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THE METRIC LINEAR SPACES L_p FOR $0 < p < 1$

by

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1. Introduction

The study of the classical function spaces $L_p (0 < p < \infty)$ is as old as functional analysis. However, the case $0 < p < 1$ has been relatively neglected. The lack of interest in this range may at least partially be explained by the failure of traditional techniques which heavily depend on local convexity and, in particular, on the Hahn-Banach theorem. In this paper we will sketch some of the recent work on the spaces L_p for $0 < p < 1$ and try to show that the 'negative' aspects of these spaces can be used to advantage to establish positive results.

In order to fix notation we define L_p as the space $L_p(0,1)$ of all measurable real-valued functions $f: (0,1) \to \mathbb{R}$ so that

$$\| f\|_p = \left\{\int_0^1 |f(t)|^p \, dt \right\}^{1/p} < \infty.$$

As usual, functions differing only on a set of measure zero are identified. Of course for $p \geq 1$, $\| \ \|_p$ is then a genuine norm, but for $0 < p < 1$ it satisfies only a weakened triangle inequality

$$\| f + g \|_p \leq 2^{1/p-1} \left(\| f \|_p + \| g \|_p \right).$$

Then L_p is an example of quasi-Banach space; i.e., a complete quasi-normed vector space.

For convenience, we will in general consider only quasi-Banach spaces in this paper. However, it is often useful and revealing to compare and contrast L_p (for $0 < p < 1$) with the extreme spaces L_0 and L_1. Here L_0 is the space of all measurable functions $f: (0,1) \to \mathbb{R}$ with the metric

$$d(f,g) = \int_0^1 \min \left\{1, |f(t) - g(t)|\right\} dt.$$

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L_0 is not a quasi-Banach space, but is an F-space (complete metrizable topological vector space).

2. Operators on L_p.

The first major result on the spaces L_p, $0 < p < 1$, is due to Day [10].

Theorem 1: For $0 < p < 1$, L_p has trivial dual, i.e., if $\phi: L_p \to \mathbb{R}$ is a continuous linear functional then $\phi = 0$.

This is usually regarded as a negative result, explaining why the spaces L_p $(p < 1)$ are not interesting. However, we shall use it as a launching point to study the properties of operators on L_p. Let us first restate Theorem 1:

Corollary 2: Suppose $0 < p < 1$. Let X be a quasi-Banach space and let $T: L \to X$ be a (continuous) linear operator. If T has finite rank then $T = 0$.

In other words, T cannot be too small, unless it is identically zero. In some sense, L_p is a space which cannot be compressed. It is natural to ask just how small a non-zero operator on L_p can be. In this context it is worth remarking that is unknown whether there exists an infinite-dimensional quasi-Banach space X so that whenever $T: X \to Y$ is a linear operator then $T(X)$ is closed; such a space X is called quotient-minimal ([11]). On a quotient-minimal space all operators would be very large; see also [22] for a discussion of this and related problems. The space $L_p (0 < p < 1)$ is not quotient-minimal as the injection $L_p \hookrightarrow L_q$ $(p > q > 0)$ shows. In fact L_p has no quotient-minimal subspace - this follows from work of Bastero [4].

We also note that there exist quasi-Banach spaces X for which the algebra of endomorphisms $\mathcal{L}(X) = \{\lambda I : \lambda \in \mathbb{R}\}$. Such spaces are called rigid and a rigid subspace of L_p $(0 < p < 1)$ is constructed in [23]. Again however, L_p itself is far from rigid; in fact L_p is transitive; i.e., given $f, g \in L_p$ with $f \neq 0$ there exists $T \in \mathcal{L}(L_p)$ with $Tf = g$ (see, for example, [33]).

Returning to Theorem 1, we find that the next development was in work of Hyers and Williamson ([13], [36]). They showed that the Fredholm theory of compact operators is not dependent on local convexity. Thus if X is a quasi-Banach space and $C:X \to X$ is a compact endomorphism of X then for $\lambda \neq 0$ the operator $A = C - \lambda I$ has closed range and dim (ker A) = dim (coker A). Williamson then observed that if X has trivial dual then X has no non-trivial subspaces of finite codimension; thus we must have dim (ker A) = dim (coker A) = 0, so that A is invertible. Thus the spectrum of C reduces to $\{0\}$. By complexifying X if necessary we can deduce:

Theorem 3: [36] *If* X *is a quasi-Banach space with trivial dual and* $C:X \to X$ *is a compact endomorphism then* C *is quasi-nilpotent*; i.e.,

$$\lim_{n \to \infty} \| C^n \|^{1/n} = 0.$$

Later Pallaschke [31] combined Williamson's argument with the fact that L_p is transitive to deduce:

Theorem 4: Suppose $0 < p < 1$. *If* $C:L_p \to L_p$ *is a compact endomorphism then* $C = 0$.

To prove this, note that if C is non-zero there exists $f \in L_p$ with $Cf \neq 0$. Now choose $T \in L(L_p)$ with $TC\,f = f$. Then TC is compact and has 1 as an eigenvalue contradicting Theorem 3.

Theorem 4 was also proved independently by Turpin [35]; both Pallaschke and Turpin obtained considerable generalizations of this result.

Of course Theorem 4 works for any transitive space X with trivial dual in place of L_p. This led Pelczyński to ask whether, if X has trivial dual and $C:X \to X$ is compact, then $C = 0$. A counterexample to this conjecture was given in [24]. In order to produce an example one only needs to find a compact operator (not an endomorphism) on a space with trivial dual. For if $X^* = \{0\}$ and $C:X \to Y$ is a non-zero compact operator, we may suppose C has dense range. Then Y has trivial dual and so has $Z = X \oplus Y$; define

$K \in L(Z)$ by $K(x,y) = (0,Cx)$. Note of course that $K^2 = 0$, so that Theorem 3 is not violated. In [24] X was constructed as a quotient space of the Hardy space H_p when $0 < p < 1$; later in [17] a very general construction of such spaces X was given. An obvious question is whether we can take $X = L_p$; the answer, however, is no:

Theorem 5: *Suppose* $0 < p < 1$ *and suppose* $C : L_p \to X$ *is a compact operator. Then* $C = 0$.

Theorem 5 improves Theorem 4, and was first established in [14]. A simple and quick proof was given in [16] and we sketch it here. Since L_p is transitive it will suffice to show that $C(1) = 0$ where 1 is the constant function identically one.

Suppose $0 < \varepsilon < 1$ and let $\{A_n\}$ be a sequence of independent sets of measure ε. Let $\{\phi_n\}$ be the associated indicator functions. Then, by passing to a subsequence, since C is compact we may suppose that there exists $u \in Y$ with

$$\| C\phi_n - u \| \leq 2^{-n}.$$

It follows that $\frac{1}{n}(C\phi_1 + C\phi_2 + \ldots + C\phi_n) \to u$ in Y. To see this it is convenient to suppose that the quasi-norm on Y is r-subadditive, where $0 < r \leq 1$; i.e.,

$$\| y_1 + y_2 \|^r \leq \| y_1 \|^r + \| y_2 \|^r \qquad y_1, y_2 \in Y.$$

Thus

$$\left\| \frac{1}{n}(C\phi_1 + \ldots + C\phi_n) - u \right\|^r \leq \frac{1}{n^r} \sum_{k=1}^{\infty} 2^{-kr}.$$

The assumption that the quasi-norm is r-subadditive can be justified by the classical Aoki-Rolewicz Theorem [3], [33].

Next we notice, by the Strong Law of Large Numbers, that the sequence $\{\frac{1}{n}(\phi_1 + \ldots + \phi_n) : n \in \mathbb{N}\}$ converges in L_p to $\varepsilon \cdot 1$. Thus

$$C(1) = \varepsilon^{-1}u$$

and hence

$$\begin{aligned} \| C(1)\| &= \varepsilon^{-1} \|u\| \\ &= \varepsilon^{-1} \lim_{n \to \infty} \| C\phi_n \| \\ &\leq \|C\| \; \varepsilon^{1/p-1} \end{aligned}$$

since $\| \phi_n\| = \varepsilon^{1/p}$. [Here we assumed the quasi-norm on Y to be continuous; again this can be justified by the Aoki-Rolewicz theorem]. Now since $\varepsilon > 0$ is arbitrary we obtain the theorem.

In fact more than Theorem 5 is true.

Theorem 6 [14]: Suppose $0 < p < 1$ and that $T:L_p \to Y$ is a non-zero operator. Then there exists a subspace H of L_p with $H \cong \ell_2$ so that T is an isomorphism on H.

In Theorem 6 we have already come some distance from Theorem 1. By now it is natural to speculate on further advances. One may wonder whether ℓ_2 is somehow special in Theorem 6. For example, if we consider the case $p = 0$ then in [9] it was shown that any non-zero operator on L_0 into an F-space preserves a copy of ω (the space of all sequences).

Again if we restrict to endomorphisms of L_p we can push Theorem 6 much further. Endomorphisms of L_p can be represented in the following form:

Theorem 7 [15]: Suppose $0 < p < 1$. Let $T:L_p \to L_p$ be any endomorphism. Then there exists Borel maps $a_n:(0,1) \to \mathbb{R}$ and $\sigma_n:(0,1) \to (0,1)$ such that

(i) For $f \in L_p$,

$$Tf(s) = \sum_{n=1}^{\infty} a_n(s)\, f(\sigma_n s) \qquad \text{a.e}$$

(ii) $\sigma_m(s) \neq \sigma_n(s) \qquad m \neq n$

(iii) $\sum_{k=1}^{\infty} |a_n(s)|^p < \infty \qquad$ a.e.

(iv) For every Borel set B

$$\sum_{n=1}^{\infty} \int_{\sigma_n^{-1}B} |a_n(s)|^p \, ds \leq \|T\|^p \, m(B)$$

The significance of Theorem 7 (which was inspired by a result of similar type for $p = 0$ proved by Kwapien [25]) is that the study of endomorphisms of L_p can be reduced, in some instances, to the study of special simple operators of the type

$$Ef(s) = a(s)\, f(\sigma s) \qquad f \in L_p.$$

Using such a reduction we obtain:

<u>Theorem 8</u> [15]: <u>Suppose</u> $0 < p < 1$. <u>Let</u> $T: L_p \to L_p$ <u>be a non-zero endomorphism. Then there is a Borel set</u> B <u>of positive measure so that</u> T <u>is an isomorphism on</u> $L_p(B)$.

Thus, in Theorem 6, ℓ_2 can be replaced by L_p itself if we restrict attention to endomorphisms. There is no hope of such an improvement in general since the injection $L_p \hookrightarrow L_q$ $(p > q > 0)$ can be shown not to preserve a copy of L_p. In the same vein ([18]) L_p has a quotient space not containing a copy of L_p. However we do have:

<u>Theorem 9</u> [18]: <u>Suppose</u> $0 < q < p < 1$. <u>Suppose</u> $T: L_p \to L_q$ <u>is a non-zero operator. Then if</u> $p < r \leq 2$ <u>there is a subspace</u> Z <u>of</u> L_p <u>with</u> $Z \cong L_r$ <u>so that</u> T <u>is an isomorphism on</u> Z.

We can replace L_q by any target space X which admits enough linear operators into L_0 to separate points. The proof of this from Theorem 9 is almost immediate if we apply a theorem of Nikishin [30]. This means that Theorem 9 applies to a very large range of target spaces - in fact almost any space which arises in practical circumstances, since most spaces arise as spaces of measurable functions. There are, however, spaces which fail to admit operators into L_0. Probably the most photogenic example is the space L_p/H_p for $0 < p < 1$ (see [1] and [17]). Here L_p is the space of complex-valued L_p-functions on the circle T, in the complex plane, with its natural Haar measure $d\theta/2\pi$, and H_p is the Hardy space H_p formed by taking the closure of the polynomials in L_p.

We conclude this section with some unsolved problems arising out of this line of thought.

Problem 1: Let X be an arbitrary quasi-Banach space and suppose $0 < p < 1$. Suppose $T:L_p \to X$ is a non-zero operator. If $p < r \leq 2$ must where exist a subspace Z of L_p with $Z \cong L_r$ so that T is an isomorphism on Z?

Problem 2: Does L_p embed into L_p/H_p when $0 < p < 1$?

See Bourgain [7] for the case $p = 1$; L_1 does embed into L_1/H_1.

3. The complemented subspaces of L_p.

For $1 < p < \infty$ it has recently been shown that L_p has uncountably many different (isomorphism classes of) complemented subspaces [8]. For $p = 1$, however, there is still hope that the complemented subspaces of L_1 can be completely classified. The major positive result here is the Lewis-Stegall Theorem [26]:

Theorem 10: Let X be an infinite-dimensional complemented subspaces of L_1 with the Radon-Nikodym property. Then $X \cong \ell_1$.

It has therefore been conjectured that the only infinite-dimensional complemented subspaces of L_1 are, up to isomorphism, ℓ_1 and L_1 [34].

For $0 < p < 1$, L_p has no non-trivial finite-dimensional complemented subspaces, nor can ℓ_p can be a complemented subspace. By analogy we ask:

Problem 3: Suppose $0 < p < 1$. Is L_p a prime space: i.e., if X is isomorphic to a complemented subspace of L_p is X isomorphic to L_p?

There is strong evidence that Problem 3 has an affirmative answer. First we note the other extreme case [15].

Theorem 11: L_0 is a prime space.

It is clear that any complemented subspace of L_p must contain a copy of L_p (see Theorem 8 above). The critical problem, however, is whether that copy must itself be complemented. Indeed, if X is a complemented subspace of L_p containing a complemented copy of L_p then $X \cong L_p$ by the Pelczyński decomposition technique. In contrast, for the case $p = 1$, the difficulty for a complemented subspace X without the Radon-Nikyodym Property is to find a subspace of X isomorphic to L_1; if this can be done a result of Enflo and Starbird [12] or [15] allows one to find a complemented subspace isomorphic to L_1 in X.

Although Problem 3 is unresolved it is possible to reduce it to just two possible cases:

Theorem 12 [19]: There are at most two complemented subspace of L_p (for $0 < p < 1$) up to isomorphism.

Of course Theorem 12 implies that there may exist at most one complemented subspace of L_p other than L_p itself. Probably this freak does not exist.

Another partial result is:

Theorem 13 [15]: For $0 < p < 1$, L_p is primary; i.e., if $L_p \cong X \oplus Y$ then either $L_p \cong X$ or $L_p \cong Y$.

Theorem 13 also holds for $1 \leq p < \infty$ ([2], [12], [15], [29]).

If X is a quasi-Banach space with a continuous quasi-norm we define $L_p(X)$ to be the space of all essentially separably-valued Borel functions $f: (0,1) \to X$ so that

$$\| f \|_p = \left\{ \int_0^1 \| f(t) \|^p \, dt \right\}^{1/p} < \infty.$$

In the case $1 \leq p < \infty$, it is easy to show that X is isomorphic to a complemented subspace of L_p if and only if $L_p(X)$ is isomorphic to L_p. This suggests that we may also attempt, for $0 < p < 1$, to classify those spaces X for which $L_p(X) \cong L_p$.

<u>Problem 4</u>: <u>Suppose</u> $0 < p < 1$. <u>Suppose</u> X <u>is a quasi-Banach space for which</u> $L_p(X) \tilde{=} L_p$. <u>Must</u> X <u>be isomorphic to a space of the form</u> $L_p(\Omega,\mu)$ <u>for some separable measure space</u> (Ω,μ)?

For $p = 0$ the answer to this question is positive [20]. For $0 < p < 1$, the complete catalogue of spaces $L_p(\Omega,\mu)$ is $\mathbb{R}^n$ $(n \geq 1)$, ℓ_p, L_p, $L_p \oplus \mathbb{R}^n$ $(n \geq 1)$, $L_p \oplus \ell_p$. Problem 4 is solved in [19] modulo the existence of the freak complemented subspace of L_p. For example we have the $p < 1$ version of the Lewis-Stegall Theorem.

<u>Theorem 14</u>: <u>Suppose</u> $0 < p < 1$ <u>and</u> X <u>is a quasi-Banach space containing no copy of</u> L_p. <u>Suppose</u> $L_p(X) \tilde{=} L_p$; <u>then either</u> X <u>is finite-dimensional</u> <u>or</u> $X \tilde{=} \ell_p$.

4. Special subspaces of L_p

Let A be any sub-σ-algebra of the Borel sets $\mathcal{B}$ on $(0,1)$. Let $L_p(A)$ be the subspace of L_p of all A-measurable functions. If $p \geq 1$ there is a natural norm-one projection of L_p onto $L_p(A)$ given by conditional expectation. For $0 \leq p < 1$ this operator is no longer continuous (or even well-defined). In fact if A if finite then there is a trivially no projection onto $L_p(A)$. For a diffuse σ-algebra A the situation is more subtle and interesting.

In]973, Berg, Peck and Porta [5] proved that there is no projection of $L_0(0,1)^2$ onto $L_0(0,1)$. Here $L_0(0,1)$ is interpreted as the subspace of $L_0(0,1)^2$ of all functions depending on the first variable. The corresponding result for $0 < p < 1$ was established by the author in [15]. In fact we have:

Theorem 15: Suppose $0 < p < 1$. $L_p(\mathcal{A})$ is complemented in L_p if and only if there exist a Borel set E and $\varepsilon > 0$ so that:

(i) $m(A \cap E) \geq \varepsilon\, m(A) \qquad A \in \mathcal{A}$

(ii) Given $B \in \mathcal{B}$ with $B \subset E$ there exists $A \in \mathcal{A}$ so that

$$m((A \cap E) \,\Delta\, B) = 0.$$

In fact Theorem 15 states that $L_p(\mathcal{A})$ can only be complemented in L_p if there exists a Borel set E of positive measure and an automorphism U of L_p so that $U\left(L_p(E)\right) = L_p(\mathcal{A})$. We see at once that if $L_p(\mathcal{A})$ is complemented then $L_p/L_p(\mathcal{A})$ is isomorphic to L_p. In general for $L_p(\mathcal{A})$ to be complemented in L_p the $\mathcal{A}$ must in some sense be large. In particular there cannot be a sub-σ-algebra $\mathcal{C}$ which is diffuse and independent of $\mathcal{A}$. Thus Theorem 15 will imply that $L_p(0,1)$ is not complemented in $L_p(0,1)^2$.

We now sketch a proof that if $\mathcal{C}$ is diffuse and independent of $\mathcal{A}$ then $L_p(\mathcal{A})$ is not complemented in L_p. Suppose $\mathcal{A}$ satisfies the conditions of Theorem 15. Pick an integer $n > \varepsilon^{-1}$. Partition $[0,1]$ into $\mathcal{C}$-measurable sets $(V_i : 1 \leq i \leq n)$ with $m(V_i) = n^{-1}$. Now for each i, there exists $A_i \in \mathcal{A}$ so that

$$m\left((A_i \cap E) \,\Delta\, (V_i \cap E)\right) = 0.$$

Then

$$\begin{aligned} m(V_i \cap E) &= m(A_i \cap V_i \cap E) \\ &\leq m(A_i)\, m(V_i) \\ &= \frac{1}{n}\, m(A_i). \end{aligned}$$

Hence

$$m(E) \leq \frac{1}{n} \sum_{i=1}^{n} m(A_i).$$

Now the A_i's are essentially disjoint since for $i \neq j, m(A_i \cap A_j \cap E) = m(V_i \cap V_j \cap E) = 0$. Hence $m(E) \leq \frac{1}{n} < \varepsilon$. However condition (i) implies $m(E) \geq \varepsilon$.

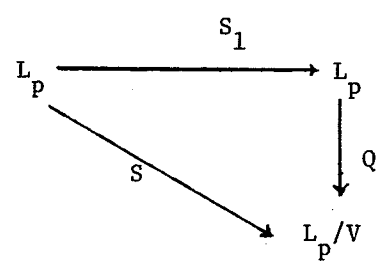
We have seen that if $L_p(A)$ is complemented in L_p then the quotient space $L_p/L_p(A) \cong L_p$. The converse to this result is rather more difficult [18]:

<u>Theorem 16</u>: <u>Suppose</u> $0 < p < 1$. <u>Then</u> $L_p(A)$ <u>is complemented in</u> L_p <u>if and only if</u> $L_p/L_p(A)$ <u>is isomorphic to</u> L_p.

To illustrate Theorem 16 take A to be the trivial algebra. Then $L_p(A)$ is a one-dimensional subspace of L_p, and $L_p/L_p(A)$ is not isomorphic to L_p. This result is in rather remarkable contrast to the situation when $p \geq 1$. In [21] a more general result is proved:

<u>Theorem 17</u>: <u>Suppose</u> $0 < p < 1$. <u>Let</u> V <u>and</u> W <u>be two finite- dimensional sub-spaces of</u> L_p. <u>Then</u> L_p/V <u>and</u> L_p/W <u>are isomorphic if and only if</u> $\dim V = \dim W$.

The proof of Theorem 17 is based on some reasonably familiar ideas from Banach space theory. The critical fact that is needed is that if $S: L_p \to L_p/V$ is a linear operator then it may be lifted to an operator $S_1: L_p \to L_p$ so that the following diagram commutes:



That this can be done depends on two facts. First L_p is locally like ℓ_p which is a projective space in the catefory of p-Banach spaces. Second, since $\dim V < \infty$, there is a necessary amount of compactness lying around. In the case $p = 1$, this type of argument (with V a dual Banach space) was first employed by Lindenstrauss [27], in showing that the kernel of a quotient map of ℓ_1 onto L_1 cannot be isomorphic to a dual space.

Now suppose L_p/W and L_p/V are isomorphic and let $S: L_p/W \to L_p/V$ be such an isomorphism.

$$\begin{array}{ccc} L_p & & L_p \\ \downarrow{\scriptstyle Q_1} & & \downarrow{\scriptstyle Q_2} \\ L_p/W & \xrightarrow{\quad S \quad} & L_p/V \end{array}$$

Lifting SQ_1 produces an operator $S_1: L_p \to L_p$; similarly lifting $S^{-1}Q_2$ produces an operator $S_2: L_p \to L_p$. Now S_2S_1 lifts the identity on L_p/W so that $S_2S_1 - I$ maps L_p into W. Now by Corollary 2 $S_2S_1 = I$ and similarly $S_1S_2 = I$. Now $S_1(W) \subset V$ and $S_2(V) \subset W$ so that $\dim V = \dim W$.

The converse direction in Theorem 17 is to establish that if $\dim V = \dim W$ then L_p/V and L_p/W are isomorphic. In fact what is required here is an automorphism U of L_p so that $U(V) = W$. This is closely related, of course, to the transitivity of L_p, and the reader is referred to [21] for the details.

The relationship between Theorem 16 and Theorem 17 suggests the possibility of replacing $L_p(A)$ in Theorem 16 by any subspace isomorphic to a space $L_p(A)$. In [24] it is shown that if $L_p/X \cong L_p$, then X has trivial dual. Hence we ask:

<u>Problem 5</u>: <u>Suppose</u> $0 < p < 1$. <u>Let</u> X <u>be a subspace of</u> L_p <u>such that</u> $L_p/X \cong L_p$. <u>Is</u> X <u>complemented in</u> L_p?

For $p = 1$, Problem 5 has a positive answer. Indeed the proof follows from simple lifting techniques as used in Theorem 17 and the fact that L_1 is complemented in its bidual.

Again the argument in Theorem 17 might suggest that Problem 5 could be resolved by moving X by an automorphism into a standard subspace $L_p(A)$.

Problem 6: Suppose $0 < p < 1$. Let X be a subspace of L_p, isomorphic to L_p. Does there exist a sub-σ-algebra A and an automorphism U of L_p so that $U(X) = L_p(A)$?

The obvious analogue of this for ℓ_p might be more tractable.

Problem 7: Suppose $0 < p < 1$. Let X and Y be two subspaces of L_p with $X \cong Y \cong \ell_p$. Does there exist an automorphism U of L_p with $U(X) = Y$?

We remark here first that if $X \cong Y \cong \ell_p$ then $L_p/X \cong L_p/Y$ implies the existence of an automorphism U of L_p with $U(X) = Y$.

Again to obtain a feeling for Problem 7 try the case $p = 1$. We clearly can take X to be complemented in L_1 and the question becomes: is every compy of ℓ_1 in L_1 complemented? This was recently answered by Bourgain [6], who gave an example of a non-complemented subspace of L_1 isomorphic to ℓ_1.

At the other extreme try $p = 0$. In place of ℓ_p we must use the space ω of all sequences. In this case Problem 7 has an affirmative answer, which was proved by Peck and Starbird [32]. Thus the evidence here is contradictory.

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