_n \Big\|^p \leq A^p \Delta^p \sum_{j=1}^{\infty} \sup_{n \in B_j} |a_n|^p.$$

Therefore,

$$\Big\| \sum_{n=1}^{\infty} a_n x_n \Big\|^p = \Big\| \sum_{j=1}^{\infty} \sum_{n \in B_j} a_n x_n \Big\|^p \leq \sum_{j=1}^{\infty} \Big\| \sum_{n \in B_j} a_n x_n \Big\|^p \leq A^p \Delta^p \sup_{n \in B_j} |a_n|^p$$

for any sequence of scalars $(a_n)_{n \in \mathbb{N}}$ finitely non zero.

In order to show inequality (2), we use Lemma 3.2 to find "large coefficients" in the coordinates of the elements x_n with respect to the canonical basis $(e_{lk})_{l,k=1}^{\infty}$. Furthermore, we find a $1-1$ map

$$\pi : \mathbb{N} \longrightarrow \mathbb{N} \times \mathbb{N}, \quad n \mapsto \pi(n) = (l, k)$$

so that $|e^*_{\pi(n)}(x_n)| > \beta$ for all $n \in \mathbb{N}$, for some positive constant β not depending on n.

This way, by Lemma 3.1, there is a constant $\rho > 0$ such that

$$\Big\| \sum_{n \in \mathbb{N}} a_n x_n \Big\| \geq \rho \Big\| \sum_{n \in \mathbb{N}} a_n e_{\pi(n)} \Big\| = \rho \Big(\sum_{j=1}^{\infty} \sup_{n \in B_j} |a_n|^p \Big)^{\frac{1}{p}}$$

for any finitely non-zero sequence of scalars $(a_n)_{n \in \mathbb{N}}$. □

Let us introduce now the other space we are dealing with in this section. For $0 < p \leq 1$ fixed, $\ell_p(\ell_2)$ is the space of infinite matrices $(x_{lk})_{l,k=1}^{\infty}$ satisfying

$$\|(x_{lk})_{l,k}\|_p = \Big(\sum_{l=1}^{\infty} \big(\sum_{k=1}^{\infty} |x_{lk}|^2 \big)^{p/2} \Big)^{1/p} < \infty.$$

For each $0 < p < 1$, $(\ell_p(\ell_2), \|\cdot\|_p)$ is a quasi-Banach space whose Banach envelope is $(\ell_1(\ell_2), \|\cdot\|_1)$ and whose dual space can be identified with $\ell_\infty(\ell_1)$, that is, the Banach space of infinite matrices $a = (a_{lk})_{l,k=1}^\infty$ satisfying

$$\|a\| = \sup_l \Big(\sum_{k=1}^\infty |a_{lk}|^2\Big)^{1/2} < \infty.$$

The lattice structure induced by the canonical basis on $\ell_p(\ell_2)$ is clearly p-convex.

Let us state the corresponding results of uniqueness of unconditional basis up to permutation for $\ell_p(\ell_2)$. The proof of this result is completely analogous to the proof of the $\ell_p(c_0)$ case:

THEOREM 5.4. (Theorem 2.10 of [**1**], Theorem 3.2 of [**2**]) *Let $(x_n)_{n=1}^\infty$ be a normalized unconditional basis for $\ell_p(\ell_2)$ $(0 < p < 1)$. Then, there exist constants Γ_1, Γ_2 and a partition of $\mathbb{N}$ into mutually disjoint subsets $(L_j)_{j=1}^\infty$ so that*

$$\Gamma_1\Big(\sum_{j=1}^\infty\big(\sum_{n\in L_j}|a_n|^2\big)^{p/2}\Big)^{1/p} \le \Big\|\sum_{n=1}^\infty a_n x_n\Big\| \le \Gamma_2\Big(\sum_{j=1}^\infty\big(\sum_{n\in L_j}|a_n|^2\big)^{p/2}\Big)^{1/p},$$

for any finitely non-zero sequence of scalars $(a_n)_{n\in\mathbb{N}}$.

6. Uniqueness of the unconditional basis of $\ell_1(\ell_p)$ and $\ell_p(\ell_1)$, $0 < p < 1$.

The main references for this section are [**1**] and [**3**].

For $0 < p < 1$ fixed, $\ell_1(\ell_p)$ is the quasi-Banach space of infinite matrices $(x_{lk})_{l,k=1}^\infty$ satisfying

$$\|(x_{lk})_{l,k}\|_{\ell_1(\ell_p)} = \sum_{l=1}^\infty\Big(\sum_{k=1}^\infty|x_{lk}|^p\Big)^{1/p} < \infty,$$

whereas $\ell_p(\ell_1)$ is the quasi-Banach space of infinite matrices $(x_{lk})_{l,k=1}^\infty$ such that

$$\|(x_{lk})_{l,k}\|_{\ell_p(\ell_1)} = \Big(\sum_{l=1}^\infty\big(\sum_{k=1}^\infty|x_{lk}|\big)^p\Big)^{1/p} < \infty.$$

The Banach envelope of both $\ell_1(\ell_p)$ and $\ell_p(\ell_1)$ $(0 < p < 1)$ is ℓ_1 and their dual spaces can be identified with ℓ_∞.

The lattice structure induced by the canonical basis, $(e_{lk})_{l,k=1}^\infty$, in $\ell_1(\ell_p)$ and $\ell_p(\ell_1)$ $(0 < p < 1)$ is p-convex.

Our goal is to show:

MAIN THEOREM. *Suppose $0 < p < 1$. Let $(x_n)_{n\in\mathbb{N}}$ be a normalized unconditional basis for $\ell_1(\ell_p)$ (respectively $\ell_p(\ell_1)$). Then $(x_n)_{n\in\mathbb{N}}$ is equivalent to a permutation of the canonical basis of $\ell_1(\ell_p)$ (respectively $\ell_p(\ell_1)$).*

The canonical basis of both $\ell_1(\ell_p)$ and $\ell_p(\ell_1)$ $(0 < p < 1)$ is also an unconditional basis of its Banach envelope, ℓ_1, where all normalized unconditional bases are equivalent and, therefore, symmetric:

THEOREM 6.1 (Uniqueness of unconditional basis of ℓ_1). ([**18**]) *Suppose $(x_n)_{n=1}^{\infty}$ is a normalized K-unconditional basis of ℓ_1. Then, there exists a constant D (depending only on K) so that*

$$D\sum_{n=1}^{N}|a_n| \leq \left\| \sum_{n=1}^{N} a_n x_n \right\| \leq \sum_{n=1}^{N}|a_n|$$

for any $(a_n)_{n=1}^{N}$ scalars and $N \in \mathbb{N}$.

Everything seemed to indicate that there might be a proof of uniqueness of the unconditional basis up to permutation for $\ell_p(\ell_1)$ $(0 < p < 1)$ similar to the ones for $\ell_p(c_0)$ and $\ell_p(\ell_2)$ and a completely different one for $\ell_1(\ell_p)$ $(0 < p < 1)$. The common point was that we were considering infinite direct sums of the only classical Banach spaces with unique normalized symmetric basis in the sense of ℓ_p $(0 < p < 1)$, and for the first two cases what really mattered was the ruling (strongly absolute) aspect of (the canonical basis of) ℓ_p. All our attempts in that direction failed.

The reason was that Theorem 6.1 does not give any information about the different behaviour of the subsets of an unconditional basis of $\ell_p(\ell_1)$ or $\ell_1(\ell_p)$ $(0 < p < 1)$ seen as unconditional basic sequences of their Banach envelope, in contrast with what happened in $c_0(\ell_p)$, $\ell_p(c_0)$ and $\ell_p(\ell_2)$, $0 < p < 1$. In these cases, the corresponding theorems of uniqueness of unconditional basis up to permutation in their Banach envelopes ([**4**]), where the canonical basis is not symmetric, were the starting point of the proofs (see [**17**], [**2**]). Since the canonical basis of ℓ_1 is symmetric we could not approach the proofs in the same way.

Eventually, we took notice of the fact that the lattice structure induced by any unconditional basis in ℓ_1, the Banach envelope of both $\ell_1(\ell_p)$ and $\ell_p(\ell_1)$ $(0 < p < 1)$, is *anti-Euclidean*, that is, ℓ_1 does not contain ℓ_2^n's as uniformly complemented sublattices. The authors had given in [**6**] a much simpler alternative proof to the uniqueness of unconditional basis up to permutation in the Banach space $c_0(\ell_1)$ using this fact (Corollary 2.5 of [**5**]). Their simplification partly depended on a useful characterization by the same authors of complemented unconditional basic sequences in Banach sequence spaces which are anti-Euclidean (see Theorem 3.5 of [**5**]). We generalized this theorem to natural quasi-Banach spaces (Theorem 3.2 of [**1**], Theorem 2.2 of [**3**]) and were highly rewarded by its importance in obtaining the following simplifications:

First simplification (Corollary 3.4 of [**1**], Corollary 2.4 of [**3**]) *Whenever we have a complemented unconditional basic sequence $(x_n)_{n=1}^{\eta}$ in either $\ell_1(\ell_p)$ or $\ell_p(\ell_1)$ $(0 < p < 1)$, we can suppose that $(x_n)_{n=1}^{\eta}$ is disjointly supported in the canonical basis (η can be either a natural number or infinity).*

Second simplification (Lemma 3.6 of [**1**], Lemma 2.6 of [**3**]) *If $(x_n)_{n=1}^{\eta}$ is a complemented unconditional basic sequence in either $\ell_1(\ell_p)$ or $\ell_p(\ell_1)$ $(0 < p < 1)$, we may further assume that supp $x_n^* \subset$ supp x_n and $x_n \geq 0$, $x_n^* \geq 0$.*

Third simplification (Lemma 3.8 of [**1**], Lemma 2.9 of [**3**]) *If $(x_n)_{n=1}^{\eta}$ is a complemented unconditional basic sequence in either $\ell_1(\ell_p)$ or $\ell_p(\ell_1)$ $(0 < p < 1)$ we may assume that all of the coordinates of x_n^* are "uniformly large".*

Basically, these results allow us to unravel the form in which any complemented, normalized unconditional basic sequence $(x_n)_{n=1}^{\eta}$ in either $\ell_1(\ell_p)$ or $\ell_p(\ell_1)$ $(0 < p < 1)$ can be written in terms of the canonical basis. Now we can establish:

THEOREM 6.2 (Main Theorem for $\ell_1(\ell_p)$). ((Theorem 3.9 of [1], Theorem 2.11 of [3]) *Let $(x_n)_{n=1}^{\eta}$ be a normalized, complemented, unconditional basic sequence in $\ell_1(\ell_p)$ $(0<p<1)$. Then, $(x_n)_{n=1}^{\eta}$ is equivalent to a subbasis of $(e_{lk})_{l,k=1}^{\infty}$.*

PROOF. First, using Lemmas 3.1 and 3.2 we prove that there is a complemented basic sequence $(y_n)_{n=1}^{\eta}$ equivalent to $(x_n)_{n=1}^{\eta}$, whose disjoint supports take an extremely simple form:

$$\begin{aligned} y_n &= \sum_{l\in F_n} e^*_{l\sigma_n(l)}(x_n)e_{l\sigma_n(l)} \\ y^*_n &= \sum_{l\in F_n} c^n_{l\sigma_n(l)}e^*_{l\sigma_n(l)} \quad (\omega^* \text{ convergence}) \end{aligned}$$

where the sets $(F_n)_{n=1}^{\eta}$ need not be mutually disjoint but they are finite. Furthermore, for each n we can assume that

$$c^n_{l\sigma_n(l)} > \frac{1}{2}, \quad \text{for all } l\in F_n.$$

By Lemma 3.1, it would be sufficient to show that $\|y_n\|_\infty > \delta$ for all n, for some positive constant δ independent of n. But, unfortunately, this does not necessarily have to be true for all n. So we proceed as follows:

Using Graph Theory we prove that there is a constant $\delta>0$ (independent of n) such that, if we consider the set

$$\mathcal{A} = \{n\in\{1,\dots,\eta\}\,;\, \|y_n\|_\infty < \delta\},$$

then $(y_n)_{n\in\mathcal{A}}$ is equivalent (in $\ell_1(\ell_p)$) to $(e_n)_{n\in\mathcal{A}}$, where $(e_n)_{n=1}^{\infty}$ denotes the canonical basis of ℓ_1. In particular, $(y_n)_{n\in\mathcal{A}}$ is equivalent to a subbasis of the canonical basis of $\ell_1(\ell_p)$.

To finish the proof we just have to observe that the basic sequence

$$\{y_n\,;\, n\notin\mathcal{A}\} = \{y_n\,;\,\|y_n\|_\infty > \delta\}$$

is equivalent to a subbasis of the canonical basis of $\ell_1(\ell_p)$. □

Next, we will see the corresponding results for $\ell_p(\ell_1)$, $0<p<1$.

THEOREM 6.3 (Main Theorem for $\ell_p(\ell_1)$). (Theorem 3.12 of [1], Theorem 2.14 of [3]) *Let $(x_n)_{n=1}^{\eta}$ be a normalized, complemented, unconditional basic sequence in $\ell_p(\ell_1)$ $(0<p<1)$. Then, $(x_n)_{n=1}^{\eta}$ is equivalent to a subbasis of $(e_{lk})_{l,k=1}^{\infty}$.*

PROOF. Using "large coefficients" techniques and the fact that the canonical basis of ℓ_p for $0<p<1$ is strongly absolute, we prove that $(x_n)_{n=1}^{\eta}$ is equivalent to a complemented, unconditional basic sequence $(y_n)_{n=1}^{\eta}$ with disjoint supports:

$$y_n = \sum_{k\in F^n_{l_n}} e^*_{l_nk}(x_n)e_{l_nk},$$

where, for each $n\in\{1,\cdots,\eta\}$ and $l\in\mathbb{N}$,

$$F^n_l = \{k\,;\, e^*_{lk}(x_n)\neq 0\}.$$

That is, F^n_l is the set consisting of the non-zero entries of the matrix $(e^*_{lk}(x_n))_{l,k=1}^{\infty}$ in the l^{th} row. Then, the support of y_n is contained in just a single row, namely l_n.

To finish the proof, we only have to observe that $(y_n)_{n=1}^{\eta}$ is equivalent to a subbasis of a normalized block basic sequence with respect to the canonical basis given by

$$u_{nl} = \frac{1}{\sum_{k \in F_l^n} e_{lk}^*(x_n)} \sum_{k \in F_l^n} e_{lk}^*(x_n) e_{lk},$$

for each $n \in \{1, \cdots, \eta\}$ and integer l for which $F_l^n \neq \emptyset$. □

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