CONTEMPORARY MATHEMATICS

Volume 85

Banach Space Theory

Proceedings of a Research Workshop
held July 5–25, 1987
with support from
the National Science Foundation

Bor-Luh Lin, Editor

AMERICAN MATHEMATICAL SOCIETY
Providence • Rhode Island

The Research Workshop on Banach Space Theory was held at the University of Iowa, Iowa City, on July 5–25, 1987 with support from the National Science Foundation, Grant DMS-8604481.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 46-06, 46B10, 46B20, 46B22, 46B25.

Library of Congress Cataloging-in-Publication Data

Research Workshop on Banach Space Theory (1987: University of Iowa)
Banach space theory: proceedings of a research workshop held July 5–25, 1987, with support from the National Science Foundation/Bor-Luh Lin, editor.
p. cm. –(Contemporary mathematics, ISSN 0271-4132; v. 85)
"The Research Workshop on Banach Space Theory was held at the University of Iowa, Iowa City"–T.p. verso.
Includes bibliographies.
ISBN 0-8218-5092-x (alk. paper)
1. Banach spaces–Congresses. I. Lin, Bor-Luh. II. American Mathematical Society. III. Title. VI. Series: Contemporary mathematics (American Mathematical Society); v. 85.
QA322.2.R47 1987 88-38106
515.7′32–dc 19 CIP

Contemporary Mathematics
Volume **85**, 1989

Independence in separable Banach spaces

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Abstract. *We answer a question of Fremlin and Sersouri concerning independence of uncountable sets in separable Banach spaces.*

Recently Fremlin and Sersouri [1] proved the following theorem:

THEOREM 1. *Let X be a separable Banach space and let G be an uncountable subset of X. Then, for any $1 < p < \infty$, there exists a sequence (g_n) of distinct points of G and a sequence of real numbers (a_n), not all zero, such that $\sum |a_n|^p < \infty$ and*

$$\sum_{n=1}^{\infty} a_n g_n = 0.$$

This theorem answered a question of Lipecki. They note that one cannot allow $p = 1$ in the above theorem, and ask if instead one can require $(a_n) \in \ell(1,\infty)$ where $\ell(1,\infty)$ denotes the space weak ℓ_1 of all sequences (a_n) such that the decreasing rearrangement (a_n^*) of $(|a_n|)$ satisfies $\sup n a_n^* < \infty$.

We show in this note that this question has an affirmative answer and, in fact, a rather stronger result is true. For convenience, we restrict attention to real Banach spaces; some obvious rewording is necessary in the complex case. Our theorem is:

[1] Research supported by NSF-grant DMS-8601401

AMS Classification: 46B15

Key words: independence, Banach spaces

THEOREM 2. *Let X be a separable Banach space and let G be an uncountable subset of X. Suppose (a_n) is a sequence of real numbers satisfying $\sum |a_n| = \infty$ and $\lim a_n = 0$. Then there is a sequence of distinct points (g_n) of G and a sequence of signs $\epsilon_n = \pm 1$ so that*

$$\sum_{n=1}^{\infty} \epsilon_n a_n g_n = 0.$$

As in [1] the first step is to reduce the problem to the case when G is dense in itself, and one may then suppose that X is the closed linear span of G. Thus Theorem 2 follows from Theorem 3 below.

THEOREM 3. *Let X be an arbitrary Banach space and let G be a subset of X. Suppose H is the set of accumulation points of G, and that X is the closed linear span of H. Then given any $x \in X$ and any sequence of real numbers (a_n) with $\sum |a_n| = \infty$ and $\lim a_n = 0$ we may find a sequence of signs ϵ_n, and a sequence of distinct elements $g_n \in G$ so that*

$$x = \sum_{n=1}^{\infty} \epsilon_n a_n g_n.$$

PROOF: We may suppose (a_n) is a sequence of nonnegative numbers satisfying the conditions of the theorem. For convenience let $b_n = \max_{i>n} |a_i|$. We shall require the following lemma:

LEMMA 4. *Suppose $\alpha \in \mathbf{R}$ and $m \in \mathbf{N}$. Then we may choose signs ϵ_i, $(i \geq m+1)$ so that if $s_m = \alpha$ and $s_k = \alpha + \sum_{i=m+1}^{k} \epsilon_i a_i$ then we have $\lim s_k = 0$ and $\sup |s_k| \leq \max(b_m, |\alpha|)$.*

PROOF OF THE LEMMA: Define ϵ_k so that $\epsilon_k s_{k-1} \leq 0$. Then the sequence s_k must change signs infinitely often since otherwise ϵ_k is eventually constant and this would imply that $\sum a_n < \infty$. If s_k and s_{k-1} have the same signs then $|s_k| \leq |s_{k-1}|$. If s_k and s_{k-1} have opposite signs then $|s_k| \leq |a_k|$. Thus we have $|s_k| \leq \max(|s_{k-1}|, |a_k|)$ and this implies the second assertion of the lemma. Since $\lim a_k = 0$ and s_k changes sign infinitely often it also implies the first assertion.

Resumption of the proof of the theorem: Let us define $F(N,\delta)$, for $N \in \mathbf{N}$, $\delta > 0$, to be the subset of X of all x with the property that for any $m \geq N$ we can find $n > m$ and $\epsilon_i = \pm 1$, $h_i \in H$, $(m+1 \leq i \leq n)$ such that

$$\|x + \sum_{i=m+1}^{n} \epsilon_i a_i h_i\| < \delta,$$

and

$$\|x + \sum_{i=m+1}^{k} \epsilon_i a_i h_i\| \leq \|x\| + \delta \qquad m+1 \leq k \leq n.$$

Note that the h_i are not required to be distinct.

Next we define $F = \cap_{\delta>0} \cup_{N \in \mathbf{N}} F(N,\delta)$. We note first that F is easily seen to be closed. Define $E = \{x : \alpha x \in F \ \forall \alpha \in \mathbf{R}\}$; E is also closed.

We show E contains H. In fact suppose $h \in H$, and $\alpha \in \mathbf{R}$. For arbitrary $\delta > 0$ we pick N so large that $b_N \|h\| < |\alpha|\|h\| + \delta$. If $m \geq N$, we pick ϵ_i according to the Lemma, applied to α and $m = N$, stopping at n where $|s_n|\|h\| < \delta$. Letting $h_i = h$, $(m+1 \leq i \leq n)$ we see that $H \subset E$.

Next we claim that E is a linear subspace. In fact, it is only necessary to show that if $x \in E$ and $y \in E$ then $x + y \in F$. Suppose $\delta > 0$. Let $M = \max(\|x\|, \|y\|)$ and then choose an integer s so large that $6M < s\delta$. Next choose N so that $s^{-1}x, s^{-1}y \in F(N, \delta/(4s))$. Now suppose $m \geq N$. Set $p_0 = m$; then we may inductively define q_k, $(1 \leq k \leq s)$, p_k, $(1 \leq k \leq s)$, $\epsilon_i, (p_0 + 1 \leq i \leq p_s)$ and $h_i \in H$, $(p_0 + 1 \leq i \leq p_s)$ so that $p_{k-1} < q_k < p_k$, $(1 \leq k \leq s)$,

$$\|\sum_{p_{k-1}+1}^{q_k} \epsilon_i a_i h_i + s^{-1}x\| < \frac{\delta}{4s}$$

$$\|\sum_{p_{k-1}+1}^{j} \epsilon_i a_i h_i + s^{-1}x\| < \frac{(4\|x\| + \delta)}{4s},$$

for $1 \leq k \leq s$ and $p_{k-1} + 1 \leq j \leq q_k$, and

$$\|\sum_{q_k+1}^{p_k} \epsilon_i a_i h_i + s^{-1}y\| < \frac{\delta}{4s}$$

$$\|\sum_{q_k+1}^{j} \epsilon_i a_i h_i + s^{-1}y\| < \frac{(4\|y\| + \delta)}{4s},$$

for $1 \le k \le s$ and $q_k + 1 \le j \le p_k$.

Then

$$\|x + y + \sum_{m+1}^{p_s} \epsilon_i a_i h_i\| < \delta.$$

If $p_{k-1} + 1 \le j \le q_k$ then

$$\|x + y + \sum_{m+1}^{j} \epsilon_i a_i h_i\| < \frac{s-k+1}{s}\|x+y\| + \frac{(k-1)\delta}{2s} + \frac{(4\|x\| + \delta)}{4s}.$$

If $q_k + 1 \le j \le p_k$ then

$$\|x + y + \sum_{m+1}^{j} \epsilon_i a_i h_i\| < \frac{s-k+1}{s}\|x+y\| + \frac{(k-1)\delta}{2s} + \frac{(4\|x\| + \delta)}{4s} + \frac{(4\|y\| + \delta)}{4s}.$$

In either case we conclude that

$$\|x + y + \sum_{m+1}^{j} \epsilon_i h_i\| < \|x+y\| + \delta$$

so that $x + y \in F(N, \delta)$.

It now follows immediately that $E = X$. Now fix any $h_0 \in H$ and let $\gamma = \|h_0\|$. Then, we claim (*) that for any $x \in X$, $m \in \mathbf{N}$ and $\delta > 0$ we can find $n > m$, $h_i \in H$, $m+1 \le i \le n$ and $\epsilon_i = \pm 1$, $m+1 \le i \le n$ so that

$$\|x + \sum_{m+1}^{n} \epsilon_i a_i h_i\| < \delta$$

and, for $m+1 \le j \le n$,

$$\|x + \sum_{m+1}^{j} \epsilon_i a_i h_i\| < \gamma b_m + \|x\| + \delta.$$

In fact there exists N so that $x \in F(N, \delta/2)$. By the Lemma, applied to $\alpha = 0$ and m, we may find ϵ_i, $m+1 \le i \le k$ where $k \ge N$ so that

$$|\sum_{m+1}^{k} \epsilon_i a_i| < \frac{\delta}{2\gamma}$$

and

$$|\sum_{m+1}^{j} \epsilon_i a_i| < b_m$$

for $m+1 \leq j \leq k$. Now choose $n > k$ and $h_i \in H$, $\epsilon_i = \pm 1$, $(k+1 \leq i \leq n)$ so that

$$\|\sum_{k+1}^{n} \epsilon_i a_i h_i + x\| < \frac{\delta}{2}$$

and

$$\|\sum_{k+1}^{j} \epsilon_i a_i h_i + x\| < \|x\| + \frac{\delta}{2}$$

for $k+1 \leq j \leq n$. We now put $h_i = h_0$ for $m+1 \leq i \leq k$ and our claim is substantiated.

We now may complete the proof of Theorem 3. Suppose $x \in X$ is fixed. Let $p_0 = 0$. Since H is the set of accumulation points of G, we may inductively choose signs ϵ_i and $g_i \in G$, $(p_{k-1}+1 \leq i \leq p_k)$ so that g_i, $(1 \leq i \leq p_k)$ are distinct,

$$\|\sum_{i=1}^{p_k} \epsilon_i a_i g_i - x\| < 2^{-k}$$

for $k \geq 1$ and if $p_{k-1}+1 \leq j \leq p_k$,

$$\begin{aligned}\|\sum_{i=1}^{j} \epsilon_i a_i g_i - x\| &< \|\sum_{i=1}^{p_{k-1}} \epsilon_i a_i g_i - x\| + 2^{-k} + \gamma b_{p_{k-1}} \\ &< 4.2^{-k} + \gamma b_{p_{k-1}}\end{aligned}$$

for $k \geq 1$. The series constructed in this way converges to x and the proof is complete.

References.

1. D. H. Fremlin and A. Sersouri, On ω-independence in separable Banach spaces, Quart. J. Math. to appear.

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