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A remark on Banach spaces isomorphic to their squares

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ABSTRACT. We prove that if a Banach space X has an unconditional basis and is "countably primary" then it is isomorphic to its square.

1. Introduction

This note originates from the work of Casazza and Lammers [**2**] on the so-called genus of a Banach space with unconditional basis. It is frequently important in the isomorphic theory of Banach spaces to know that a Banach space X is isomorphic to its square $X^2 = X \oplus X$. For example a well-known variant of the Pełczyński decomposition technique states that if Y is isomorphic to a complemented subspace of X, X is isomorphic to a complemented subspace of Y and both X and Y are isomorphic to their squares then X is isomorphic to Y (see [**1**]).

It is well-known that, even when X has an unconditional basis it need not be isomorphic to its square. The first example was discovered by Figiel [**3**] but there is a more recent spectacular example due to Gowers [**4**] of a space X with unconditional basis which is not isomorphic to any proper subspace of itself. However, in this note we show that if X has an unconditional basis and, in a certain sense, rather few complemented subspaces then X is isomorphic to its square (and also to its hyperplanes). The hypothesis needed is that X is countably primary, i.e. there is a countable family of Banach spaces $(W_n)_{n=1}^{\infty}$ so that if $X = Y \oplus Z$ then either Y or Z is isomorphic to one of the spaces (W_n). This includes certain important special cases. For example X could be countably prime, i.e. have at most countably many non-isomorphic complemented subspaces, or it could be primary (so that $X = Y \oplus Z$ implies $X \approx Y$ or $X \approx Z$.)

The proofs of these results depend on simple measure-theoretic ideas and very little is needed from Banach space theory. We have given here a very general statement in terms of unconditional Schauder decompositions in place of unconditional bases because we feel there may be potential applications in that setting. We refer the reader to [**2**] for some immediate applications.

We take the opportunity to spell out some connections with the so-called Schroeder-Bernstein property. We also note without proof that very similar results can be given for equivalence of bases (see Section 3).

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2. The main results

Let us start with a measure theoretic proposition. We will identify the collection of all subsets $\mathcal{P}\mathbb{N}$ of $\mathbb{N}$ with the Cantor set $\Delta = \{0,1\}^{\mathbb{N}}$ under its usual product topology. Thus we identify a subset $\mathbb{A}$ of $\mathbb{N}$ with the element $\eta = (\eta_i)_{i=1}^{\infty}$ where $\mathbb{A} = \{i : \eta_i = 1\}$.

Now suppose X is a separable Banach space with an unconditional Schauder decomposition $(E_i)_{i=1}^{\infty}$. For each $\eta \in \Delta$ we define $X_\eta = \sum_{\eta_i=1} E_i$.

LEMMA 2.1. *Under the above hypotheses, suppose Y is any fixed separable Banach space. Then the set $\{\eta : X_\eta \approx Y\}$ is an analytic subset of Δ.*

PROOF. Consider the spaces $\mathcal{L}_1 = \mathcal{L}(X,Y)$ and $\mathcal{L}_2 = \mathcal{L}(Y,X)$ with the strong operator toplogies. Let $\mathcal{V}$ be the subset of $\Delta \times \mathcal{L}_1 \times \mathcal{L}_2$ of all (η, S, T) such that $STy = y$ for $y \in Y$ and $TS(e) = \eta_i e$ if $e \in E_i$. Now $\mathcal{V} = \cup_{n\in\mathbb{N}}\mathcal{V}_n$ where $\mathcal{V}_n = \{(\eta, S, T) \in \mathcal{V} : \ \|S\|, \|T\| \le n\}$. Each $\mathcal{V}_n$ is easily seen to be closed, and since the strong operator topology coincides with a Polish topology on bounded sets (for separable Banach spaces) this implies each $\mathcal{V}_n$ is Polish and so $\mathcal{V}$ is analytic. Now let π be the projection on the first co-ordinate. It is clear that $\pi(\mathcal{V})$ is the set of η such that $X_\eta \approx Y$. $\square$

Now suppose $0 < p \le 1$ and let μ_p be the probability measure on Δ such that the co-ordinate maps $\eta \to \eta_i$ are mutually independent and $\int \eta_i d\mu_p = p$. Thus the measure μ_p is the product measure induced by randomly selecting each i with probability p. We also introduce the notation $X_{\eta,n} = \sum_{i\le n} E_i \oplus \sum_{i>n} \eta_i E_i$. It will be convenient to say that a statement S_n holds *eventually* if there exists n_0 so that S_n holds for all $n \ge n_0$.

LEMMA 2.2. *Suppose $0 < p < 1$ and that there is a Banach space Y so that $\mu_p\{X_\eta \approx Y\} > 0$. There is a Banach space Z of the form $Z \approx Y \oplus \sum_{i\in\mathbb{F}} E_i$ where $\mathbb{F}$ is finite, such that $Z \oplus E_n \approx Z$ eventually and so that with μ_p−probability one we have $X_{\eta,n} \approx Z$ eventually.*

Remark. Of course we use here the fact from Lemma 2.1 that each of the sets listed is universally measurable.

PROOF. Let $\tau = \min(p/(1-p), (1-p)/p)$, and let $\sigma = (1+\tau)^{-1} < 1$. We argue first that there exists N and a choice $\delta_i \in \{0,1\}$ for $1 \le i \le N$ so that the conditional probability

$$\mu_p\{X_\eta \approx Y | \eta_i = \delta_i, \ 1 \le i \le N\} > \sigma = (1+\tau)^{-1}.$$

Indeed if we define the functions f_k by

$$f_k(\xi) = \mu_p\{X_\eta \approx Y | \ \eta_i = \xi_i, \ 1 \le i \le k\}$$

then the functions (f_k) form a μ_p−martingale converging a.e. to the characteristic function of the set $\{\eta : X_\eta \approx Y\}$. It follows that there exists N so that if $k \ge N$ then $\max f_k = \alpha_k > \sigma$. We then choose $\delta \in \Delta$ so that $f_N(\delta) = \alpha_N$ and this determines our choice of $\delta_1, \cdots, \delta_N$.

Now define $Z = \sum_{\delta_i=0} E_i \oplus Y$. We will argue first that $Z \oplus E_n \approx Z$ if $n > N$. Indeed it suffices to show that $Y \oplus E_n \approx Y$ if $n > N$. Fix $n > N$ and define η' by $\eta'_i = \eta_i$ if $i \ne n$ and $\eta'_n = 1 - \eta_n$. We have the conditional probability

$$\mu_p\{X_{\eta'} \approx Y | \ \eta_i = \delta_i, \ 1 \le i \le N\} \ge \tau\mu_p\{X_\eta \approx Y | \ \eta_i = \delta_i, \ 1 \le i \le N\}.$$

Since $(1+\tau)\sigma = 1$ there exists η so that $X_\eta \approx X_{\eta'} \approx Y$ so that $Y \oplus E_n \approx E_n$.

Note that $\mu_p(X_{\eta,N} \approx Z) \geq \alpha_N$. It follows from the preceding that the sets $\{\eta : X_{\eta,n} \approx Z\}$ increase in n for $n \geq N$. Hence $\mu_p(X_{\eta,n} \approx Z) \geq \alpha_N$ for $n \geq N$. However we can repeat the reasoning for each $n \geq N$ to produce a space Z_n so that $\mu_p(X_{\eta,n} \approx Z_n) \geq \alpha_n$. Since $\alpha_n, \alpha_N > 1/2$ we have $Z_n \approx Z$ for all n and so $\lim_{n\to\infty} \mu_p(X_{\eta,n} \approx Z) = 1$. The Lemma now follows. $\square$

We now define a Banach space X to be *countably primary* if there is a countable set Γ of (equivalence classes, up to isomorphism) of Banach spaces so that if $X = Y \oplus Z$ then either $Y \in \Gamma$ or $Z \in \Gamma$. Let us also define X to be *countably prime* if it has only countably many distinct complemented subspaces (up to isomorphism).

If X is a Banach space with an unconditional Schauder decomposition (E_n) we will say that (E_n) is a *countably primary* (resp. *countably prime*) decomposition if there is a countable set Γ of Banach spaces so that if $\mathbb{A} \subset \mathbb{N}$ then either $\sum_{n\in\mathbb{A}} E_i$ or $\sum_{n\notin\mathbb{A}} E_i \in \Gamma$ (resp. $\sum_{i\in\mathbb{A}} E_i \in \Gamma$). Clearly any unconditional Schauder decomposition of a countably primary (resp. countably prime) space is countably primary (resp. countably prime).

THEOREM 2.3. *Suppose X is a separable Banach space with a countably primary unconditional Schauder decomposition (E_n). Then :*
(1) $X \oplus E_n \approx X$ eventually.
(2) $X \oplus \sum_{i=n+1}^{\infty} E_i \approx X$ eventually.
(3) For every $\frac{1}{2} < p < 1$ we have $X_{\eta,n} \approx X$ eventually μ_p–a.e.

Under the stronger hypotheses that either X is countably primary or (E_n) is a countably prime decomposition, we have:
(4) $X_{\eta,n} \approx X$ eventually μ_p-a.e. for $0 < p < 1$.

PROOF. We start by observing that we can suppose that the set Γ contains each E_n and X and is closed under finite direct sums.

If $\eta \in \Delta$ let $\tilde{\eta}_i = 1-\eta_i$. Then $X \approx X_\eta \oplus X_{\tilde{\eta}}$. Hence either $X_\eta \in \Gamma$ or $X_{\tilde{\eta}} \in \Gamma$. For any $0 < p < 1$ we thus have either $\mu_p(X_\eta \in \Gamma) > 0$ or $\mu_p(X_{\tilde{\eta}} \in \Gamma) > 0$. The latter inequality is equivalent to $\mu_{1-p}(X_\eta \in \Gamma) > 0$. If we let $S = \{p : \mu_p(X_\eta \in \Gamma) > 0\}$ then for any p either p or $1-p \in S$. Note that $S \subset (0,1)$. Unfortunately it is not immediate that S is Lebesgue measurable, but we can easily see that its outer Lebesgue measure $\lambda^*(S) \geq \frac{1}{2}$.

For each $p \in S$ there exists $Z_p \in \Gamma$ so that $X_{\eta,n} \approx Z_p$ eventually μ_p–a.e. We also have $Z_p \oplus E_n \approx Z_p$ eventually. Since Z_p is complemented in X this implies that $X \oplus E_n \approx X$ eventually. This proves the first claim (1).

For $W \in \Gamma$ let $S_W = \{p : Z_p \approx W\} \subset S$. We first observe that for each $W \in \Gamma$ the sets $\{\eta : X_\eta \approx W\}$ are analytic and disjoint. Hence by the Borel separation theorem (see [**7**] Proposition 4.4.4, p. 148) there are disjoint Borel sets $\{\mathcal{B}_W : W \in \Gamma\}$ so that $X_\eta \approx W$ implies $\eta \in \mathcal{B}_W$. If $\eta \in \Delta$, let us define $\eta^{[n]}$ by $\eta_i^{[n]} = 1$ if $i \leq n$ and $\eta_i^{[n]} = \eta_i$ otherwise. If we then define $\tilde{\mathcal{B}}_W$ as the set of η so there exists n_0 such that $\eta^{[n]} \in \mathcal{B}_W$ for all $n \geq n_0$. then the sets $\tilde{\mathcal{B}}_W$ are also disjoint and Borel and contain S_W. If $\mu_p(\tilde{\mathcal{B}}_W) = 1$ then either $p \notin S$ or $p \in S_W$. The maps $r \to \mu_r(\tilde{\mathcal{B}}_W)$ are Borel (this follows easily from the fact that the map $r \to \mu_r$ is weak*-continuous) and so the sets $B_W = \{r : \mu_r(\tilde{\mathcal{B}}_W) = 1\}$ are Borel subsets of $(0,1)$. Note that $S_W \subset B_W$ and $B_W \cap (1 - B_V) \subset S_W \cup (1 - S_V)$ by the above remarks.

We will now show that $X_{\eta,n} \approx X$ eventually μ_p–a.e. if $\frac{1}{2} < p \leq 1$.

Note that $I = (1-p,p) \subset \cup_{W\in\Gamma} S_W \cup \cup_{V\in\Gamma}(1-S_V)$. It follows that there exists $W \in \Gamma$ so that $\lambda^*(S_W \cap I) > 0$. Now either $S_W \cap I$ is Lebesgue measurable (and so $I \cap (S_W \cup (1-S_W))$ contains a measurable set of positive measure, or $\lambda^*(I \cap (B_W \setminus S_W)) > 0$. In the latter case there exists $V \in \Gamma$ so that $\lambda^*(I \cap ((B_W \setminus S_W) \cap (1-S_V))) > 0$. In this latter case $I \cap B_W \cap (1-B_V)$ has positive measure and so $S_W \cup (1-S_V)$ contains a measurable set of positive measure. Thus we can assume the existence of $W, V \in \Gamma$ so that $I \cap (S_W \cup (1-S_V))$ contains a Lebesgue measurable set T of positive measure.

We now use a slight variant of a well-known theorem of Steinhaus: thus there exists $\epsilon > 0$ so that if $0 < r < \epsilon$ there exists $t \in T$ so that $t \pm r \in T$. (This can be proved by considering the continuous function $r \to \int \chi_T(t)\chi_T(t+r)\chi_T(t-r)dt$.) Now it is clear that at least two of the points $t-r, t, t+r$ belong to the same set S_W or $1-S_V$. Hence with $d = r$ or $d = 2r$ we have that either there exists $q \in (1-p,p)$ with either $q, q+d \in S_W$ or $q, q+d \in S_V$.

Now since this can be done for all small enough r we may suppose that $(1-p)/d = m \in \mathbb{N}$. Now let θ_i be a sequence of independent random variables on some probability space (Ω, P) with common uniform distribution $P(\theta_i \in B) = \lambda(B \cap [0,1])$.

We observe that with probability one we have eventually,

$$\sum_{i\le n} E_i \oplus \sum_{\substack{i>n\\ \theta_i<q}} E_i \approx \sum_{i\le n} E_i \oplus \sum_{\substack{i>n\\ \theta_i<q+d}} E_i$$

so that for $1 \le s \le m$, we have eventually,

$$\sum_{i\le n} E_i \oplus \sum_{\substack{i>n\\ \theta_i<q}} E_i \approx \sum_{i\le n} E_i \oplus \sum_{\substack{i>n\\ \theta_i<q}} E_i \oplus \sum_{\substack{i>n\\ p+(s-1)d\le\theta_i<p+sd}} E_i.$$

Iterating, since d divides $1-p$ we have

$$\sum_{i\le n} E_i \oplus \sum_{\substack{i>n\\ \theta_i<q}} E_i \oplus \sum_{\substack{i>n\\ \theta_i\ge p}} E_i \approx \sum_{i\le n} E_i \oplus \sum_{\substack{i>n\\ \theta_i<q}} E_i$$

eventually with probability one. Hence, by adding the same factor to both sides to have:

$$\sum_{i\le n} E_i \oplus \sum_{\substack{i>n\\ \theta_i<p}} E_i \approx X$$

eventually with probability one. This implies that $X_{\eta,n} \approx X$ eventually μ_p-a.e. as claimed in (3).

Now using the same notation we observe that with probability one we have eventually (using the case $p = \frac{2}{3}$),

$$X \oplus \sum_{i>n} E_i \approx \sum_{i\le n} E_i \oplus \sum_{\substack{i>n\\ \theta_i\le\frac{2}{3}}} E_i \oplus \sum_{i>n} E_i$$

and hence also

$$X \oplus \sum_{i>n} E_i \approx \sum_{i\le n} E_i \oplus \sum_{\substack{i>n\\ \theta_i\le\frac{2}{3}}} E_i \oplus \sum_{\substack{i>n\\ \theta_i\ge\frac{1}{3}}} E_i.$$

However it is also clear that (again using the case $p = \frac{2}{3}$)

$$X \approx \sum_{i \leq n} E_i \oplus \sum_{\substack{i>n \\ \theta_i \leq \frac{1}{3}}} E_i \oplus \sum_{\substack{i>n \\ \theta_i \geq \frac{2}{3}}} E_i$$

eventually with probability one. Thus we have $X \oplus \sum_{i>n} E_i \approx X$ eventually, i.e. we have (2).

To prove the final claim (4) notice that under either stronger hypothesis we have that if $\sum_{i \in \mathbb{A}} E_i \approx X$ then $(E_n)_{n \in \mathbb{A}}$ is countably primary. We will deduce that if $X_{\eta,n} \approx X$ eventually μ_p-a.e. then $X_{\eta,n} \approx X$ eventually μ_q-a.e. where $q = p^2$. Indeed suppose we consider $\Delta \times \Delta$ with the product measure $\mu_p \times \mu_p$. Then, denoting a typical element of the product by (ξ, η) we observe that $X_{\xi,n} \approx X$ eventually a.e. and so using Fubini's theorem $X_{\xi\eta,n} \approx X$ eventually a.e. (where $\xi\eta$ is the pointwise product). This implies that $X_{\eta,n} \approx X$ eventually μ_q−a.e. and so the proposition is established. □

Remark. Let us point out that the preceding proof can be simplified in the special case when (E_n) is countably prime. The main point is then that the sets $\{\eta : X_\eta \approx W\}$ are both analytic and co-analytic and hence are Borel sets. This means that the sets S_W are already Borel sets and one only needs to use the fact that if S_W has positive Lebesgue measure then $S_W - S_W$ contains a neighborhood of the origin.

THEOREM 2.4. *Suppose X is a separable Banach space with a countably primary unconditional Schauder decomposition (E_n). Then there exists N so that if $\mathbb{A}$ is any subset of $\{N+1, N+2, \cdots\}$ we have*

$$X \oplus \sum_{i \in \mathbb{A}} E_i \approx X.$$

PROOF. By Theorem 2.3 there exists N so that $X \oplus E_n \approx X$ if $n > N$ and $X \oplus \sum_{i>N} E_i \approx X$. Suppose $\mathbb{A} \subset \{N+1, N+2, \cdots\}$. Then consider the unconditional Schauder decomposition $\{\sum_{i \in \mathbb{A}} E_i\} \cup \{E_j\}_{j \notin \mathbb{A}}$. This also is countably primary and so there is a finite subset $\mathbb{F}$ of $\mathbb{N} \setminus \mathbb{A}$ so that $X \oplus \sum_{i \notin (\mathbb{A} \cup \mathbb{F})} E_i \approx X$. We may assume $\mathbb{F}$ contains $\{1, 2, \cdots, N\}$. Now we deduce $X \approx X \oplus \sum_{i>N} E_i \approx X \oplus \sum_{i \in \mathbb{A}} E_i$. □

Our main applications concern the case when X has an unconditional basis.

THEOREM 2.5. *Suppose X is a separable Banach space with a countably primary unconditional basis $(u_i)_{i=1}^\infty$. Then:*
(1) $X \approx X \oplus \mathbb{R}$.
(2) $X \approx X^2$.
(3) If Y is a subspace of X spanned by a subsequence $(u_i)_{i \in \mathbb{A}}$ then $X \oplus Y \approx X$.

PROOF. Of course (1) is immediate from Theorem 2.3 (1) and then (2) follows from (2) of the same theorem. (3) follows from Theorem 2.4. □

THEOREM 2.6. *Suppose X is a separable Banach space with an unconditional basis (u_i) so that either X is countably primary or (u_i) is countably prime. Then we may partition $\mathbb{N}$ into sets $(\mathbb{A}_k)_{k=1}^\infty$ so that each $[u_i]_{i \in \mathbb{A}_k} \approx X$ and for any nonempty subset $\mathbb{B}$ of $\mathbb{N}$ we have $[u_i : i \in \cup_{k \in \mathbb{B}} \cup_{i \in \mathbb{A}_k}] \approx X$.*

PROOF. Let us again consider the sequence of independent random variables (θ_i) each uniformly distributed on $[0,1]$. Applying Theorem 2.3 and Theorem 2.5 we see that with probability one we have for $k \in \mathbb{N}$, $[u_i]_{2^{-k} \le \theta_i < 2^{1-k}} \approx X$. It follows that we can choose the sets $\mathbb{A}_k$ so that $[u_i : i \in \mathbb{A}_k] \approx X$.

Now notice that if $\mathbb{D} \supset \mathbb{A}_k$ for some k then if $Y = [u_i]_{i \in \mathbb{D}}$ we have $Y \approx X \oplus Z$ for some Z and hence $Y \approx X \oplus Y \approx X$ by Theorem 2.5. □

Let us give some applications. We recall that a Banach space X has the *Schroeder-Bernstein property* if whenever Y is a complemented subspace of X which contains a further complemented subspace isomorphic to X then $Y \approx X$. See [**1**] for a discussion of this property and [**5**] for an example of a space X failing the Schroeder-Bernstein property. There is apparently no known example of a space X with an unconditional basis failing the Schroeder-Bernstein property. Note that if $X \approx X^2$ then X has the Schroeder-Bernstein property if and only if $X \oplus Y \approx X$ for every complemented subspace Y of X.

If X has an unconditional finite-dimensional decomposition (UFDD) we will say that X has the *restricted Schroeder-Bernstein property* if whenever Y is complemented in X, has a (UFDD) and contains a complemented subspace isomorphic to X then $X \approx Y$. Let us remark that there is no known example of a complemented subspace of a Banach space with unconditional basis which fails to have a (UFDD). There is an unpublished example of Read [**6**] of a Banach space with a (UFDD) and a complemented subspace without a (UFDD).

PROPOSITION 2.7. *Let X be a Banach space with a (UFDD). Then the following are equivalent:*
(1) $X \approx X^2$ and X has the restricted Schroeder-Bernstein property.
(2) $X \oplus Y \approx X$ for every complemented subspace Y of X with a (UFDD).
(3) $X \oplus Y \approx X$ for every complemented subspace Y of X so that X/Y has a (UFDD).

PROOF. We leave the elementary proof that (1) and (2) are equivalent to the reader. Let us prove (2) and (3) are equivalent. Obviously both imply that $X \approx X^2$. Now suppose $X \approx Y \oplus Z$. We claim that if $X \oplus Y \approx X$ then $X \oplus Z \approx X$. Indeed $X \approx X^2 \approx X \oplus Y \oplus Z \approx X \oplus Z$. This clearly establishes the equivalence of (2) and (3). □

THEOREM 2.8. *Let X be a countably primary Banach space with an unconditional basis. Then X has the restricted Schroeder-Bernstein property.*

PROOF. Of course X satisfies the conclusions of Theorem 2.5. Let $X \approx Y \oplus Z$ where Y has a (UFDD) (E_n). Then $\{Z, (E_n)_{n=1}^{\infty}\}$ forms an unconditional Schauder decomposition of $X^2 \approx X$. Hence by Theorem 2.3 we can deduce that $X \approx X \oplus Y_0$ where Y_0 has finite codimension in Y. But then $X \oplus Y \approx X \oplus V$ where V is finite-dimensional and so by Theorem 2.5 $X \approx X \oplus Y$. □

THEOREM 2.9. *Let X be a Banach space with an unconditional basis. Suppose there are at most countably many non-isomorphic Banach spaces Y which are complemented in X^2 and contain a complemented subspace isomorphic to X. Then $X^2 \approx X^3$ and X^2 has the restricted Schroeder-Bernstein property.*

PROOF. First note that $X^2 = X \oplus X$ has an unconditional Schauder decomposition consisting of $X \oplus \{0\}$ and the unconditional basic sequence $(0, u_n)$. This is countably primary and so by Theorem 2.3 we obtain both $X^2 \approx X^2 \oplus R$ and $X^2 \approx X^2 \oplus [u_n]_{n \geq N+1}$ for some N. These combine to give $X^2 \approx X^3$. In particular $(X^2)^2 \approx X^2$.

Now suppose Y is a complemented subspace of X^2 with a (UFDD). Then $X \oplus Y$ has a countably primary Schauder decomposition (E_n) with $E_1 \approx X$ and E_n finite-dimensional for $n \geq 2$. Hence by Theorem 2.3 there is a finite-codimensional subspace Y_0 of Y so that $X \oplus Y \oplus Y_0 \approx X \oplus Y$. Thus $X^2 \oplus Y^2 \approx X^2 \oplus Y$. It follows that $(X^2 \oplus Y)^2 \approx X^2 \oplus Y$ and hence the usual decomposition argument (cf. [**1**]) gives $X^2 \oplus Y \approx X^2$. By Proposition 2.7 we are done. □

3. Equivalence of unconditional bases

In this short final section we point out that very similar ideas can be applied to equivalence of unconditional bases in place of isomorphisms. If $(u_n)_{n=1}^{\infty}$ is an unconditional basis of X and $(v_n)_{n=1}^{\infty}$ an unconditional basis of Y we will say that (u_n) and (v_n) are *permutatively equivalent* if there is an isomorphism $T : X \to Y$ and a permutation π of $\mathbb{N}$ so that $Tu_i = v_{\pi(i)}$. This definition extends naturally to an unconditional basis indexed by any countable set in place of $\mathbb{N}$.

One easily proves the following lemma similarly to Lemma 2.1.

LEMMA 3.1. *Let (u_n) be an unconditional basis of a Banach space X and let (v_n) be an unconditional basis of a Banach space Y. The set of $\eta \in \Delta$ such that $(u_i)_{\eta_i=1}$ is permutatively equivalent to (v_n) is analytic.*

Now arguments very similar to those for Theorem 2.4 establish that:

THEOREM 3.2. *Let (u_n) be an unconditional basis of a Banach space X and suppose that there is a countable family of unconditional bases Γ so that for every infinite subset $\mathbb{A}$ of $\mathbb{N}$ either $(u_n)_{n\in\mathbb{A}}$ or $(u_n)_{n\notin\mathbb{A}}$ is permutatively equivalent to a basis from the collection Γ. Then:*
(1) $(u_n)_{n\geq 1}$ is permutatively equivalent to $(u_n)_{n\geq 2}$.
(2) $(u_n)_{n\geq 1}$ is permutatively equivalent to its square $(u_n)^2$.

Note that these hypotheses apply particularly to the case when (u_n) has only countably many non-permutatively equivalent subsequences. For more on this subject see [**2**].

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