1. Sequence spaces and ideals. Let $\xi \in c_0$. Then its decreasing rearrangement $\xi^*$ is given by

$$\xi^*_n = \inf \{ \lambda > 0 : \{ k : |\xi_k| > \lambda \} < n \}.$$ 

A symmetric sequence space $E$ is a vector subspace of $c_0$ such that $\xi \in E$, $\eta \in c_0$ with $\eta^* \leq \xi^* \Rightarrow \eta \in E$.

If $E$ is a symmetric sequence space then $S_E$ is the ideal of compact operators $T$ on a separable Hilbert space $\mathcal{H}$ whose singular values satisfy

$$\{ s_n(T) \}_{n=1}^{\infty} \in E$$

$E \rightarrow S_E$ defines a correspondence between symmetric sequence spaces and ideals of compact operators. For example $\ell^p$ corresponds to the Schatten class $S_p$. 

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2. Traces and symmetric functionals. If $S_E$ is an ideal of compact operators then a \textit{trace} on $S_E$ is any linear functional $\tau : S_E \to \mathbb{C}$ such that

$$\tau(AB) = \tau(BA), \quad A \in S_E, \quad B \in \mathcal{B}(\mathcal{H}).$$

$\tau$ is called positive if $\tau(P) \geq 0$ for all positive $P \in S_E$. If $\tau$ is a trace then we can define a linear functional on $E$ by

$$\phi(\xi) = \tau(\text{diag}(\xi_1, \xi_2, \ldots)).$$

$\phi$ is then a \textit{symmetric} functional, i.e. $\phi(\xi) = \phi(\eta)$ if $\eta$ is a permutation of $\xi$.

(Figiel) There is a correspondence between traces on $S_E$ and symmetric functionals on $E$; precisely if $\phi$ is a symmetric functional on $E$ there is a unique trace $\tau$ on $S_E$ such that

$$\phi\left(\{s_n(P)\}_{n=1}^{\infty}\right) = \tau(P), \quad P \geq 0, \quad P \in S_E.$$

3. Nonstandard positive traces. The first construction of a nonstandard positive trace was by Dixmier. He takes $E$ to be a Marcinkiewicz space $M_\psi$ where $\psi$ is a concave function with $\psi(0) = 0$ and $\lim_{x \to \infty} \psi(x) = \infty$. $M_\psi$ consists of all sequences such that

$$\|\xi\|_{M_\psi} = \sup_n \frac{1}{\psi(n)} \sum_{k=1}^{n} \xi_k^s < \infty.$$

We are particularly interested in the case $\psi(x) = \log(1 + x)$ when we write $M_\psi = M_{\log}$.

We will write $M_\psi$ for the ideal $S_{M_\psi}$. $M_{\log}$ is the dual of the Matsaev ideal.

4. Dixmier traces

\textbf{Theorem 1} (Dixmier 1966). If

$$\lim_{n \to \infty} \frac{\psi(2n)}{\psi(n)} = 1$$

then there is a positive trace $\tau$ on $M_\psi$ defined by

$$\tau(P) = \omega\left(\frac{1}{\psi(2^n)} \sum_{k=1}^{2^n} s_k(P)\right), \quad P \in M_\psi, \quad P \geq 0.$$ 

Here $\omega$ is a (translation invariant) Banach limit.

Traces of this type are called \textit{Dixmier traces}.

If $\psi(x) = \log(1 + x)$ then $\tau(\text{diag}\{1/n\}_{n=1}^{\infty}) = 1$ for every Dixmier trace.

5. Positive traces II

\textbf{Theorem 2} (Dodds, de Pagter, Semenov and Sukochev 1988). The condition

$$\liminf_{n \to \infty} \frac{\psi(2n)}{\psi(n)} = 1$$

is necessary and sufficient for the existence of a positive trace on $M_\psi$.

Dixmier traces play a significant role in noncommutative geometry via the \textit{Connes trace theorem}. Let us explain (in a very simplified case) how this goes.
6. The Connes trace theorem. Let $X$ be a compact Riemannian manifold of dimension $d$. (For example $X = \mathbb{T}^d$.) Let $\Delta$ denote the negative Laplacian on $X$, and let $dx$ denote the standard volume measure on $X$. For $f \in L_\infty(X)$ we define

$$T_f(g) = fg \quad \text{for} \quad g \in L_2(X).$$

We now state “toy” version of the Connes trace formula.

**Theorem 3 (Connes 1988).** Then for any $f \in C^\infty(X)$, and any Dixmier trace on $M_{\log}$ we have $T_f(1 + \Delta)^{-d/2} \in M_{\log}$ and

$$\tau(T_f(1 + \Delta)^{-d/2}) = \frac{\Omega_d}{d(2\pi)^d} \int_X f(x) \, dx.$$

Here $\Omega_d$ is the surface area of the $(d-1)$-dimensional sphere.

The correct statement is for classical pseudo-differential operators and considers integration over the cosphere bundle.

7. General traces. We now consider general (not necessarily positive or continuous) traces. If $E$ is any symmetric sequence space we define the commutator subspace of $S_E$, $\text{Com} S_E$, to be the linear span of all commutators $[A, B] = AB - BA$ for $A \in S_E$ and $B \in B(H)$. A linear functional $\tau$ is a trace if and only if it annihilates $\text{Com} S_E$.

The problem of existence of non-trivial traces on $S_p$ was first considered by Pearcy and Topping in 1971 who showed that $\text{Com} S_p = S_p$ when $p > 1$, so that the only trace on these ideals is the zero trace. If $p < 1$, Anderson (1986) showed that

$$\text{Com} S_p = \{ T \in S_p : \text{tr} T = d \}$$

so that the only traces are multiples of the standard trace.

The case $p = 1$ is more tricky. Weiss (1980) showed that there are discontinuous traces so that $\text{Com} S_1 \neq \{ T \in S_1 : \text{tr} T = 0 \}$. Kalton (1989) gave a complete characterization of $\text{Com} S_1$.

8. General traces II. In fact in the middle of the 1990’s there appeared a complete characterization of $\text{Com} \mathcal{J}$ for any ideal $\mathcal{J}$. This was not published till 2004.

From now if $T$ is a compact operator we write $\{ \lambda_n(T) \}_{n=1}^{\infty}$ for the eigenvalues of $T$, repeated according to algebraic multiplicity and arranged in (some) order of decreasing absolute value so that

$$|\lambda_1(T)| \geq |\lambda_2(T)| \geq \ldots .$$

If there are only finitely many eigenvalues the list is completed with zeros.

**Theorem 4 (Dykema, Figiel, Weiss, Wodzicki 1995, 2004).** Let $\mathcal{J} = S_E$. Let $T \in \mathcal{J}$ be a normal operator. Then $T \in \text{Com} \mathcal{J}$ if and only if

$$\left\{ \frac{\lambda_1 + \ldots + \lambda_n}{n} \right\}_{n=1}^{\infty} \in E$$

where $\lambda_j = \lambda_j(T)$. 
9. General traces III. The DFWW theorem implies that $\text{Com} \mathcal{J} = \mathcal{J}$ (or, equivalently, $\mathcal{J}$ admits no nonzero traces) if $\mathcal{J} = \mathcal{S}_E$ where $\xi \in E$ implies that $\{\frac{1}{n}(\xi_1^* + \ldots + \xi_n^*)\}_{n=1}^{\infty} \in E$.

In the same paper the authors show that $T \in \text{Com} \mathcal{J}$ if and only if $T$ can be expressed as the sum of at most 3 commutators $[A_j, B_j]$ where $A_j \in \mathcal{J}$ and $B_j \in \mathcal{B}(\mathcal{H})$.

10. Formulas for nonstandard traces. Suppose $\tau$ is a trace on $\mathcal{J} = \mathcal{S}_E$. The DFWW theorem implies that to calculate $\tau(T)$ we should write $T = H + iK$ where $H, K$ are hermitian and then

$$\tau(T) = \tau(\text{diag}\{\lambda_n(H) + i\lambda_n(K)\}_{n=1}^{\infty}).$$

Are there other natural formulas?

**Theorem 5 (Kalton 1998).** Suppose $E$ is a Banach or quasi-Banach sequence space. Then

$$\tau(T) = \tau(\text{diag}\{\lambda_n(T)\}_{n=1}^{\infty}).$$

This theorem fails for arbitrary ideals (Dykema–Kalton 1998). This can be expressed by saying that if $E$ is quasi-Banach then a form of Lidskii’s theorem holds, i.e. all traces are spectral.

11. Taking the diagonal. When can we compute $\tau$ just from the diagonal of the operator? Let $(e_n)_{n=1}^{\infty}$ be an orthonormal basis of $\mathcal{H}$; can we compute

$$\tau(T) = \tau(\text{diag}\{(Te_n, e_n)\}_{n=1}^{\infty})?$$

Not in general of course.

**Theorem 6 (Kalton, Lord, Potapov, Sukochev 2010).** Let $A$ be a positive compact operator on $\mathcal{H}$. Let $(e_n)_{n=1}^{\infty}$ be an orthonormal basis of eigenvectors for $\mathcal{H}$. Let $\tau$ be any trace on $\mathcal{J}_A$ (the smallest ideal containing $A$). Then for any $T \in \mathcal{B}(\mathcal{H})$ we have

$$\tau(TA) = \tau(AT) = \tau(\text{diag}\{(T Ae_n, e_n)\}_{n=1}^{\infty}).$$

12. Idea of the proof. We can take $T$ to be hermitian. Let $(f_n)_{n=1}^{\infty}$ be an orthonormal basis of eigenvectors for $H = \frac{1}{2}(TA + AT)$ arranged so that $H f_n = \mu_n f_n$ with $|\mu_n|$ decreasing. We similarly suppose that $A e_n = \lambda_n e_n$ with $\lambda_n$ decreasing.

The key is to estimate the difference

$$\sum_{k=1}^{n} (H f_k, f_k) - \sum_{k=1}^{n} (He_k, e_k).$$

To do this we let $E_n$ be the span of $\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$ and $P_n$ be the orthogonal projection on $E_n$. Then

$$\left|\sum_{k=1}^{n} (H f_k, f_k) - \text{tr}(P_n H)\right|, \left|\sum_{k=1}^{n} (He_k, e_k) - \text{tr}(P_n H)\right| \leq n\lambda_n \|T\|.$$

These estimates (and a little more) and the DFWW theorem allow us to show that $\text{diag}\{(H f_k, f_k) - (He_k, e_k)\}_{k=1}^{\infty} \in \text{Com} \mathcal{J}_A$. 

13. **Back to the Connes trace formula.** Let $X$ be a compact Riemannian manifold of dimension $d$. (For example $X = \mathbb{T}^d$). Let $\Delta$ denote the Laplacian on $X$, and let $dx$ denote the standard volume measure on $X$. For $f \in L_\infty(X)$ we define $T_{f g} = f g$ for $g \in L_2(X)$.

**Theorem 7 (Connes 1988).** For any $f \in C^\infty(X)$ and any Dixmier trace on $\mathcal{M}_{\log}$ we have $T_f(1 + \Delta)^{-d/2} \in \mathcal{M}_{\log}$ and

$$\tau(T_f(1 + \Delta)^{-d/2}) = \frac{\Omega_d}{d(2\pi)^d} \int_X f(x) \, dx.$$ 

Here $\Omega_d$ is the surface area of the $(d - 1)$-dimensional sphere.

Let us take $X = \mathbb{T}^d$ and use the preceding ideas.

14. **The Connes trace formula for the torus.** In this case $A = (1 + \Delta)^{-d/2}$ and so $\mathcal{A}_A = S_{1,\infty}$ is associated with the weak-$\ell^1$ sequence space $\ell_{1,\infty} = \{ \xi : \xi^* = O(1/n) \}$. This is strictly contained in the Marcinkiewicz space $\mathcal{M}_{\log}$.

The basis of eigenvectors is given by $e_n(2\pi)^{-d/2} e^{i(n,\theta)}$ for $n \in \mathbb{Z}^d$. Here $n = (n_1, \ldots, n_d) \in \mathbb{Z}^d$ and $\theta = (\theta_1, \ldots, \theta_d) \in (-\pi, \pi]^d$.

If $T = T_f$ is a multiplication operator with $f \in L_\infty(\mathbb{T}^d)$ then

$$(T_f e_n, e_n) = \frac{1}{(2\pi)^d} \int f(x) \, dx$$

Hence if $|n| = (n_1^2 + \ldots + n_d^2)^{1/2}$,

$$(T_f(1 + \Delta)^{-d/2} e_n, e_n) = \frac{1}{(2\pi)^d} (1 + |n|^2)^{-d/2} \int f(x) \, dx.$$ 

15. **The Connes trace formula for the torus II.** Thus if $\tau$ is any trace on $S_{1,\infty}$ we have

$$\tau(T_f(1 + \Delta)^{-d/2}) = \frac{1}{(2\pi)^d} \tau(\{ (1 + |n|^2)^{-d/2} \}_{n \in \mathbb{Z}^d}) \int f(x) \, dx.$$ 

If $\tau$ is normalized so that $\tau(\{1/n\}_{n=1}^\infty) = 1$ (e.g. a Dixmier trace) one can evaluate

$$\tau(\{ (1 + |n|^2)^{-d/2} \}_{n \in \mathbb{Z}^d}) = d^{-1} \Omega_d.$$ 

This proves the Connes trace formula for $L_\infty$-functions (not just $C^\infty$-functions) and for every (perhaps discontinuous) trace on $S_{1,\infty}$ (not just Dixmier traces).

For Dixmier traces the extension to $L_\infty$-functions was proved in Lord-Potapov-Sukochev 2010.

16. **Eigenvalues.** Let us interpret this extension of Connes’s trace formula in terms of the eigenvalues of $S = T_f(1 + \Delta)^{-d/2}$. Let $\lambda_n = \lambda_n(S)$.

**Theorem 8 (Kalton, Lord, Potapov, Sukochev 2010).** There exists a constant $C$ such that

$$\left| \sum_{k=1}^n \lambda_k - \frac{\Omega_d \log n}{d(2\pi)^d} \int f(x) \, dx \right| \leq C.$$ 

Note that $\ell_{1,\infty}$ is quasi-Banach so all traces are spectral.
17. An improvement

**Theorem 9** (Kalton, Lord, Potapov, Sukochev 2010). Let \((e_n)_{n=1}^\infty\) be an orthonormal basis of \(H\) and let \(T : H \to H\) be a linear operator such that for some constant \(C\) we have
\[
\sum_{k=n+1}^\infty \|Te_k\|^2 \leq C.
\]

Then:
- \(T \in S_1, \infty\),
- \(\sup_n \left| \sum_{k=1}^n \lambda_k(T) - \sum_{k=1}^n (Te_k, e_k) \right| < \infty\),
- \(\tau(T) = \tau(\{(Te_k, e_k)\}_{k=1}^\infty)\) for every trace \(\tau\) on \(S_1, \infty\).

18. Operators on \(L_2(\mathbb{R}^d)\). We consider operators \(T : L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)\) of the form
\[
Tf(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i(x,\xi)p_T(x,\xi)} \hat{f}(\xi) \, d\xi.
\]

\(T\) is compactly supported if there is a compact set \(K\) so that \(p_T(x,\xi) = 0\) for \(x \notin K\) and \(Tf = 0\) if \(f = 0\) on \(K\).

\(T\) is pseudo-differential operator of order \(m\) if \(p_T\) is \(C^\infty\) and satisfies estimates of the type
\[
|\partial_x^\alpha \partial_\xi^\beta p_T(x,\xi)| \leq C_{\alpha,\beta} (1 + |\xi|)^{m-|\beta|}.
\]

However we do not need to consider such smooth kernels . . . We assume only that \(p_T\) is measurable.

19. Operators on \(L_2(\mathbb{R}^d)\) II

**Theorem 10** (Kalton, Lord, Potapov, Sukochev 2010). Suppose for some constant \(C\)
\[
\int_{\mathbb{R}^d} \int_{|\xi| \geq t} |p_T(x,\xi)|^2 \, d\xi \, dx \leq C t^{-d}
\]
and
\[
Tf(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i(x,\xi)p_T(x,\xi)} \hat{f}(\xi) \, d\xi
\]
is compactly supported. For example \(T\) could be a pseudo-differential operator of order \(-d\). Then
- \(T \in S_1, \infty\),
- \(\sup_n \left| \sum_{k=1}^n \lambda_k(T) - \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \int_{|\xi| \leq n^{1/d}} p_T(x,\xi) \, d\xi \, dx \right| < \infty\).

20. The Connes trace formula again

**Theorem 11** (Kalton, Lord, Potapov, Sukochev 2010). Let \(X\) be a compact Riemannian manifold and consider the operator \(S = Tf(1 + \Delta)^{-d/2}\) where \(f \in L_2(X)\). Then \(S \in S_1, \infty\). If \(\tau\) is a normalized trace on \(S_1, \infty\) we have
\[
\tau(S) = \frac{\Omega_d}{d(2\pi)^d} \int_X f(x) \, dx
\]
or, equivalently
\[
\sup_n \left| \sum_{k=1}^{n} \lambda_k(S) - \frac{\Omega_d \log n}{d(2\pi)^d} \int_X f(x) \, dx \right| < \infty.
\]

Notice now we require \( f \in L_2(X) \) (not \( f \in L_\infty(X) \)) so that \( T_f \) is potentially unbounded.

21. \( L_p \) for \( p < 2 \)? One cannot expect \( T_f(1 + \Delta)^{-d/2} \) be necessarily even a bounded operator if \( p < 2 \). However one can consider \( (1 + \Delta)^{-d/4} T_f(1 + \Delta)^{-d/4} \). All the previous results would have worked in this case.

**Theorem 12** (Lord, Potapov and Sukochev 2010). *If \( f \in L_p \) where \( p > 1 \) then \( (1 + \Delta)^{-d/4} T_f(1 + \Delta)^{-d/4} \in \mathcal{M}_{\log} \) and if \( \tau \) is a Dixmier trace on \( \mathcal{M}_{\log} \) one still has the trace formula.*

However for \( f \in L_1 \) it is not necessarily true that \( (1 + \Delta)^{-d/4} T_f(1 + \Delta)^{-d/4} \) is even a bounded operator.

22. \( p < 2 \)

**Theorem 13** (Kalton, Lord, Potapov, Sukochev 2010). *Suppose \( X = S^d \) or \( X = T^d \). Suppose \( f \in L(\log L)^2(\log \log L)(X) \), i.e.*
\[
\int |f(x)| |(\log_+ |f(x)|)|^2 \log_+ \log_+ |f(x)| \, dx < \infty.
\]
*Then \( (1 + \Delta)^{-d/4} T_f(1 + \Delta)^{-d/4} \in \mathcal{M}_{\log} \) and for every normalized trace \( \tau \) on \( \mathcal{M}_{\log} \), we have \( \tau((1 + \Delta)^{-d/4} T_f(1 + \Delta)^{-d/4}) = \frac{\Omega_d}{d(2\pi)^d} \int f(x) \, dx \).*