A remark on Banach spaces isomorphic to their squares

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Abstract. We prove that if a Banach space $X$ has an unconditional basis and is "countably primary" then it is isomorphic to its square.

1. Introduction

This note originates from the work of Casazza and Lammers [2] on the so-called genus of a Banach space with unconditional basis. It is frequently important in the isomorphic theory of Banach spaces to know that a Banach space $X$ is isomorphic to its square $X^2 = X \oplus X$. For example a well-known variant of the Pelczyński decomposition technique states that if $Y$ is isomorphic to a complemented subspace of $X$, $X$ is isomorphic to a complemented subspace of $Y$ and both $X$ and $Y$ are isomorphic to their squares then $X$ is isomorphic to $Y$ (see [1]).

It is well-known that, even when $X$ has an unconditional basis it need not be isomorphic to its square. The first example was discovered by Figiel [3] but there is a more recent spectacular example due to Gowers [4] of a space $X$ with unconditional basis which is not isomorphic to any proper subspace of itself. However, in this note we show that if $X$ has an unconditional basis and, in a certain sense, rather few complemented subspaces then $X$ is isomorphic to its square (and also to its hyperplanes). The hypothesis needed is that $X$ is countably primary, i.e. there is a countable family of Banach spaces $(W_n)_{n=1}^\infty$ so that if $X = Y \oplus Z$ then either $Y$ or $Z$ is isomorphic to one of the spaces $(W_n)$. This includes certain important special cases. For example $X$ could be countably prime, i.e. have at most countably many non-isomorphic complemented subspaces, or it could be primary (so that $X = Y \oplus Z$ implies $X \approx Y$ or $X \approx Z$).

The proofs of these results depend on simple measure-theoretic ideas and very little is needed from Banach space theory. We have given here a very general statement in terms of unconditional Schauder decompositions in place of unconditional bases because we feel there may be potential applications in that setting. We refer the reader to [2] for some immediate applications.

We take the opportunity to spell out some connections with the so-called Schroeder-Bernstein property. We also note without proof that very similar results can be given for equivalence of bases (see Section 3).
2. The main results

Let us start with a measure theoretic proposition. We will identify the collection of all subsets \( P \mathbb{N} \) of \( \mathbb{N} \) with the Cantor set \( \Delta = \{0,1\}^{\mathbb{N}} \) under its usual product topology. Thus we identify a subset \( A \) of \( \mathbb{N} \) with the element \( \eta = (\eta_i)_{i=1}^{\infty} \) where \( A = \{ i : \eta_i = 1 \} \).

Now suppose \( X \) is a separable Banach space with an unconditional Schauder decomposition \((E_i)_{i=1}^{\infty}\). For each \( \eta \in \Delta \) we define \( X_\eta = \sum_{\eta_i=1} E_i \).

**Lemma 2.1.** Under the above hypotheses, suppose \( Y \) is any fixed separable Banach space. Then the set \( \{ \eta : X_\eta \approx Y \} \) is an analytic subset of \( \Delta \).

**Proof.** Consider the spaces \( L_1 = L(X,Y) \) and \( L_2 = L(Y,X) \) with the strong operator topologies. Let \( \mathcal{V} \) be the subset of \( \Delta \times L_1 \times L_2 \) of all \( (\eta, S, T) \) such that \( STy = y \) for \( y \in Y \) and \( TS(e) = \eta e \) if \( e \in E_i \). Now \( \mathcal{V} = \bigcup_{n \in \mathbb{N}} \mathcal{V}_n \) where \( \mathcal{V}_n = \{ (\eta, S, T) \in \mathcal{V} : \| S \|, \| T \| \leq n \} \). Each \( \mathcal{V}_n \) is easily seen to be closed, and since the strong operator topology coincides with a Polish topology on bounded sets (for separable Banach spaces) this implies each \( \mathcal{V}_n \) is Polish and so \( \mathcal{V} \) is analytic. Now let \( \pi \) be the projection on the first co-ordinate. It is clear that \( \pi(\mathcal{V}) \) is the set of \( \eta \) such that \( X_\eta \approx Y \). \( \square \)

Now suppose \( 0 < p \leq 1 \) and let \( \mu_p \) be the probability measure on \( \Delta \) such that \( \mu_p \{ \eta : X_\eta \approx Y \} > 0 \). There is a Banach space \( Z \) of the form \( Z \approx Y \oplus \sum_{i \in F} E_i \) where \( F \) is finite, such that \( Z \oplus E_n \approx Z \) eventually and so that with \( \mu_p \)-probability one we have \( X_{\eta_n} \approx Z \) eventually.

**Remark.** Of course we use here the fact from Lemma 2.1 that each of the sets listed is universally measurable.

**Proof.** Let \( \tau = \min(p/(1-p), (1-p)/p) \), and let \( \sigma = (1+\tau)^{-1} < 1 \). We argue first that there exists \( N \) and a choice \( \delta_i \in \{0,1\} \) for \( 1 \leq i \leq N \) so that the conditional probability

\[
\mu_p \{ X_\eta \approx Y | \eta_i = \delta_i, \ 1 \leq i \leq N \} > \sigma = (1+\tau)^{-1}.
\]

Indeed if we define the functions \( f_k \) by

\[
f_k(\xi) = \mu_p \{ X_\eta \approx Y | \eta_i = \xi_i, \ 1 \leq i \leq k \}
\]

then the functions \((f_k)\) form a \( \mu_p \)-martingale converging a.e. to the characteristic function of the set \( \{ \eta : X_\eta \approx Y \} \). It follows that there exists \( N \) so that if \( k \geq N \) then \( \max f_k = \alpha_k > \sigma \). We then choose \( \delta \in \Delta \) so that \( f_N(\delta) = \alpha_N \) and this determines our choice of \( \delta_1, \ldots, \delta_N \).

Now define \( Z = \sum_{\delta_i=\delta} E_i \oplus Y \). We will argue first that \( Z \oplus E_n \approx Z \) if \( n > N \). Indeed it suffices to show that \( Y \oplus E_n \approx Y \) if \( n > N \). Fix \( n > N \) and define \( \eta' \) by \( \eta'_i = \eta_i \) if \( i \neq n \) and \( \eta'_n = 1 - \eta_n \). We have the conditional probability

\[
\mu_p \{ X_{\eta'} \approx Y | \eta_i = \delta_i, \ 1 \leq i \leq N \} \geq \tau \mu_p \{ X_\eta \approx Y | \eta_i = \delta_i, \ 1 \leq i \leq N \}.
\]
Since \((1 + \tau)\sigma = 1\) there exists \(\eta\) so that \(X_\eta \approx X_{\eta'} \approx Y\) so that \(Y \oplus E_n \approx E_n\).

Note that \(\mu_p(X_{\eta,n} \approx Z) \geq \alpha_n\). It follows from the preceding that the sets \(\{\eta : X_{\eta,n} \approx Z\}\) increase in \(n\) for \(n \geq N\). Hence \(\mu_p(X_{\eta,n} \approx Z) \geq \alpha_N\) for \(n \geq N\). However we can repeat the reasoning for each \(n \geq N\) to produce a space \(Z_n\) so that \(\mu_p(X_{\eta,n} \approx Z_n) \geq \alpha_n\). Since \(\alpha_n, \alpha_n > 1/2\) we have \(Z_n \approx Z\) for all \(n\) and so \(\lim_{n \to \infty} \mu_p(X_{\eta,n} \approx Z) = 1\). The Lemma now follows. \(\square\)

We now define a Banach space \(X\) to be countably primary if there is a countable set \(\Gamma\) of (equivalence classes, up to isomorphism) of Banach spaces so that if \(X = Y \oplus Z\) then either \(Y \in \Gamma\) or \(Z \in \Gamma\). Let us also define \(X\) to be countably prime if it has only countably many distinct complements (up to isomorphism).

If \(X\) is a Banach space with an unconditional Schauder decomposition \((E_n)\) we will say that \((E_n)\) is a countably primary (resp. countably prime) decomposition if there is a countable set \(\Gamma\) of Banach spaces so that if \(A \subset N\) then either \(\sum_{e \in A} E_e\) or \(\sum_{e \notin A} E_e \in \Gamma\) (resp. \(\sum_{e \in A} E_e \in \Gamma\)). Clearly any unconditional Schauder decomposition of a countably primary (resp. countably prime) space is countably primary (resp. countably prime).

**Theorem 2.3.** Suppose \(X\) is a separable Banach space with a countably primary unconditional Schauder decomposition \((E_n)\). Then:

1. \(X \oplus E_n \approx X\) eventually.
2. \(X \oplus \sum_{i=n+1}^{\infty} E_i \approx X\) eventually.
3. For every \(\frac{1}{2} < p < 1\) we have \(X_{\eta,n} \approx X\) eventually \(\mu_p\)-a.e.

Under the stronger hypotheses that either \(X\) is countably primary or \((E_n)\) is a countably prime decomposition, we have:

4. \(X_{\eta,n} \approx X\) eventually \(\mu_p\)-a.e. for \(0 < p < 1\).

**Proof.** We start by observing that we can suppose that the set \(\Gamma\) contains each \(E_n\) and \(X\) and is closed under finite direct sums.

If \(\eta \in \Delta\) let \(\tilde{\eta}_i = 1 - \eta_i\). Then \(X \approx X_\eta \oplus X_{\tilde{\eta}}\). Hence either \(X_\eta \in \Gamma\) or \(X_{\tilde{\eta}} \in \Gamma\). For any \(0 < p < 1\) we thus have either \(\mu_p(X_\eta \in \Gamma) > 0\) or \(\mu_p(X_{\tilde{\eta}} \in \Gamma) > 0\). The latter inequality is equivalent to \(\mu_{1-p}(X_\eta \in \Gamma) > 0\). If we let \(S = \{\eta : \mu_p(X_\eta \in \Gamma) > 0\}\) then for any \(p\) either \(p\) or \(1 - p - S\). Note that \(S \subset (0,1)\). Unfortunately it is not immediate that \(S\) is Lebesgue measurable, but we can easily see that its outer Lebesgue measure \(\lambda^*(S) \geq \frac{1}{2}\).

For each \(p \in S\) there exists \(Z_p \in \Gamma\) so that \(X_{\eta,n} \approx Z_p\) eventually \(\mu_p\)-a.e. We also have \(Z_p \oplus E_n \approx Z_p\) eventually. Since \(Z_p\) is complemented in \(X\) this implies that \(X \oplus E_n \approx X\) eventually. This proves the first claim (1).

For \(W \in \Gamma\) let \(S_W = \{\eta : Z_p \approx W\} \subset S\). We first observe that for each \(W \in \Gamma\) the sets \(\{\eta : X_\eta \approx W\}\) are analytic and disjoint. Hence by the Borel separation theorem (see [7] Proposition 4.4.4, p. 148) there are disjoint Borel sets \(\{\tilde{B}_W : W \in \Gamma\}\) so that \(X_\eta \approx W\) implies \(\eta \in \tilde{B}_W\). If \(\eta \in \Delta\), let us define \(\eta^{[n]}\) by \(\eta_i^{[n]} = 1\) if \(i \leq n\) and \(\eta_i^{[n]} = \tilde{\eta}_i\) otherwise. If we then define \(\tilde{B}_W\) as the set of \(\eta\) so there exists \(n_0\) such that \(\eta^{[n]} \in B_W\) for all \(n \geq n_0\). Then the sets \(\tilde{B}_W\) are also disjoint and Borel and contain \(S_W\). If \(\mu_p(\tilde{B}_W) = 1\) then either \(p \notin S\) or \(p \in S\). The maps \(r \rightarrow \mu_r(\tilde{B}_W)\) are Borel (this follows easily from the fact that the map \(r \rightarrow \mu_r\) is weak*-continuous) and so the sets \(B_W = \{r : \mu_r(\tilde{B}_W) = 1\}\) are Borel subsets of \((0,1)\). Note that \(S_W \subset B_W\) and \(B_W \cap (1 - B_V) \subset S_W \cup (1 - S_V)\) by the above remarks.

We will now show that \(X_{\eta,n} \approx X\) eventually \(\mu_p\)-a.e. if \(\frac{1}{2} < p \leq 1\).
Note that \( I = (1 - p, p) \subset \cup_{W \in \Gamma} S_W \cup \cup_{V \in \Gamma} (1 - S_V) \). It follows that there exists \( W \in \Gamma \) so that \( \lambda^*(S_W \cap I) > 0 \). Now either \( S_W \cap I \) is Lebesgue measurable (and so \( I \cap (S_W \cup (1 - S_W)) \) contains a measurable set of positive measure, or \( \lambda^*(I \cap (B_W \setminus S_W)) > 0 \). In the latter case there exists \( V \in \Gamma \) so that \( \lambda^*(I \cap ((B_W \setminus S_W) \cap (1 - S_V))) > 0 \). In this latter case \( I \cap B_W \cap (1 - B_V) \) has positive measure and so \( S_W \cup (1 - S_V) \) contains a measurable set of positive measure. Thus we can assume the existence of \( W, V \in \Gamma \) so that \( I \cap (S_W \cup (1 - S_V)) \) contains a Lebesgue measurable set \( T \) of positive measure.

We now use a slight variant of a well-known theorem of Steinhaus: thus there exists \( \epsilon > 0 \) so that if \( 0 < r < \epsilon \) there exists \( t \in T \) so that \( t \pm r \in T \). (This can be proved by considering the continuous function \( r \to \int \chi_T(t)\chi_T(t+r)\chi_T(t-r)dt \).)

Now it is clear that at least two of the points \( t - r, t, t + r \) belong to the same set \( S_W \) or \( 1 - S_V \). Hence with \( d = r \) or \( d = 2r \) we have that either there exists \( q \in (1 - p, p) \) with either \( q, q + d \in S_W \) or \( q, q + d \in S_V \).

Now since this can be done for all small enough \( r \) we may suppose that \( (1 - p)/d = m \in \mathbb{N} \). Now let \( \theta_i \) be a sequence of independent random variables on some probability space \( (\Omega, P) \) with common uniform distribution \( P(\theta_i \in B) = \lambda(B \cap [0, 1]) \).

We observe that with probability one we have eventually,

\[
\sum_{i \leq n} E_i \oplus \sum_{i > n, \theta_i < q} E_i \approx \sum_{i \leq n} E_i \oplus \sum_{i > n, \theta_i < q + d} E_i
\]

so that for \( 1 \leq s \leq m \), we have eventually,

\[
\sum_{i \leq n} E_i \oplus \sum_{i > n, \theta_i < q} E_i \approx \sum_{i \leq n} E_i \oplus \sum_{i > n, \theta_i < q} E_i \oplus \sum_{p+(s-1)d \leq \theta_i < p+sd} E_i.
\]

Iterating, since \( d \) divides \( 1 - p \) we have

\[
\sum_{i \leq n} E_i \oplus \sum_{i > n, \theta_i < q} E_i \oplus \sum_{i > n, \theta_i \geq p} E_i \approx \sum_{i \leq n} E_i \oplus \sum_{i > n, \theta_i < q} E_i
\]

eventually with probability one. Hence, by adding the same factor to both sides to have:

\[
\sum_{i \leq n} E_i \oplus \sum_{i > n, \theta_i < p} E_i \approx X
\]

eventually with probability one. This implies that \( X_{n,n} \approx X \) eventually \( \mu_p \)-a.e. as claimed in (3).

Now using the same notation we observe that with probability one we have eventually (using the case \( p = \frac{1}{2} \)),

\[
X \oplus \sum_{i > n} E_i \approx \sum_{i \leq n} E_i \oplus \sum_{i > n, \theta_i \leq \frac{1}{2}} E_i \oplus \sum_{i > n, \theta_i > \frac{1}{2}} E_i
\]

and hence also

\[
X \oplus \sum_{i > n} E_i \approx \sum_{i \leq n} E_i \oplus \sum_{i > n, \theta_i \leq \frac{1}{2}} E_i \oplus \sum_{i > n, \theta_i \geq \frac{1}{2}} E_i.
\]
However it is also clear that (again using the case $p = \frac{2}{3}$)

$$X \approx \sum_{i \leq n} E_i \oplus \sum_{i > n}^{\theta_i \leq \frac{2}{3}} E_i \oplus \sum_{i > n}^{\theta_i \geq \frac{2}{3}} E_i$$

eventually with probability one. Thus we have $X \oplus \sum_{i > n} E_i \approx X$ eventually, i.e. we have (2).

To prove the final claim (4) notice that under either stronger hypothesis we have that if $\sum_{i \in A} E_i \approx X$ then $(E_n)_{n \in A}$ is countably primary. We will deduce that if $X_{\eta,n} \approx X$ eventually $\mu_p$-a.e. then $X_{\eta,n} \approx X$ eventually $\mu_q$-a.e. where $q = p^2$. Indeed suppose we consider $\Delta \times \Delta$ with the product measure $\mu_p \times \mu_p$. Then, denoting a typical element of the product by $(\xi, \eta)$ we observe that $X_{\xi,\eta,n} \approx X$ eventually a.e. and so using Fubini’s theorem $X_{\xi,\eta,n} \approx X$ eventually a.e. (where $\xi$ is the pointwise product). This implies that $X_{\eta,n} \approx X$ eventually $\mu_q$-a.e. and so the proposition is established.

Remark. Let us point out that the preceding proof can be simplified in the special case when $(E_n)$ is countably prime. The main point is then that the sets $\{\eta : X_\eta \approx W\}$ are both analytic and co-analytic and hence are Borel sets. This means that the sets $S_W$ are already Borel sets and one only needs to use the fact that if $S_W$ has positive Lebesgue measure then $S_W - S_W$ contains a neighborhood of the origin.

**Theorem 2.4.** Suppose $X$ is a separable Banach space with a countably primary unconditional Schauder decomposition $(E_n)$. Then there exists $N$ so that if $A$ is any subset of $\{N+1, N+2, \cdots\}$ we have

$$X \oplus \sum_{i \in A} E_i \approx X.$$

**Proof.** By Theorem 2.3 there exists $N$ so that $X \oplus E_n \approx X$ if $n > N$ and $X \oplus \sum_{i > N} E_i \approx X$. Suppose $A \subseteq \{N+1, N+2, \cdots\}$. Then consider the unconditional Schauder decomposition $\{\sum_{i \in A} E_i\} \cup \{E_j\}_{j \notin A}$. This also is countably primary and so there is a finite subset $F$ of $\mathbb{N} \setminus A$ so that $X \oplus \sum_{i \notin (A \cup F)} E_i \approx X$. We may assume $F$ contains $\{1, 2, \cdots, N\}$. Now we deduce $X \approx X \oplus \sum_{i > N} E_i \approx X \oplus \sum_{i \in A} E_i$. \qed

Our main applications concern the case when $X$ has an unconditional basis.

**Theorem 2.5.** Suppose $X$ is a separable Banach space with a countably primary unconditional basis $(u_i)_{i=1}^\infty$. Then:

1. $X \approx X \oplus \mathbb{R}$.
2. $X \approx X^2$.
3. If $Y$ is a subspace of $X$ spanned by a subsequence $(u_i)_{i \in A}$ then $X \oplus Y \approx X$.

**Proof.** Of course (1) is immediate from Theorem 2.3 (1) and then (2) follows from (2) of the same theorem. (3) follows from Theorem 2.4. \qed

**Theorem 2.6.** Suppose $X$ is a separable Banach space with an unconditional basis $(u_i)$ so that either $X$ is countably primary or $(u,i)$ is countably prime. Then we may partition $\mathbb{N}$ into sets $(A_k)_{k=1}^\infty$ so that each $[u_i]_{i \in A_k} \approx X$ and for any nonempty subset $B$ of $\mathbb{N}$ we have $[u_i : i \in \bigcup_k B \cup \bigcup_i \in A_k] \approx X$. 


PROOF. Let us again consider the sequence of independent random variables \((\theta_i)\) each uniformly distributed on \([0, 1]\). Applying Theorem 2.3 and Theorem 2.5 we see that with probability one we have for \(k \in \mathbb{N}\), \([u_i]_{i=k}^{2e\log_2(\theta_i)+1-k} \approx X\). It follows that we can choose the sets \(A_k\) so that \([u_i : i \in A_k]\) \(\approx X\).

Now notice that if \(D \supseteq A_k\) for some \(k\) then if \(Y = [u_i]_{i \in D}\) we have \(Y \approx X \oplus Z\) for some \(Z\) and hence \(Y \approx X \oplus Y \approx X\) by Theorem 2.5.

Let us give some applications. We recall that a Banach space \(X\) has the *Schroeder-Bernstein property* if whenever \(Y\) is a complemented subspace of \(X\) which contains a further complemented subspace isomorphic to \(X\) then \(Y \approx X\). See [1] for a discussion of this property and [5] for an example of a space \(X\) failing the Schroeder-Bernstein property. There is apparently no known example of a space \(X\) with an unconditional basis failing the Schroeder-Bernstein property. Note that if \(X \approx X^2\) then \(X\) has the Schroeder-Bernstein property if and only if \(X \oplus Y \approx X\) for every complemented subspace \(Y\) of \(X\).

If \(X\) has an unconditional finite-dimensional decomposition (UFDD) we will say that \(X\) has the *restricted Schroeder-Bernstein property* if whenever \(Y\) is complemented in \(X\), has a (UFDD) and contains a complemented subspace isomorphic to \(X\) then \(X \approx Y\). Let us remark that there is no known example of a complemented subspace of a Banach space with unconditional basis which fails to have a (UFDD). There is an unpublished example of Read [6] of a Banach space with a (UFDD) and a complemented subspace without a (UFDD).

**Proposition 2.7.** Let \(X\) be a Banach space with a (UFDD). Then the following are equivalent:

1. \(X \approx X^2\) and \(X\) has the restricted Schroeder-Bernstein property.
2. \(X \oplus Y \approx X\) for every complemented subspace \(Y\) of \(X\) with a (UFDD).
3. \(X \oplus Y \approx X\) for every complemented subspace \(Y\) of \(X\) so that \(X/Y\) has a (UFDD).

**PROOF.** We leave the elementary proof that (1) and (2) are equivalent to the reader. Let us prove (2) and (3) are equivalent. Obviously both imply that \(X \approx X^2\). Now suppose \(X \approx Y \oplus Z\). We claim that if \(X \oplus Y \approx X\) then \(X \oplus Z \approx X\). Indeed \(X \approx X^2 \approx X \oplus Y \oplus Z \approx X \oplus Z\). This clearly establishes the equivalence of (2) and (3). \(\Box\)

**Theorem 2.8.** Let \(X\) be a countably primary Banach space with an unconditional basis. Then \(X\) has the restricted Schroeder-Bernstein property.

**PROOF.** Of course \(X\) satisfies the conclusions of Theorem 2.5. Let \(X \approx Y \oplus Z\) where \(Y\) has a (UFDD) \((E_n)\). Then \(\{Z, (E_n)\}_{n=1}^{\infty}\) forms an unconditional Schauder decomposition of \(X^2 \approx X\). Hence by Theorem 2.3 we can deduce that \(X \approx X \oplus Y_0\) where \(Y_0\) has finite codimension in \(Y\). But then \(X \oplus Y \approx X \oplus V\) where \(V\) is finite-dimensional and so by Theorem 2.5 \(X \approx X \oplus Y\). \(\Box\)

**Theorem 2.9.** Let \(X\) be a Banach space with an unconditional basis. Suppose there are at most countably many non-isomorphic Banach spaces \(Y\) which are complemented in \(X^2\) and contain a complemented subspace isomorphic to \(X\). Then \(X^2 \approx X^3\) and \(X^2\) has the restricted Schroeder-Bernstein property.

**PROOF.** First note that \(X^2 = X \oplus X\) has an unconditional Schauder decomposition consisting of \(X \oplus \{0\}\) and the unconditional basic sequence \((0, u_n)\). This
is countably primary and so by Theorem 2.3 we obtain both $X^2 \cong X^2 \oplus R$ and $X^2 \cong X^2 \oplus [u_n]_{n \geq N + 1}$ for some $N$. These combine to give $X^2 \cong X^3$. In particular $(X^2)^2 \cong X^2$.

Now suppose $Y$ is a complemented subspace of $X^2$ with a (UFDD). Then $X \oplus Y$ has a countably primary Schauder decomposition $(E_n)$ with $E_1 \cong X$ and $E_n$ finite-dimensional for $n \geq 2$. Hence by Theorem 2.3 there is a finite-codimensional subspace $Y_0$ of $Y$ so that $X \oplus Y \oplus Y_0 \cong X \oplus Y$. Thus $X^2 \oplus Y^2 \cong X^2 \oplus Y$. It follows that $(X^2 \oplus Y)^2 \cong X^2 \oplus Y$ and hence the usual decomposition argument (cf. [1]) gives $X^2 \oplus Y \cong X^2$. By Proposition 2.7 we are done. \hfill\Box

3. Equivalence of unconditional bases

In this short final section we point out that very similar ideas can be applied to equivalence of unconditional bases in place of isomorphisms. If $(u_n)_{n=1}^\infty$ is an unconditional basis of $X$ and $(v_n)_{n=1}^\infty$ an unconditional basis of $Y$ we will say that $(u_n)$ and $(v_n)$ are permutatively equivalent if there is an isomorphism $T : X \to Y$ and a permutation $\pi$ of $N$ so that $Tu_i = v_{\pi(i)}$. This definition extends naturally to an unconditional basis indexed by any countable set in place of $N$.

One easily proves the following lemma similarly to Lemma 2.1.

**Lemma 3.1.** Let $(u_n)$ be an unconditional basis of a Banach space $X$ and let $(v_n)$ be an unconditional basis of a Banach space $Y$. The set of $\eta \in \Delta$ such that $(u_{\eta})_{\eta \in \Delta}$ is permutatively equivalent to $(v_n)$ is analytic.

Now arguments very similar to those for Theorem 2.4 establish that:

**Theorem 3.2.** Let $(u_n)$ be an unconditional basis of a Banach space $X$ and suppose that there is a countable family of unconditional bases $\Gamma$ so that for every infinite subset $A$ of $N$ either $(u_n)_{n \in A}$ or $(u_n)_{n \notin A}$ is permutatively equivalent to a basis from the collection $\Gamma$. Then:

1. $(u_n)_{n \geq 1}$ is permutatively equivalent to $(u_n)_{n \geq 2}$.
2. $(u_n)_{n \geq 1}$ is permutatively equivalent to its square $(u_n)^2$.

Note that these hypotheses apply particularly to the case when $(u_n)$ has only countably many non-permutatively equivalent subsequences. For more on this subject see [2].

**References**


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